



0.4 TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Many phenomena encountered in your daily life are **waves**. For instance, music is transmitted from radio stations in the form of electromagnetic waves. Your radio receiver decodes these electromagnetic waves and causes a thin membrane inside the speakers to vibrate. This vibration, in turn, creates pressure waves in the air. When these waves reach your ears, you hear the music from your radio (see Figure 0.48). Each of these waves is **periodic**, meaning that the basic shape of the wave is repeated over and over again. The mathematical description of such phenomena involves periodic functions, the most familiar of which are the trigonometric functions. First, we remind you of a basic definition.

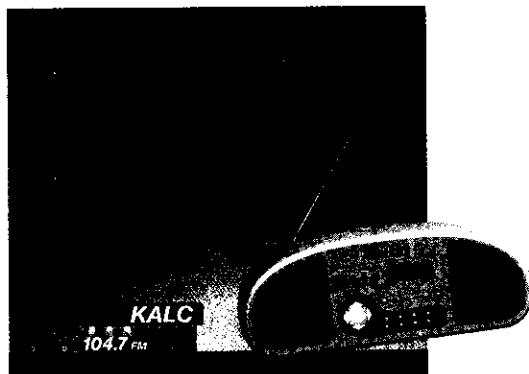


FIGURE 0.48
Radio and sound waves.

NOTES

When we discuss the period of a function, we most often focus on the fundamental period.

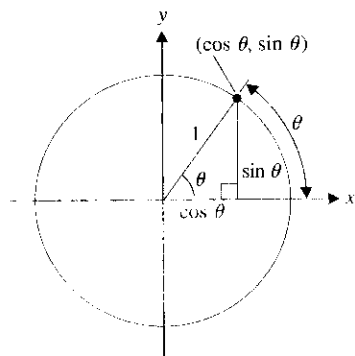


FIGURE 0.49
Definition of $\sin \theta$ and $\cos \theta$: $\cos \theta = x$ and $\sin \theta = y$.

DEFINITION 4.1

A function f is **periodic** of **period** T if

$$f(x + T) = f(x)$$

for all x such that x and $x + T$ are in the domain of f . The smallest such number $T > 0$ is called the **fundamental period**.

There are several equivalent ways of defining the sine and cosine functions. Regardless of how you were originally taught these functions, we want to emphasize a simple definition from which you can easily reproduce many of the basic properties of these functions. Referring to Figure 0.49, begin by drawing the unit circle $x^2 + y^2 = 1$. Let θ be the angle measured (counterclockwise) from the positive x -axis to the line segment joining the point (x, y) on the circle and the origin. Here, we measure θ in **radians**, where the radian measure of the angle θ is the length of the arc indicated in the figure. Again referring to Figure 0.49, we define $\sin \theta$ to be the y -coordinate of the point on the circle and $\cos \theta$ to be the x -coordinate of the point. Notice that from this definition, it follows that $\sin \theta$ and $\cos \theta$ are defined for all values of θ . That is, the domains of $f(\theta) = \sin \theta$ and of $g(\theta) = \cos \theta$ are both $-\infty < \theta < \infty$ (the entire real line). Observe that the range for each of these functions is the interval $[-1, 1]$.

Note that since the circumference of a circle ($C = 2\pi r$) of radius 1 is 2π , we have that 360° corresponds to 2π radians. Similarly, 180° corresponds to π radians, 90° corresponds to $\pi/2$ radians, and so on. In the table that follows, we list some common angles as measured in degrees together with the corresponding radian measure.

REMARK 4.1

Unless otherwise noted, we always measure angles in radians.

Angle in degrees	0°	30°	45°	60°	90°	135°	180°	270°	360°
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

THEOREM 4.1

The functions $f(\theta) = \sin \theta$ and $g(\theta) = \cos \theta$ are periodic, of period 2π .

PROOF

Since a complete circle is 2π radians, adding 2π to any angle takes you all the way around the circle and back to the same point (x, y) . This says that

$$\sin(\theta + 2\pi) = \sin \theta$$

and

$$\cos(\theta + 2\pi) = \cos \theta,$$

for all values of θ . Furthermore, 2π is the smallest angle for which this is true. ■

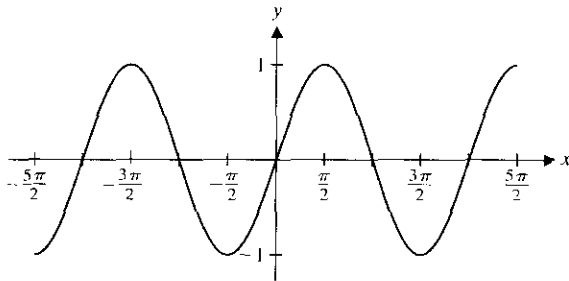


FIGURE 0.50a
 $y = \sin x$.

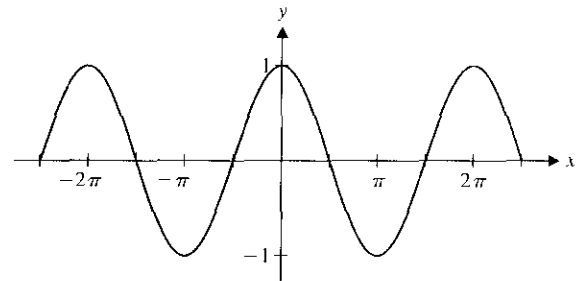


FIGURE 0.50b
 $y = \cos x$.

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{7\pi}{6}$	$-\frac{1}{2}$	0
2π	0	1

You are likely already familiar with the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ shown in Figures 0.50a and 0.50b, respectively.

Notice that you could slide the graph of $y = \sin x$ slightly to the left or right and get an exact copy of the graph of $y = \cos x$. Specifically, you should observe that

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x.$$

REMARK 4.2

Instead of writing $(\sin \theta)^2$ or $(\cos \theta)^2$, we usually use the notation $\sin^2 \theta$ and $\cos^2 \theta$, respectively.

The accompanying table lists some common values of sine and cosine. Notice that many of these can be found by closely examining Figure 0.49.

EXAMPLE 4.1 Solving Equations Involving Sines and Cosines

Find all solutions of the equations (a) $2 \sin x - 1 = 0$ and (b) $\cos^2 x - 3 \cos x + 2 = 0$.

Solution For (a), notice that $2 \sin x - 1 = 0$ if $2 \sin x = 1$ or $\sin x = \frac{1}{2}$. From the unit circle, we find that $\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Since $\sin x$ has period 2π , additional solutions are $\frac{\pi}{6} + 2\pi$, $\frac{5\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$ and so on. A convenient way of indicating that any integer multiple of 2π can be added to either solution is to write: $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, for any integer n . Part (b) may look rather difficult at first. However, notice that it looks like a quadratic equation using $\cos x$ instead of x . With this clue, you can factor the left-hand side to get

$$0 = \cos^2 x - 3 \cos x + 2 = (\cos x - 1)(\cos x - 2),$$

from which it follows that either $\cos x = 1$ or $\cos x = 2$. Since $-1 \leq \cos x \leq 1$ for all x , the equation $\cos x = 2$ has no solution. However, we get $\cos x = 1$ if $x = 0, 2\pi$ or any integer multiple of 2π . Notice that we can summarize all the solutions by writing $x = 2n\pi$, for any integer n . ■

We now give definitions of the remaining four trigonometric functions.

DEFINITION 4.2

The **tangent** function is defined by $\tan x = \frac{\sin x}{\cos x}$.

The **cotangent** function is defined by $\cot x = \frac{\cos x}{\sin x}$.

The **secant** function is defined by $\sec x = \frac{1}{\cos x}$.

The **cosecant** function is defined by $\csc x = \frac{1}{\sin x}$.

We give graphs of these functions in Figures 0.51a, 0.51b, 0.51c and 0.51d. Notice in each graph the locations of the vertical asymptotes. For the “co” functions $\cot x$ and $\csc x$, the division by $\sin x$ causes vertical asymptotes at $0, \pi, 2\pi$ and so on (where $\sin x = 0$). For $\tan x$ and $\sec x$, the division by $\cos x$ produces vertical asymptotes at $\pi/2, 3\pi/2, 5\pi/2$ and so on (where $\cos x = 0$). Once you have the vertical asymptotes in place, the graphs are relatively easy to draw.

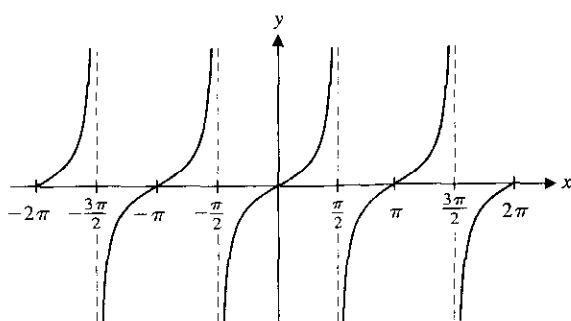


FIGURE 0.51a
 $y = \tan x$.

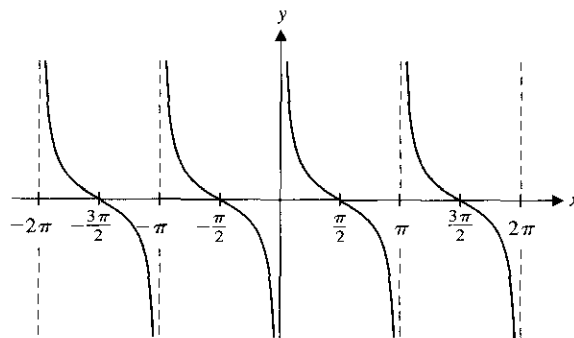


FIGURE 0.51b
 $y = \cot x$.

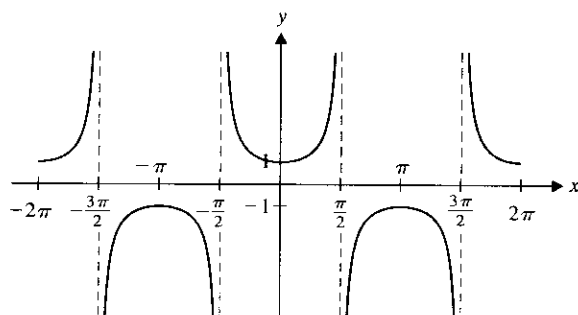


FIGURE 0.51c
 $y = \sec x$.

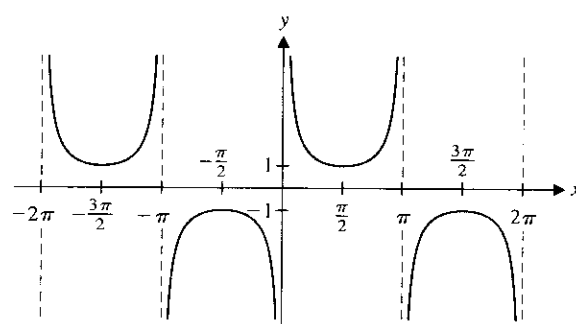


FIGURE 0.51d
 $y = \csc x$.

Notice that $\tan x$ and $\cot x$ are periodic, of period π , while $\sec x$ and $\csc x$ are periodic, of period 2π .

REMARK 4.3

Most calculators have keys for the functions $\sin x$, $\cos x$, and $\tan x$, but not for the other three trigonometric functions. This reflects the central role that $\sin x$, $\cos x$ and $\tan x$ play in applications. To calculate function values for the other three trigonometric functions, you can simply use the identities

$$\cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \csc x = \frac{1}{\sin x}.$$

It is important to learn the effect of slight modifications of these functions. We present a few ideas here and in the exercises.

EXAMPLE 4.2 Altering Amplitude and Period

Graph $y = 2 \sin x$ and $y = \sin 2x$ and describe how each differs from the graph of $y = \sin x$ (see Figure 0.52a).

Solution The graph of $y = 2 \sin x$ is given in Figure 0.52b. Notice that this graph is similar to the graph of $y = \sin x$, except that the y -values oscillate between -2 and 2

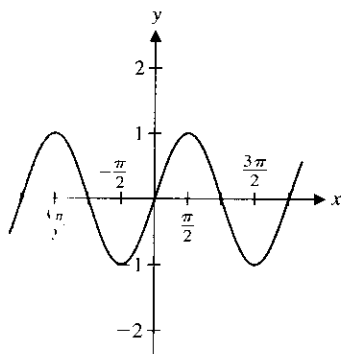


FIGURE 0.52a
 $y = \sin x$.

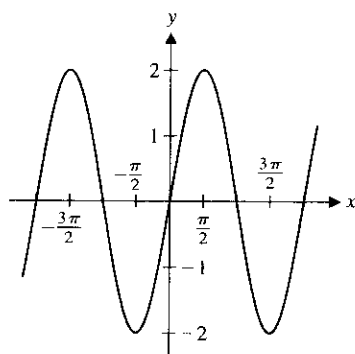


FIGURE 0.52b
 $y = 2 \sin x$.

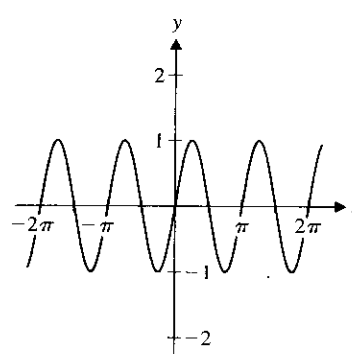


FIGURE 0.52c
 $y = \sin(2x)$.

instead of -1 and 1 . Next, the graph of $y = \sin 2x$ is given in Figure 0.52c. In this case, the graph is similar to the graph of $y = \sin x$, except that the period is π instead of 2π . (The oscillations occur twice as fast.) ■

The results in example 4.2 can be generalized. For $A > 0$, the graph of $y = A \sin x$ oscillates between $y = -A$ and $y = A$. In this case, we call A the **amplitude** of the sine curve. Notice that for any positive constant c , the period of $y = \sin cx$ is $2\pi/c$. Similarly, for the function $A \cos cx$, the amplitude is A and the period is $2\pi/c$.

The sine and cosine functions can be used to model sound waves. A pure tone (think of a single flute note) is a pressure wave described by the sinusoidal function $A \sin ct$. (Here, we are using the variable t , since the air pressure is a function of *time*.) The amplitude A determines how loud the tone is perceived to be, and the period determines the pitch of the note. In this setting, it is usually convenient to talk about the **frequency** $f = c/2\pi$ rather than the period. As you may know, the higher the frequency is, the higher the pitch of the note will be. (Frequency is measured in hertz, where 1 hertz equals 1 cycle per second.) Note that the frequency is simply the reciprocal of the period.

EXAMPLE 4.3 Finding Amplitude, Period, and Frequency

Find the amplitude, period, and frequency of (a) $f(x) = 4 \cos 3x$ and (b) $g(x) = 2 \sin(x/3)$.

Solution (a) For $f(x)$, the amplitude is 4, the period is $2\pi/3$ and the frequency is $3/(2\pi)$ (see Figure 0.53a). (b) For $g(x)$, the amplitude is 2, the period is $2\pi/(1/3) = 6\pi$ and the frequency is $1/(6\pi)$ (see Figure 0.53b).

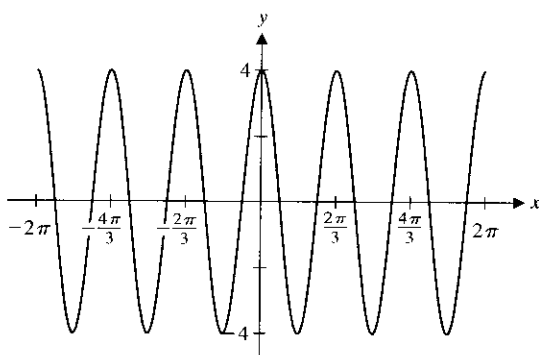


FIGURE 0.53a
 $y = 4 \cos 3x$.

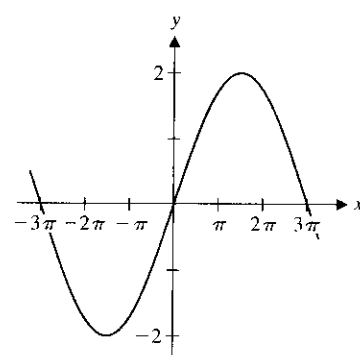


FIGURE 0.53b
 $y = 2 \sin(x/3)$.

As you will recall from your earlier study of trigonometry, there are numerous formulas, or **identities**, that are helpful in manipulating the trigonometric functions. You should observe from the definition of $\sin \theta$ and $\cos \theta$ (see Figure 0.49) that the Pythagorean Theorem gives us the familiar identity

$$\sin^2 \theta + \cos^2 \theta = 1,$$

since the hypotenuse of the indicated triangle is 1. This holds for any angle θ .

We list several important identities in Theorem 4.2.

THEOREM 4.2

For any real numbers α and β , the following identities hold:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha \quad (4.1)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (4.2)$$

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha) \quad (4.3)$$

$$\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha). \quad (4.4)$$

From the basic identities summarized in Theorem 4.2, numerous other useful identities can be derived. We derive two of these in example 4.4.

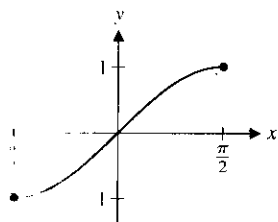
EXAMPLE 4.4 Deriving New Trigonometric Identities

Derive the identities $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$.

Solution These can be obtained from formulas (4.1) and (4.2), respectively, by substituting $\alpha = \theta$ and $\beta = \theta$. Alternatively, the identity for $\cos 2\theta$ can be obtained by subtracting equation (4.3) from equation (4.4). ■

□ The Inverse Trigonometric Functions

We can expand the set of functions available to you by defining inverses to the trigonometric functions. To get started, let's again look at a graph of $y = \sin x$ (see Figure 0.50a). Notice that we cannot define an inverse function, since $\sin x$ is not one-to-one. Although the sine function as we have defined it does not have an inverse function, we can define one by modifying the domain of the sine. We do this by choosing a portion of the sine curve that passes the horizontal line test. In particular, we select an interval of x -values on which the sine increases from -1 to 1 , without repeating any of those values. If we restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $y = \sin x$ is one-to-one there (see Figure 0.54) and hence, has an inverse. We thus define the **inverse sine function** by

**FIGURE 0.54**

$y = \sin^{-1} x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}. \quad (4.5)$$

It is convenient to think of this definition as follows. If $y = \sin^{-1} x$, then y is the angle (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) for which $\sin y = x$. You should note that we could have selected any interval on which $\sin x$ is one-to-one, but $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is the most convenient. To see that these are indeed inverse functions, you should observe that

$$\sin(\sin^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\sin^{-1}(\sin x) = x, \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad (4.6)$$

Read equation (4.6) very carefully. It *does not* say that $\sin^{-1}(\sin x) = x$ for *all* x , but rather, *only* for those in the restricted domain, $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For instance, $\sin^{-1}(\sin \pi) \neq \pi$, since

$$\sin^{-1}(\sin \pi) = \sin^{-1}(0) = 0.$$

REMARK 4.4

Mathematicians often use the notation **arcsin x** in place of $\sin^{-1} x$. People will read $\sin^{-1} x$ interchangeably as “inverse sine of x ” or “arcsine of x .”

EXAMPLE 4.5 Evaluating the Inverse Sine Function

Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution For (a), we look for the angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for which $\sin \theta = \frac{\sqrt{3}}{2}$. Note that since $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. For (b), note that $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$ and $-\frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

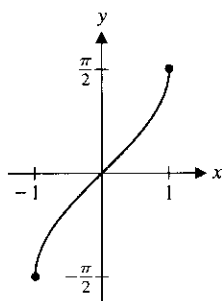


FIGURE 0.55
 $y = \sin^{-1} x$.

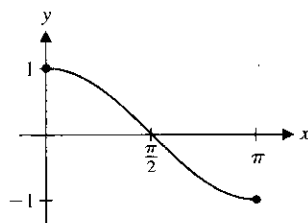


FIGURE 0.56
 $y = \cos x$ on $[0, \pi]$.

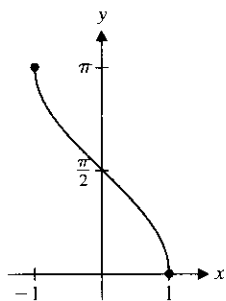


FIGURE 0.57
 $y = \cos^{-1} x$.

Judging by the preceding two examples, you might think that (4.5) is a roundabout way of defining a function. If so, you've got the idea exactly. In fact, we want to emphasize that what we know about the inverse sine function is principally through reference to the sine function. We will not have any other definition of arcsine, nor are there any algebraic formulas for this function. (These things are true of most inverse functions.) Further, you should recall from our discussion in section 0.3 that we can draw a graph of $y = \sin^{-1} x$ simply by reflecting the graph of $y = \sin x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (from Figure 0.54) through the line $y = x$ (see Figure 0.55).

Turning to $y = \cos x$, can you think of how to restrict the domain to make the function one-to-one? Notice that restricting the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, as we did for the inverse sine function will not work here. (Why not?) The simplest way to do this is to restrict its domain to the interval $[0, \pi]$ (see Figure 0.56). Consequently, we define the **inverse cosine** function by

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \quad \text{and} \quad 0 \leq y \leq \pi.$$

Note that here, we have

$$\cos(\cos^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\cos^{-1}(\cos x) = x, \quad \text{for all } x \in [0, \pi].$$

As with the definition of arcsine, it is helpful to think of $\cos^{-1} x$ as that angle θ in $[0, \pi]$ for which $\cos \theta = x$. As with $\sin^{-1} x$, it is common to use $\cos^{-1} x$ and $\arccos x$ interchangeably.

EXAMPLE 4.6 Evaluating the Inverse Cosine Function

Evaluate (a) $\cos^{-1}(0)$ and (b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution For (a), you will need to find that angle θ in $[0, \pi]$ for which $\cos \theta = 0$. It's not hard to see that $\cos^{-1}(0) = \frac{\pi}{2}$. If you calculate this on your calculator and get 90, your calculator is in degrees mode. In this event, you should immediately change it to radians mode. For (b), look for the angle $\theta \in [0, \pi]$ for which $\cos \theta = -\frac{\sqrt{2}}{2}$. Notice that $\cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$ and $\frac{3\pi}{4} \in [0, \pi]$. Consequently,

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}.$$

Once again, we obtain the graph of this inverse function by reflecting the graph of $y = \cos x$ on the interval $[0, \pi]$ (seen in Figure 0.56) through the line $y = x$ (see Figure 0.57).

We can define inverses for each of the four remaining trigonometric functions in similar ways. For $y = \tan x$, we restrict the domain to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Think about why the endpoints of this interval are not included (see Figure 0.58). Having done this, you should readily see that we define the **inverse tangent** function by

$$y = \tan^{-1} x \quad \text{if and only if } \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The graph of $y = \tan^{-1} x$ is then as seen in Figure 0.59, found by reflecting the graph in Figure 0.58 through the line $y = x$.

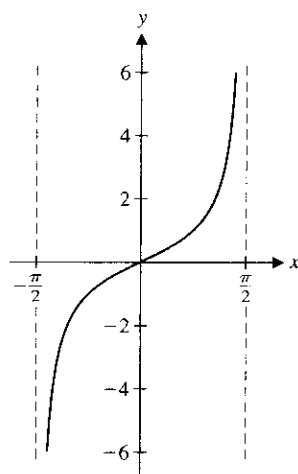


FIGURE 0.58
 $y = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

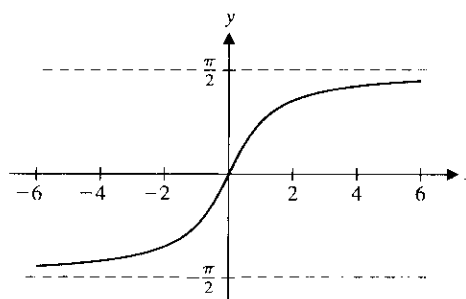


FIGURE 0.59
 $y = \tan^{-1} x$.

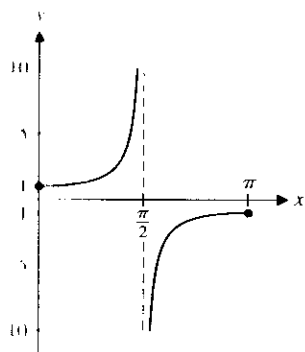


FIGURE 0.60
 $y = \sec x$ on $[0, \pi]$.

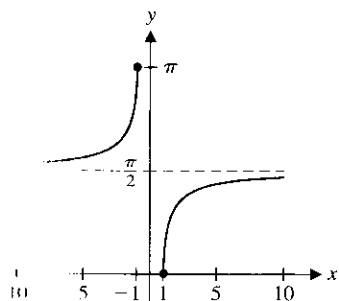


FIGURE 0.61
 $y = \sec^{-1} x$.

EXAMPLE 4.7 Evaluating an Inverse Tangent

Evaluate $\tan^{-1}(1)$.

Solution You must look for the angle, θ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ for which $\tan \theta = 1$. This is easy enough. Since $\tan(\frac{\pi}{4}) = 1$ and $\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have that $\tan^{-1}(1) = \frac{\pi}{4}$. ■

We now turn to defining an inverse for $\sec x$. First, we must issue a disclaimer. As we have indicated, there are any number of ways to suitably restrict the domains of the trigonometric functions in order to make them one-to-one. With the first three we've seen, there has been an obvious choice of how to do this and there is general agreement among mathematicians on the choice of these intervals. In the case of $\sec x$, this is *not* true. There are several reasonable ways in which to suitably restrict the domain and different authors restrict these differently. We have (arbitrarily) chosen to restrict the domain to be $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. You might initially think that this looks strange. Why not use all of $[0, \pi]$? You need only think about the definition of $\sec x$ to see why we needed to exclude the point $x = \frac{\pi}{2}$. See Figure 0.60 for a graph of $\sec x$ on this domain. (Note the vertical asymptote at $x = \frac{\pi}{2}$.) Consequently, we define the **inverse secant** function by

$$y = \sec^{-1} x \quad \text{if and only if } \sec y = x \text{ and } y \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi].$$

A graph of $\sec^{-1} x$ is shown in Figure 0.61.

REMARK 4.5

We can likewise define inverses to $\cot x$ and $\csc x$. As these functions are used only infrequently, we will omit them here and examine them in the exercises.

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

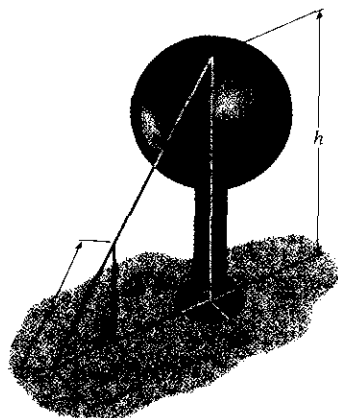


FIGURE 0.62
Height of a tower.

EXAMPLE 4.8 Evaluating an Inverse Secant

Evaluate $\sec^{-1}(-\sqrt{2})$.

Solution You must look for the angle θ with $\theta \in [0, \frac{\pi}{2}] \cup (\frac{\pi}{2}, \pi]$, for which $\sec \theta = -\sqrt{2}$. Notice that if $\sec \theta = -\sqrt{2}$, then $\cos \theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. Recall from example 4.6 that $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$. Further, the angle $\frac{3\pi}{4}$ is in the interval $(\frac{\pi}{2}, \pi]$ and so, $\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$. ■

Calculators do not usually have built-in functions for $\sec x$ or $\sec^{-1} x$. In this case, you must convert the desired secant value to a cosine value and use the inverse cosine function, as we did in example 4.8.

We summarize the three main inverse trigonometric functions in the margin.

In many applications, we need to calculate the length of one side of a right triangle using the length of another side and an **acute** angle (i.e., an angle between 0 and $\frac{\pi}{2}$ radians). We can do this rather easily, as in example 4.9.

EXAMPLE 4.9 Finding the Height of a Tower

A person 100 feet from the base of a water tower measures an angle of 60° from the ground to the top of the tower (see Figure 0.62). (a) Find the height of the tower. (b) What angle is measured if the person is 200 feet from the base?

Solution For (a), we first convert 60° to radians:

$$60^\circ = 60 \frac{\pi}{180} = \frac{\pi}{3} \text{ radians.}$$

We are given that the base of the triangle in Figure 0.62 is 100 feet. We must now compute the height of the tower h . Using the similar triangles indicated in Figure 0.62, we have that

$$\frac{\sin \theta}{\cos \theta} = \frac{h}{100},$$

so that the height of the tower is

$$h = 100 \frac{\sin \theta}{\cos \theta} = 100 \tan \theta = 100 \tan \frac{\pi}{3} = 100\sqrt{3} \approx 173 \text{ feet.}$$

For part (b), the similar triangles in Figure 0.62 give us

$$\tan \theta = \frac{h}{200} = \frac{100\sqrt{3}}{200} = \frac{\sqrt{3}}{2}.$$

Since $0 < \theta < \frac{\pi}{2}$, we have that

$$\theta = \tan^{-1} \left(\frac{\sqrt{3}}{2} \right) \approx 0.7137 \text{ radians (about 41 degrees).}$$

In example 4.10, we simplify expressions involving both trigonometric and inverse trigonometric functions.

EXAMPLE 4.10 Simplifying Expressions Involving Inverse Trigonometric Functions

Simplify (a) $\sin(\cos^{-1} x)$ and (b) $\tan(\cos^{-1} x)$.

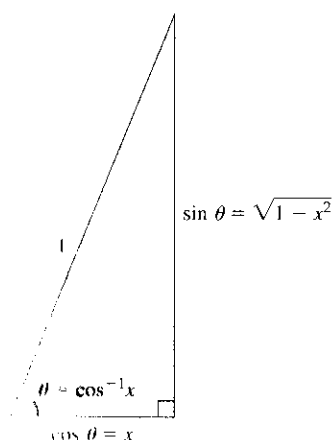


FIGURE 0.63

 $\theta = \cos^{-1} x$.

Solution Do not look for some arcane formula to help you out. *Think* first: $\cos^{-1} x$ is the angle (call it θ) for which $x = \cos \theta$. First, consider the case where $x > 0$. Looking at Figure 0.63, we have drawn a right triangle, with hypotenuse 1 and adjacent angle θ . From the definition of the sine and cosine, then, we have that the base of the triangle is $\cos \theta = x$ and the altitude is $\sin \theta$, which by the Pythagorean Theorem is

$$\sin(\cos^{-1} x) = \sin \theta = \sqrt{1 - x^2}.$$

Wait! We have not yet finished part (a). Figure 0.63 shows $0 < \theta < \frac{\pi}{2}$, but by definition, $\theta = \cos^{-1} x$ could range from 0 to π . Does our answer change if $\frac{\pi}{2} < \theta < \pi$? To see that it doesn't change, note that if $0 \leq \theta \leq \pi$, then $\sin \theta \geq 0$. From the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$, we get

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = \pm \sqrt{1 - x^2}.$$

Since $\sin \theta \geq 0$, we must have

$$\sin \theta = \sqrt{1 - x^2},$$

for all values of x . For part (b), you can read from Figure 0.63 that

$$\tan(\cos^{-1} x) = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{1 - x^2}}{x}.$$

Note that this last identity is valid, regardless of whether $x = \cos \theta$ is positive or negative. ■



EXERCISES 0.4

WRITING EXERCISES

- Many students are comfortable using degrees to measure angles and don't understand why they must learn radian measures. As discussed in the text, radians directly measure distance along the unit circle. Distance is an important aspect of many applications. In addition, we will see later that many calculus formulas are simpler in radians form than in degrees. Aside from familiarity, discuss any and all advantages of degrees over radians. On balance, which is better?
- A student graphs $f(x) = \cos x$ on a graphing calculator and gets what appears to be a straight line at height $y = 1$ instead of the usual cosine curve. Upon investigation, you discover that the calculator has graphing window $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ and is in degrees mode. Explain what went wrong and how to correct it.
- Inverse functions are necessary for solving equations. The restricted range we had to use to define inverses of the trigonometric functions also restricts their usefulness in equation-solving. Explain how to use $\sin^{-1} x$ to find all solutions of the equation $\sin u = x$.
- Discuss how to compute $\sec^{-1} x$, $\csc^{-1} x$ and $\cot^{-1} x$ on a calculator that only has built-in functions for $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.
- In example 4.3, $f(x) = 4 \cos 3x$ has period $2\pi/3$ and $g(x) = 2 \sin(x/3)$ has period 6π . Explain why the sum $h(x) = 4 \cos 3x + 2 \sin(x/3)$ has period 6π .
- Give a different range for $\sec^{-1} x$ than that given in the text. For which x 's would the value of $\sec^{-1} x$ change? Using the calculator discussion in exercise 4, give one reason why we might have chosen the range that we did.

In exercises 1 and 2, convert the given radians measure to degrees.

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{4\pi}{3}$
- (a) $\frac{3\pi}{5}$ (b) $\frac{\pi}{7}$ (c) 2 (d) 3

In exercises 3 and 4, convert the given degrees measure to radians.

- (a) 180° (b) 270° (c) 120° (d) 30°
- (a) 40° (b) 80° (c) 450° (d) 390°

In exercises 5–14, find all solutions of the given equation.

- $2 \cos x - 1 = 0$
- $2 \sin x + 1 = 0$
- $\sqrt{2} \cos x - 1 = 0$
- $2 \sin x - \sqrt{3} = 0$

9. $\sin^2 x - 4 \sin x + 3 = 0$ 10. $\sin^2 x - 2 \sin x - 3 = 0$
 11. $\sin^2 x + \cos x - 1 = 0$ 12. $\sin 2x - \cos x = 0$
 13. $\cos^2 x + \cos x = 0$ 14. $\sin^2 x - \sin x = 0$

T In exercises 15–24, sketch a graph of the function.

15. $f(x) = \sin 2x$ 16. $f(x) = \cos 3x$
 17. $f(x) = \tan 2x$ 18. $f(x) = \sec 3x$
 19. $f(x) = 3 \cos(x - \pi/2)$ 20. $f(x) = 4 \cos(x + \pi)$
 21. $f(x) = \sin 2x - 2 \cos 2x$ 22. $f(x) = \cos 3x - \sin 3x$
 23. $f(x) = \sin x \sin 12x$ 24. $f(x) = \sin x \cos 12x$

In exercises 25–32, identify the amplitude, period, and frequency.

25. $f(x) = 3 \sin 2x$ 26. $f(x) = 2 \cos 3x$
 27. $f(x) = 5 \cos 3x$ 28. $f(x) = 3 \sin 5x$
 29. $f(x) = 3 \cos(2x - \pi/2)$ 30. $f(x) = 4 \sin(3x + \pi)$
 31. $f(x) = -4 \sin x$ 32. $f(x) = -2 \cos 3x$

In exercises 33–36, prove that the given trigonometric identity is true.

33. $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$
 34. $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
 35. (a) $\cos(2\theta) = 2 \cos^2 \theta - 1$
 (b) $\cos(2\theta) = 1 - 2 \sin^2 \theta$
 36. (a) $\sec^2 \theta = \tan^2 \theta + 1$
 (b) $\csc^2 \theta = \cot^2 \theta + 1$

In exercises 37–46, evaluate the inverse function by sketching a unit circle and locating the correct angle on the circle.

37. $\cos^{-1} 0$ 38. $\tan^{-1} 0$
 39. $\sin^{-1}(-1)$ 40. $\cos^{-1}(1)$
 41. $\sec^{-1} 1$ 42. $\tan^{-1}(-1)$
 43. $\sec^{-1} 2$ 44. $\csc^{-1} 2$
 45. $\cot^{-1} 1$ 46. $\tan^{-1} \sqrt{3}$
 47. Prove that, for some constant β ,

$$4 \cos x - 3 \sin x = 5 \cos(x + \beta).$$

Then, estimate the value of β .

48. Prove that, for some constant β ,

$$2 \sin x + \cos x = \sqrt{5} \sin(x + \beta).$$

Then, estimate the value of β .

In exercises 49–52, determine whether or not the function is periodic. If it is periodic, find the smallest (fundamental) period.

49. $f(x) = \cos 2x + 3 \sin \pi x$
 50. $f(x) = \sin x - \cos \sqrt{2}x$
 51. $f(x) = \sin 2x - \cos 5x$
 52. $f(x) = \cos 3x - \sin 7x$

In exercises 53–56, use the range for θ to determine the indicated function value.

53. $\sin \theta = \frac{1}{3}$, $0 \leq \theta \leq \frac{\pi}{2}$, find $\cos \theta$.
 54. $\cos \theta = \frac{4}{5}$, $0 \leq \theta \leq \frac{\pi}{2}$, find $\sin \theta$.
 55. $\sin \theta = \frac{1}{2}$, $\frac{\pi}{2} \leq \theta \leq \pi$, find $\cos \theta$.
 56. $\sin \theta = \frac{1}{2}$, $\frac{\pi}{2} \leq \theta \leq \pi$, find $\tan \theta$.

In exercises 57–64, use a triangle to simplify each expression. Where applicable, state the range of x 's for which the simplification holds.

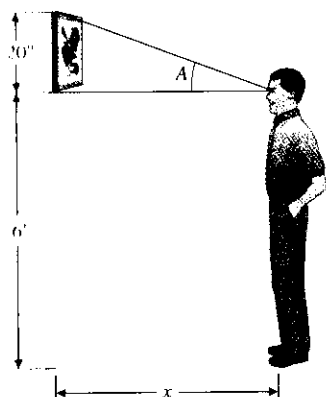
57. $\cos(\sin^{-1} x)$ 58. $\cos(\tan^{-1} x)$
 59. $\tan(\sec^{-1} x)$ 60. $\cot(\cos^{-1} x)$
 61. $\sin(\cos^{-1} \frac{1}{2})$ 62. $\cos(\sin^{-1} \frac{1}{2})$
 63. $\tan(\cos^{-1} \frac{3}{5})$ 64. $\csc(\sin^{-1} \frac{2}{3})$

T In exercises 65–68, use a graphing calculator or computer to determine the number of solutions of each equation and numerically estimate the solutions (x is in radians).

65. $2 \cos x = 2 - x$ 66. $3 \sin x = x$
 67. $\cos x = x^2 - 2$ 68. $\sin x = x^2$
 69. A person sitting 2 miles from a rocket launch site measures 20° up to the current location of the rocket. How high up is the rocket?
 70. A person who is 6 feet tall stands 4 feet from the base of a light pole and casts a 2-foot-long shadow. How tall is the light pole?
 71. A surveyor stands 80 feet from the base of a building and measures an angle of 50° to the top of the steeple on top of the building. The surveyor figures that the center of the steeple lies 20 feet inside the front of the structure. Find the height of the steeple.

T 72. Suppose that the surveyor of exercise 71 estimates that the center of the steeple lies between 20' and 21' inside the front of the structure. Determine how much the extra foot would change the calculation of the height of the building.

T 73. A picture hanging in an art gallery has a frame 20 inches high and the bottom of the frame is 6 feet above the floor. A person who is 6 feet tall stands x feet from the wall. Let A be the angle formed by the ray from the person's eye to the bottom of the frame and the ray from the person's eye to the top of the frame. Write A as a function of x and graph $y = A(x)$.



74. In golf, the goal is to hit a ball into a hole of diameter 4.5 inches. Suppose a golfer stands x feet from the hole trying to putt the ball into the hole. A first approximation of the margin of error in a putt is to measure the angle A formed by the ray from the ball to the right edge of the hole and the ray from the ball to the left edge of the hole. Find A as a function of x .

75. In an AC circuit, the voltage is given by $v(t) = v_p \sin 2\pi ft$, where v_p is the peak voltage and f is the frequency in Hz. A voltmeter actually measures an average (called the **root-mean-square**) voltage, equal to $v_p/\sqrt{2}$. If the voltage has amplitude 170 and period $\pi/30$, find the frequency and meter voltage.

76. An old-fashioned LP record player rotated records at $33\frac{1}{3}$ rpm (revolutions per minute). What is the period (in minutes) of the rotation? What is the period for a 45 rpm single?

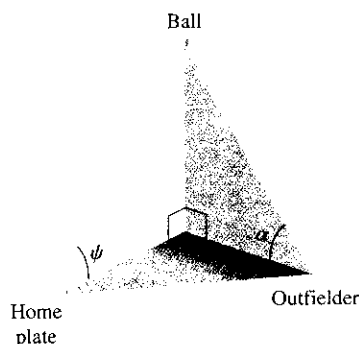
77. Suppose that the ticket sales of an airline (in thousands of dollars) is given by $s(t) = 110 + 2t + 15 \sin(\frac{1}{6}\pi t)$, where t is measured in months. What real-world phenomenon might cause the fluctuation in ticket sales modeled by the sine term? Based on your answer, what month corresponds to $t = 0$? Disregarding seasonal fluctuations, by what amount is the airline's sales increasing annually?

78. Piano tuners sometimes start by striking a tuning fork and then the corresponding piano key. If the tuning fork and piano note each have frequency 8, then the resulting sound is $\sin 8t + \sin 8t$. Graph this. If the piano is slightly out-of-tune at frequency 8.1, the resulting sound is $\sin 8t + \sin 8.1t$. Graph this and explain how the piano tuner can hear the small difference in frequency.

79. Give precise definitions of $\csc^{-1} x$ and $\cot^{-1} x$.

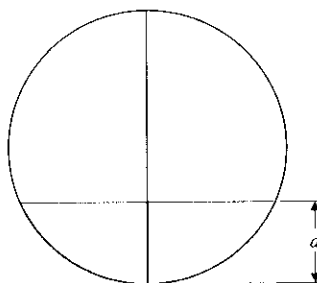
80. In baseball, outfielders are able to easily track down and catch fly balls that have very long and high trajectories. Physicists have argued for years about how this is done. A recent explanation involves the following geometry.

The player can catch the ball by running to keep the angle ψ constant (this makes it appear that the ball is moving in a straight line). Assuming that all triangles shown are right triangles, show that $\tan \psi = \frac{\tan \alpha}{\tan \beta}$ and then solve for ψ .

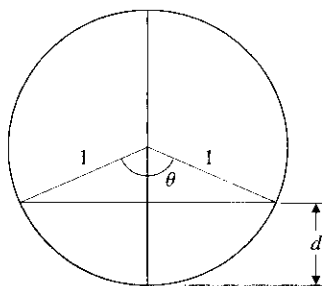


EXPLORATORY EXERCISES

- In his book and video series *The Ring of Truth*, physicist Philip Morrison performed an experiment to estimate the circumference of the earth. In Nebraska, he measured the angle to a bright star in the sky, drove 370 miles due south into Kansas, and measured the new angle to the star. Some geometry shows that the difference in angles, about 5.02° , equals the angle from the center of the earth to the two locations in Nebraska and Kansas. If the earth is perfectly spherical (it's not) and the circumference of the portion of the circle measured out by 5.02° is 370 miles, estimate the circumference of the earth. This experiment was based on a similar experiment by the ancient Greek scientist Eratosthenes. The ancient Greeks and the Spaniards of Columbus' day knew that the earth was round; they just disagreed about the circumference. Columbus argued for a figure about half of the actual value, since a ship couldn't survive on the water long enough to navigate the true distance.
- An oil tank with circular cross sections lies on its side. A stick is inserted in a hole at the top and used to measure the depth d of oil in the tank. Based on this measurement, the goal is to compute the percentage of oil left in the tank.

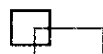


To simplify calculations, suppose the circle is a unit circle with center at $(0, 0)$. Sketch radii extending from the origin to the top of the oil. The area of oil at the bottom equals the area of the portion of the circle bounded by the radii minus the area of the triangle formed above.



Start with the triangle, which has area one-half base times height. Explain why the height is $1 - d$. Find a right triangle in the figure (there are two of them) with hypotenuse 1 (the radius of the circle) and one vertical side of length $1 - d$. The horizontal side has length equal to one-half the base of the larger triangle. Show that this equals $\sqrt{1 - (1 - d)^2}$. The area of the portion of the circle equals $\pi\theta/2\pi = \theta/2$, where θ is the angle at the top of the triangle. Find this angle as a function of d . (Hint: Go back to the right triangle used above with upper angle $\theta/2$.) Then find the area filled with oil and divide by π to get the portion of the tank filled with oil.

- T** 3. Computer graphics can be misleading. This exercise works best using a “disconnected” graph (individual dots, not connected). Graph $y = \sin x^2$ using a graphing window for which each pixel represents a step of 0.1 in the x - or y -direction. You should get the impression of a sine wave that oscillates more and more rapidly as you move to the left and right. Next, change the graphing window so that the middle of the original screen (probably $x = 0$) is at the far left of the new screen. You will probably see what appears to be a random jumble of dots. Continue to change the graphing window by increasing the x -values. Describe the patterns or lack of patterns that you see. You should find one pattern that looks like two rows of dots across the top and bottom of the screen; another pattern looks like the original sine wave. For each pattern that you find, pick adjacent points with x -coordinates a and b . Then change the graphing window so that $a \leq x \leq b$ and find the portion of the graph that is missing. Remember that, whether the points are connected or not, computer graphs always leave out part of the graph; it is part of your job to know whether the missing part is important or not.



0.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Some bacteria reproduce very quickly, as you may have discovered if you have ever had an infected cut or strep throat. Under the right circumstances, the number of bacteria in certain cultures will double in as little as an hour (or even less). In this section, we discuss some functions that can be used to model such rapid growth.

For instance, suppose that initially there are 100 bacteria at a given site, and the population doubles every hour. Call the population function $P(t)$, where t represents time (in hours) and start the clock running at time $t = 0$. The initial population of 100 means that $P(0) = 100$. After 1 hour, the population will double to 200, so that $P(1) = 200$. After another hour, the population will have doubled again to 400, making $P(2) = 400$. After a third hour, the population will have doubled again to 800, making $P(3) = 800$ and so on.

Let's compute the population after 10 hours. (Try guessing this now. Most people guess poorly on this type of problem.) You could calculate the population at 4 hours, 5 hours, and so on, or you could use the following shortcut. To find $P(1)$, you double the initial population, so that $P(1) = 2 \cdot 100$. To find $P(2)$, you double the population at time $t = 1$, so that $P(2) = 2 \cdot 2 \cdot 100 = 2^2 \cdot 100$. Similarly, $P(3) = 2^3 \cdot 100$. This pattern leads us to

$$P(10) = 2^{10} \cdot 100 = 102,400.$$

Depending on the organism, this is now a population that could cause some trouble!

The pattern just discovered suggests that the population can be modeled by the function

$$P(t) = 2^t \cdot 100.$$

We call $P(t)$ an **exponential** function because the variable t is in the exponent. There is a subtle question here: what is the domain of this function? We have so far used only integer values of t , but for what other values of t does $P(t)$ make sense? Certainly, rational powers

make sense, as in $P(1/2) = 2^{1/2} \cdot 100$, where you recognize that $2^{1/2} = \sqrt{2}$. This says that the number of bacteria in the culture after a half hour is

$$P(1/2) = 2^{1/2} \cdot 100 = \sqrt{2} \cdot 100 \approx 141.$$

In this context, the function is not intended to give the exact number of bacteria at a given time, but rather, to predict an approximate number.

In any case, it's a simple matter to interpret fractional powers as roots. For instance, $x^{1/2} = \sqrt{x}$, $x^{1/3} = \sqrt[3]{x}$, $x^{1/4} = \sqrt[4]{x}$, and so on. Further, we can write

$$x^{2/3} = \sqrt[3]{x^2} = (\sqrt[3]{x})^2,$$

$$x^{7/4} = \sqrt[4]{x^7} = (\sqrt[4]{x})^7,$$

$$x^{3.1} = x^{31/10} = \sqrt[10]{x^{31}}$$

and so on. But, what about irrational powers? These are somewhat harder to define, but they work exactly the way you would want them to. For instance, since π is between 3.14 and 3.15, 2^π is between $2^{3.14}$ and $2^{3.15}$. In this way, we define 2^x for x irrational to fill in the gaps in the graph of $y = 2^x$ for x rational. That is, if x is irrational and $a < x < b$, for rational numbers a and b , then $2^a < 2^x < 2^b$. This is the logic behind the definition of irrational powers.

If for some reason, you want to find the population after π hours, you can use your calculator or computer to obtain the approximate population:

$$P(\pi) = 2^\pi \cdot 100 \approx 882.$$

For your convenience, we summarize the usual rules of exponents next.

RULES OF EXPONENTS

- For any integers m and n ,

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m.$$

- For any real number p ,

$$x^{-p} = \frac{1}{x^p}.$$

- For any real numbers p and q ,

$$(x^p)^q = x^{p \cdot q}.$$

- For any real numbers p and q ,

$$x^p \cdot x^q = x^{p+q}.$$

Throughout your calculus course, you will need to be able to quickly convert back and forth between exponential form and fractional or root form.

EXAMPLE 5.1 Converting Expressions to Exponential Form

Convert each to exponential form: (a) $3\sqrt{x^5}$, (b) $\frac{5}{\sqrt[3]{x}}$, (c) $\frac{3x^2}{2\sqrt{x}}$, and (d) $(2^x \cdot 2^{3+x})^2$.

Solution For (a), simply leave the 3 alone and convert the power:

$$3\sqrt{x^5} = 3x^{5/2}.$$

For (b), use a negative exponent to write x in the numerator:

$$\frac{5}{\sqrt[3]{x}} = 5x^{-1/3}.$$

For (c), first separate the constants from the variables and then simplify:

$$\frac{3x^2}{2\sqrt{x}} = \frac{3}{2} \frac{x^2}{x^{1/2}} = \frac{3}{2} x^{2-1/2} = \frac{3}{2} x^{3/2}.$$

For (d), first work inside the parentheses and then square:

$$(2^x \cdot 2^{3+x})^2 = (2^{x+3+x})^2 = (2^{2x+3})^2 = 2^{4x+6}.$$

The function in part (d) of example 5.1 is called an exponential function with a *base* of 2.

DEFINITION 5.1

For any positive constant $b \neq 1$, the function $f(x) = b^x$ is called an **exponential function**. Here, b is called the **base** and x is the **exponent**.

Be careful to distinguish between algebraic functions like $f(x) = x^3$ and $g(x) = x^{2/3}$ and exponential functions. Confusing these types of functions is a very common error. Notice that for exponential functions like $h(x) = 2^x$, the variable is in the exponent (hence the name), instead of in the base. Also, notice that the domain of an exponential function is the entire real line, $(-\infty, \infty)$, while the range is the open interval $(0, \infty)$.

While any positive real number can be used as a base for an exponential function, three bases are the most commonly used in practice. Base 2 arises naturally when analyzing processes that double at regular intervals (such as the bacteria at the beginning of this section). Also, most computers perform their calculations using base 2 arithmetic. Our standard counting system is base 10, so this base is commonly used. However, far and away the most useful base is the irrational number e . Like π , the number e has a surprising tendency to occur in important calculations. We define e by the following:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n. \quad (5.1)$$

Note that equation (5.1) has at least two serious shortcomings. First, we have not yet said what the notation $\lim_{n \rightarrow \infty}$ means. (In fact, we won't define this until Chapter 1.) Second, it's unclear why anyone would ever define a number in such a strange way. We will not be in a position to answer the second question until Chapter 6 (but the answer is worth the wait).

It suffices for the moment to say that equation (5.1) means that e can be approximated by calculating values of $(1 + 1/n)^n$ for large values of n and that the larger the value of n , the closer the approximation will be to the actual value of e . In particular, if you look at the sequence of numbers $(1 + 1/2)^2, (1 + 1/3)^3, (1 + 1/4)^4$ and so on, they will get progressively closer and closer to (i.e., home-in on) the irrational number e (named by the famous mathematician Leonhard Euler).

To get an idea of the value of e , we compute several of these numbers:

$$\left(1 + \frac{1}{10} \right)^{10} = 2.5937 \dots,$$

$$\left(1 + \frac{1}{1000}\right)^{1000} = 2.7169\dots,$$

$$\left(1 + \frac{1}{10,000}\right)^{10,000} = 2.7181\dots$$

and so on. You can compute enough of these values to convince yourself that the first few digits of the decimal representation of e ($e \approx 2.718281828459\dots$) are correct.

EXAMPLE 5.2 Computing Values of Exponentials

Approximate e^4 , $e^{-1/5}$ and e^0 .

Solution From a calculator, we find that

$$e^4 = e \cdot e \cdot e \cdot e \approx 54.598.$$

From the usual rules of exponents,

$$e^{-1/5} = \frac{1}{e^{1/5}} = \frac{1}{\sqrt[5]{e}} \approx 0.81873.$$

(On a calculator, it is convenient to replace $-1/5$ with -0.2 .) Finally, $e^0 = 1$. ■

The graphs of the exponential functions summarize many of their important properties.

EXAMPLE 5.3 Sketching Graphs of Exponentials

Sketch the graphs of the exponential functions $y = 2^x$, $y = e^x$, $y = e^{2x}$, $y = e^{x/2}$, $y = (1/2)^x$ and $y = e^{-x}$.

Solution Using a calculator or computer, you should get graphs similar to those that follow.

Notice that each of the graphs in Figures 0.64a, 0.64b, 0.65a and 0.65b starts very near the x -axis (reading left to right), passes through the point $(0, 1)$ and then rises steeply. This is true for all exponentials with base greater than 1 and with a positive coefficient in the exponent. Note that the larger the base ($e > 2$) or the larger the coefficient in the exponent ($2 > 1 > 1/2$), the more quickly the graph rises to the right (and drops to the left). Note that the graphs in Figures 0.66a and 0.66b (on the following page) are the mirror images in the y -axis of Figures 0.64a and 0.64b, respectively. The graphs rise as you move to the left and drop toward the x -axis as you move to the right. It's worth noting that by the rules of exponents, $(1/2)^x = 2^{-x}$ and $(1/e)^x = e^{-x}$.

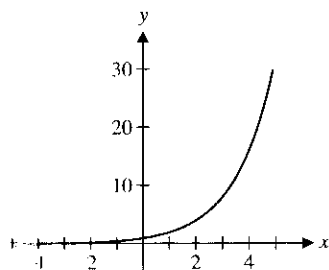


FIGURE 0.64a

$y = 2^x$

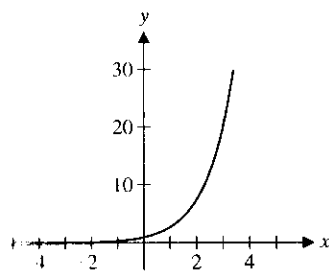


FIGURE 0.64b

$y = e^x$

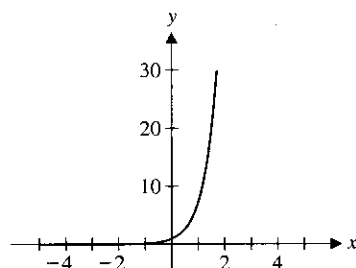


FIGURE 0.65a

$y = e^{2x}$

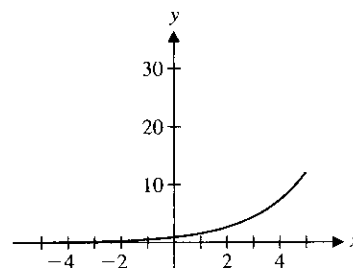


FIGURE 0.65b

$y = e^{x/2}$

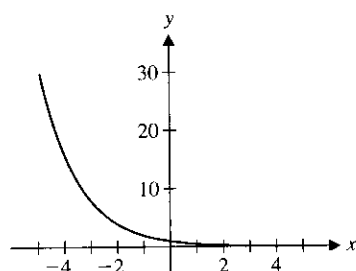


FIGURE 0.66a

$$y = (1/2)^x.$$

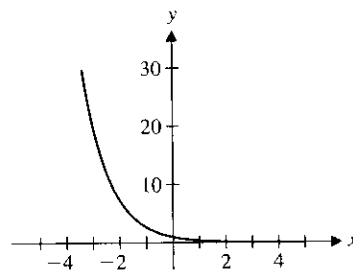


FIGURE 0.66b

$$y = e^{-x}.$$

Notice in Figures 0.64–0.66 that each exponential function is one-to-one and hence, has an inverse function. We define the logarithmic functions to be inverses of the exponential functions.

DEFINITION 5.2

For any positive number $b \neq 1$, the **logarithm** function with base b , written $\log_b x$, is defined by

$$y = \log_b x \text{ if and only if } x = b^y.$$

That is, the logarithm $\log_b x$ gives the exponent to which you must raise the base b to get the given number x . For example,

$$\log_{10} 10 = 1 \text{ (since } 10^1 = 10),$$

$$\log_{10} 100 = 2 \text{ (since } 10^2 = 100),$$

$$\log_{10} 1000 = 3 \text{ (since } 10^3 = 1000)$$

and so on. The value of $\log_{10} 45$ is less clear than the preceding three values, but the idea is the same: you need to find the number y such that $10^y = 45$. The answer is between 1 and 2 (why?), but to be more precise, you will need to employ trial and error. (Of course, you can always use your calculator or computer to compute an approximate value, but that won't help you to understand what logarithms are all about.) You should get $\log_{10} 45 \approx 1.6532$. Later in the course, we introduce you to a powerful method for accurately approximating the values of such functions.

Observe from Definition 5.2 that for any base $b > 0$ ($b \neq 1$), if $y = \log_b x$, then $x = b^y > 0$. That is, the domain of $f(x) = \log_b x$ is the interval $(0, \infty)$. Likewise, the range of f is the entire real line, $(-\infty, \infty)$.

As with exponential functions, the most useful bases turn out to be 2, 10 and e . We usually abbreviate $\log_{10} x$ by $\log x$. Similarly, $\log_e x$ is usually abbreviated $\ln x$ (for **natural logarithm**).

EXAMPLE 5.4 Evaluating Logarithms

Without using your calculator, determine $\log(1/10)$, $\log(0.001)$, $\ln e$ and $\ln e^3$.

Solution Since $1/10 = 10^{-1}$, $\log(1/10) = -1$. Similarly, since $0.001 = 10^{-3}$, we have that $\log(0.001) = -3$. Since $\ln e = \log_e e^1$, $\ln e = 1$. Similarly, $\ln e^3 = 3$. ■

We want to emphasize the inverse relationship defined by Definition 5.2. That is, b^x and $\log_b x$ are inverse functions for any $b > 0$.

In particular, for the base e , we have

$$e^{\ln x} = x \quad \text{for any } x > 0 \quad \text{and} \quad \ln(e^x) = x \quad \text{for any } x. \quad (5.2)$$

We demonstrate this as follows. Let

$$y = \ln x = \log_e x.$$

By Definition 5.2, we have that

$$x = e^y = e^{\ln x}.$$

We can use this relationship between natural logarithms and exponentials to solve equations involving logarithms and exponentials, as in examples 5.5 and 5.6.

EXAMPLE 5.5 Solving a Logarithmic Equation

Solve the equation $\ln(x + 5) = 3$ for x .

Solution Taking the exponential of both sides of the equation and writing things backward (for convenience), we have

$$e^3 = e^{\ln(x+5)} = x + 5,$$

from (5.2). Subtracting 5 from both sides gives us

$$e^3 - 5 = x. \quad \blacksquare$$

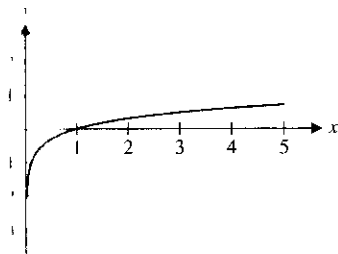


FIGURE 0.67a

$y = \log x.$

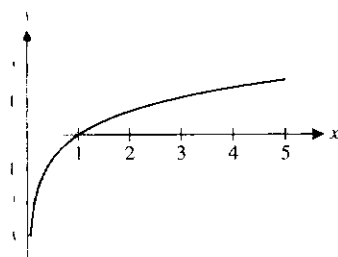


FIGURE 0.67b

$y = \ln x.$

EXAMPLE 5.6 Solving an Exponential Equation

Solve the equation $e^{x+4} = 7$ for x .

Solution Taking the natural logarithm of both sides and writing things backward (for simplicity), we have from (5.2) that

$$\ln 7 = \ln(e^{x+4}) = x + 4.$$

Subtracting 4 from both sides yields

$$\ln 7 - 4 = x. \quad \blacksquare$$

As always, graphs provide excellent visual summaries of the important properties of a function.

EXAMPLE 5.7 Sketching Graphs of Logarithms

Sketch graphs of $y = \log x$ and $y = \ln x$ and briefly discuss the properties of each.

Solution From a calculator or computer, you should obtain graphs resembling those in Figures 0.67a and 0.67b. Notice that both graphs appear to have a vertical asymptote at $x = 0$ (why would that be?), cross the x -axis at $x = 1$ and very gradually increase as x increases. Neither graph has any points to the left of the y -axis, since $\log x$ and $\ln x$ are defined only for $x > 0$. The two graphs are very similar, although not identical. \blacksquare

The properties just described graphically are summarized in Theorem 5.1.

THEOREM 5.1

For any positive base $b \neq 1$,

- (i) $\log_b x$ is defined only for $x > 0$,
- (ii) $\log_b 1 = 0$ and
- (iii) if $b > 1$, then $\log_b x < 0$ for $0 < x < 1$ and $\log_b x > 0$ for $x > 1$.

PROOF

- (i) Note that since $b > 0$, $b^y > 0$ for any y . So, if $\log_b x = y$, then $x = b^y > 0$.
- (ii) Since $b^0 = 1$ for any number $b \neq 0$, $\log_b 1 = 0$ (i.e., the exponent to which you raise the base b to get the number 1 is 0).
- (iii) We leave this as an exercise. ■

All logarithms share a set of defining properties, as stated in Theorem 5.2.

THEOREM 5.2

For any positive base $b \neq 1$ and positive numbers x and y , we have

- (i) $\log_b(xy) = \log_b x + \log_b y$,
- (ii) $\log_b(x/y) = \log_b x - \log_b y$
- (iii) $\log_b(x^y) = y \log_b x$.

As with most algebraic rules, each of these properties can dramatically simplify calculations when they apply.

EXAMPLE 5.8 Simplifying Logarithmic Expressions

Write each as a single logarithm: (a) $\log_2 27^x - \log_2 3^x$ and (b) $\ln 8 - 3 \ln(1/2)$.

Solution First, note that there is more than one order in which to work each problem. For part (a), we have $27 = 3^3$ and so, $27^x = (3^3)^x = 3^{3x}$. This gives us

$$\begin{aligned}\log_2 27^x - \log_2 3^x &= \log_2 3^{3x} - \log_2 3^x \\ &= 3x \log_2 3 - x \log_2 3 = 2x \log_2 3 = \log_2 3^{2x}.\end{aligned}$$

For part (b), note that $8 = 2^3$ and $1/2 = 2^{-1}$. Then,

$$\begin{aligned}\ln 8 - 3 \ln(1/2) &= 3 \ln 2 - 3(-\ln 2) \\ &= 3 \ln 2 + 3 \ln 2 = 6 \ln 2 = \ln 2^6 = \ln 64. \quad \blacksquare\end{aligned}$$

In some circumstances, it is beneficial to use the rules of logarithms to expand a given expression, as in the following example.

EXAMPLE 5.9 Expanding a Logarithmic Expression

Use the rules of logarithms to expand the expression $\ln \left(\frac{x^3 y^4}{z^5} \right)$.

Solution From Theorem 5.2, we have that

$$\begin{aligned}\ln\left(\frac{x^3 y^4}{z^5}\right) &= \ln(x^3 y^4) - \ln(z^5) = \ln(x^3) + \ln(y^4) - \ln(z^5) \\ &= 3 \ln x + 4 \ln y - 5 \ln z. \quad \blacksquare\end{aligned}$$

Using the rules of exponents and logarithms, notice that we can rewrite any exponential as an exponential with base e , as follows. For any base $a > 0$, we have

$$a^x = e^{\ln(a^x)} = e^{x \ln a}. \quad (5.3)$$

This follows from Theorem 5.2 (iii) and the fact that $e^{\ln y} = y$, for all $y > 0$.

EXAMPLE 5.10 Rewriting Exponentials as Exponentials with Base e

Rewrite the exponentials 2^x , 5^x and $(2/5)^x$ as exponentials with base e .

Solution From (5.3), we have

$$\begin{aligned}2^x &= e^{\ln(2^x)} = e^{x \ln 2}, \\ 5^x &= e^{\ln(5^x)} = e^{x \ln 5}\end{aligned}$$

and

$$\left(\frac{2}{5}\right)^x = e^{\ln[(2/5)^x]} = e^{x \ln(2/5)}. \quad \blacksquare$$

Just as we were able to use the relationship between the natural logarithm and exponentials to rewrite an exponential with any positive base in terms of an exponential with base e , we can use these same properties to rewrite any logarithm in terms of natural logarithms, as follows. For any positive base b ($b \neq 1$), we will show that

$$\log_b x = \frac{\ln x}{\ln b}. \quad (5.4)$$

Let $y = \log_b x$. Then by Definition 5.2, we have that $x = b^y$. Taking the natural logarithm of both sides of this equation, we get by Theorem 5.2 (iii) that

$$\ln x = \ln(b^y) = y \ln b.$$

Dividing both sides by $\ln b$ (since $b \neq 1$ and $\ln b \neq 0$) gives us

$$y = \frac{\ln x}{\ln b},$$

establishing (5.4).

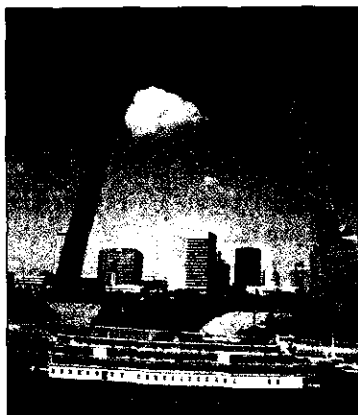
One use you will find for equation (5.4) is for computing logarithms with bases other than e or 10. More than likely, your calculator has keys only for $\ln x$ and $\log x$. We illustrate this idea in example 5.11.

EXAMPLE 5.11 Approximating the Value of Logarithms

Approximate the value of $\log_7 12$.

Solution From (5.4), we have

$$\log_7 12 = \frac{\ln 12}{\ln 7} \approx 1.2769894. \quad \blacksquare$$



Saint Louis Gateway Arch.

□ Hyperbolic Functions

There are two special combinations of exponential functions, called the **hyperbolic sine** and **hyperbolic cosine** functions, that have important applications. For instance, the Gateway Arch in Saint Louis was built in the shape of a hyperbolic cosine graph. (See the photograph in the margin.) The hyperbolic sine function is denoted by $\sinh(x)$ and the hyperbolic cosine function is denoted by $\cosh(x)$. These are defined to be

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Graphs of these functions are shown in Figures 0.68a and 0.68b. The hyperbolic functions (including the hyperbolic tangent, $\tanh x$, defined in the obvious way) are often convenient to use when solving equations. For now, we verify several basic properties that the hyperbolic functions satisfy in parallel with their trigonometric counterparts.

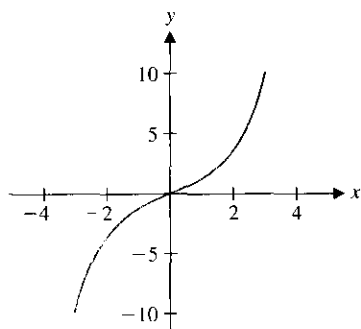


FIGURE 0.68a
 $y = \sinh x$.

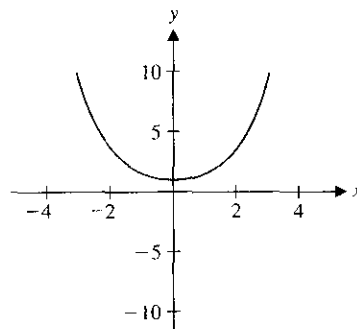


FIGURE 0.68b
 $y = \cosh x$.

EXAMPLE 5.12 Computing Values of Hyperbolic Functions

Compute $f(0)$, $f(1)$ and $f(-1)$ and determine how $f(x)$ and $f(-x)$ compare for each function: (a) $f(x) = \sinh x$ and (b) $f(x) = \cosh x$.

Solution For part (a), we have $\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1 - 1}{2} = 0$. Note that this means that $\sinh 0 = \sin 0 = 0$. Also, we have $\sinh 1 = \frac{e^1 - e^{-1}}{2} \approx 1.18$, while $\sinh(-1) = \frac{e^{-1} - e^1}{2} \approx -1.18$. Notice that $\sinh(-1) = -\sinh 1$. In fact, for any x ,

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = \frac{-(e^x - e^{-x})}{2} = -\sinh x.$$

[Recall that the same rule holds for the corresponding trigonometric function: $\sin(-x) = -\sin x$.] For part (b), we have $\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1 + 1}{2} = 1$. Note that this means that $\cosh 0 = \cos 0 = 1$. Also, we have $\cosh 1 = \frac{e^1 + e^{-1}}{2} \approx 1.54$, while $\cosh(-1) = \frac{e^{-1} + e^1}{2} \approx 1.54$. Notice that $\cosh(-1) = \cosh 1$. In fact, for any x ,

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

[Again, the same rule holds for the corresponding trigonometric function: $\cos(-x) = \cos x$.] ■

□ Fitting a Curve to Data

You are familiar with the idea that two points determine a straight line. As we see in example 5.13, two points will also determine an exponential function.

EXAMPLE 5.13 Matching Data to an Exponential Curve

Find the exponential function of the form $f(x) = ae^{bx}$ that passes through the points $(0, 5)$ and $(3, 9)$.

Solution Notice that you can't simply solve this problem by inspection. That is, you can't just read off appropriate values for a and b . Instead, we solve for a and b , using the properties of logarithms and exponentials. First, for the graph to pass through the point $(0, 5)$, this means that

$$5 = f(0) = ae^{b \cdot 0} = a,$$

so that $a = 5$. Next, for the graph to pass through the point $(3, 9)$, we must have

$$9 = f(3) = ae^{3b} = 5e^{3b}.$$

To solve for the b in the exponent, we divide both sides of the last equation by 5 and take the natural logarithm of both sides, which yields

$$\ln\left(\frac{9}{5}\right) = \ln e^{3b} = 3b \ln e = 3b,$$

since $\ln e = 1$. Finally, dividing by 3 gives us the value for b :

$$b = \frac{1}{3} \ln\left(\frac{9}{5}\right).$$

Thus, $f(x) = 5e^{(1/3)\ln(9/5)x}$. ■

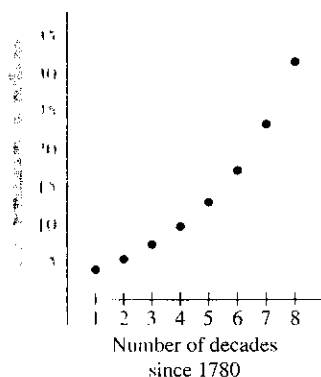


FIGURE 0.69a
U.S. Population 1790–1860.

Year	U.S. Population
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
1830	12,866,020
1840	17,069,453
1850	23,191,876
1860	31,443,321

Consider the population of the United States from 1790 to 1860, found in the accompanying table. A plot of these data points can be seen in Figure 0.69a (where the vertical scale represents the population in millions). This shows that the population was increasing, with larger and larger increases each decade. If you sketch an imaginary curve through these points, you will probably get the impression of a parabola or perhaps the right half of a cubic or exponential. And that's the question: is this data best modeled by a quadratic function, a cubic function, a fourth-order polynomial, an exponential function or what?

We can use the properties of logarithms from Theorem 5.2 to help determine whether a given set of data is modeled better by a polynomial or an exponential function, as follows. Suppose that the data actually comes from an exponential, say $y = ae^{bx}$ (i.e., the data points lie on the graph of this exponential). Then,

$$\ln y = \ln(ae^{bx}) = \ln a + \ln e^{bx} = \ln a + bx.$$

If you draw a new graph, where the horizontal axis shows values of x and the vertical axis corresponds to values of $\ln y$, then the graph will be the line $\ln y = bx + c$ (where the

constant $c = \ln a$). On the other hand, suppose the data actually came from a polynomial. If $y = bx^n$ (for any n), then observe that

$$\ln y = \ln(bx^n) = \ln b + \ln x^n = \ln b + n \ln x.$$



In this case, a graph with horizontal and vertical axes corresponding to x and $\ln y$, respectively, will look like the graph of a logarithm, $\ln y = n \ln x + c$. Such a **semi-log graph** (i.e., graphing $\ln y$ versus x) lets us distinguish the graph of an exponential from that of a polynomial: graphs of exponentials become straight lines, while graphs of polynomials (of degree ≥ 1) become logarithmic curves. Scientists and engineers frequently use semi-log graphs to help them gain an understanding of physical phenomena represented by some collection of data.

EXAMPLE 5.14 Using a Semi-Log Graph to Identify a Type of Function

Determine if the population of the United States from 1790 to 1860 was increasing exponentially or as a polynomial.

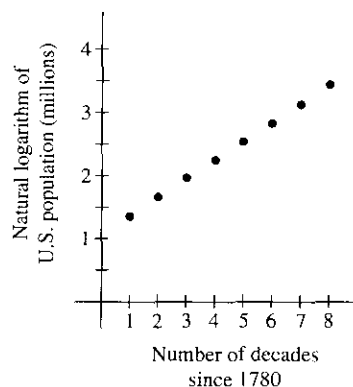
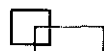


FIGURE 0.69b
Semi-log plot of U.S. population.

Solution As already indicated, the trick is to draw a semi-log graph. That is, instead of plotting $(1, 3.9)$ as the first data point, plot $(1, \ln 3.9)$ and so on. A semi-log plot of this data set is seen in Figure 0.69b. Although the points are not exactly colinear (how would you prove this?), the plot is very close to a straight line with $\ln y$ -intercept of 1 and slope 0.35. You should conclude that the population is well-modeled by an exponential function. (Keep in mind that here, as with most real problems, the data is somewhat imprecise and so, the points in the semi-log plot need *not* be perfectly colinear for you to conclude that the data is modeled quite well by an exponential.) The exponential model would be $y = P(t) = ae^{bt}$, where t represents the number of decades since 1780. Here, b is the slope and $\ln a$ is the $\ln y$ -intercept of the line in the semi-log graph. That is, $b = 0.35$ and $\ln a = 1$ (why?), so that $a = e$. The population is then modeled by

$$P(t) = e \cdot e^{0.35t} \text{ million.} \blacksquare$$



EXERCISES 0.5

WRITING EXERCISES

- Starting from a single cell, a human being is formed by 50 generations of cell division. Explain why after n divisions there are 2^n cells. Guess how many cells will be present after 50 divisions, then compute 2^{50} . Briefly discuss how rapidly exponential functions increase.
- Explain why the graphs of $f(x) = 2^{-x}$ and $g(x) = (\frac{1}{2})^x$ are the same.
- Compare $f(x) = x^2$ and $g(x) = 2^x$ for $x = \frac{1}{2}, x = 1, x = 2, x = 3$ and $x = 4$. In general, which function is bigger for large values of x ? For small values of x ?
- Compare $f(x) = 2^x$ and $g(x) = 3^x$ for $x = -2, x = -\frac{1}{2}$,

$x = \frac{1}{2}$ and $x = 2$. In general, which function is bigger for negative values of x ? For positive values of x ?

In exercises 1–6, convert each exponential expression into fractional or root form.

- | | | |
|--------------|--------------|---------------|
| 1. 2^{-3} | 2. 4^{-2} | 3. $3^{1/2}$ |
| 4. $6^{2/5}$ | 5. $5^{2/3}$ | 6. $4^{-2/3}$ |

In exercises 7–12, convert each expression into exponential form.

- | | | |
|---------------------|---------------------------|-----------------------------|
| 7. $\frac{1}{x^2}$ | 8. $\sqrt[3]{x^2}$ | 9. $\frac{2}{x^3}$ |
| 10. $\frac{4}{x^2}$ | 11. $\frac{1}{2\sqrt{x}}$ | 12. $\frac{3}{2\sqrt{x^3}}$ |

In exercises 13–16, find the integer value of the given expression without using a calculator.

13. $4^{3/2}$ 14. $8^{2/3}$ 15. $\frac{\sqrt{8}}{2^{1/2}}$ 16. $\frac{2}{(1/3)^2}$

T In exercises 17–20, use a calculator or computer to estimate each value.

17. $2e^{-1/2}$ 18. $4e^{-2/3}$ 19. $\frac{12}{e}$ 20. $\frac{14}{\sqrt{e}}$

T In exercises 21–30, sketch a graph of the given function.

21. $f(x) = e^{2x}$ 22. $f(x) = e^{3x}$
 23. $f(x) = 2e^{x/4}$ 24. $f(x) = e^{-x^2}$
 25. $f(x) = 3e^{-2x}$ 26. $f(x) = 10e^{-x/3}$
 27. $f(x) = \ln 2x$ 28. $f(x) = \ln x^2$
 29. $f(x) = e^{2 \ln x}$ 30. $f(x) = e^{-x/4} \sin x$

In exercises 31–40, solve the given equation for x .

31. $e^{2x} = 2$ 32. $e^{4x} = 3$
 33. $e^x(x^2 - 1) = 0$ 34. $xe^{-2x} + 2e^{-2x} = 0$
 35. $\ln 2x = 4$ 36. $2 \ln 3x = 1$
 37. $4 \ln x = -8$ 38. $x^2 \ln x - 9 \ln x = 0$
 39. $e^{2 \ln x} = 4$ 40. $\ln(e^{2x}) = 6$

In exercises 41 and 42, use the definition of logarithm to determine the value.

41. (a) $\log_3 9$ (b) $\log_4 64$ (c) $\log_3 \frac{1}{27}$
 42. (a) $\log_4 \frac{1}{16}$ (b) $\log_4 2$ (c) $\log_9 3$

T In exercises 43 and 44, use equation (5.4) to approximate the value.

43. (a) $\log_3 7$ (b) $\log_4 60$ (c) $\log_3 \frac{1}{24}$
 44. (a) $\log_4 \frac{1}{10}$ (b) $\log_4 3$ (c) $\log_9 8$

In exercises 45–50, rewrite the expression as a single logarithm.

45. $\ln 3 - \ln 4$ 46. $2 \ln 4 - \ln 3$
 47. $\frac{1}{2} \ln 4 - \ln 2$ 48. $3 \ln 2 - \ln \frac{1}{2}$
 49. $\ln \frac{3}{4} + 4 \ln 2$ 50. $\ln 9 - 2 \ln 3$

In exercises 51–54, find a function of the form $f(x) = ae^{bx}$ with the given function values.

51. $f(0) = 2, f(2) = 6$ 52. $f(0) = 3, f(3) = 4$
 53. $f(0) = 4, f(2) = 2$ 54. $f(0) = 5, f(1) = 2$

T 55. A fast-food restaurant gives every customer a game ticket. With each ticket, the customer has a 1-in-10 chance of winning a free meal. If you go 10 times, estimate your chances of winning at least one free meal. The exact probability is $1 - \left(\frac{9}{10}\right)^{10}$. Compute this number and compare it to your guess.

T 56. In exercise 55, if you had 20 tickets with a 1-in-20 chance of winning, would you expect your probability of winning to increase or decrease? Compute the probability $1 - \left(\frac{19}{20}\right)^{20}$ to find out.

57. In general, if you have n chances of winning with a 1-in- n chance on each try, the probability of winning at least once is $1 - \left(1 - \frac{1}{n}\right)^n$. As n gets larger, what number does this probability approach? (Hint: There is a very good reason that this question is in this section!)

58. If $y = a \cdot x^m$, show that $\ln y = \ln a + m \ln x$. If $v = \ln y$, $u = \ln x$ and $b = \ln a$, show that $v = mu + b$. Explain why the graph of v as a function of u would be a straight line. This graph is called the **log-log plot** of y and x .

T 59. For the given data, compute $v = \ln y$ and $u = \ln x$ and plot points (u, v) . Find constants m and b such that $v = mu + b$ and use the results of exercise 58 to find a constant a such that $y = a \cdot x^m$.

x	2.2	2.4	2.6	2.8	3.0	3.2
y	14.52	17.28	20.28	23.52	27.0	30.72

T 60. Repeat exercise 59 for the given data.

x	2.8	3.0	3.2	3.4	3.6	3.8
y	9.37	10.39	11.45	12.54	13.66	14.81

T 61. Construct a log-log plot (see exercise 58) of the U.S. population data in example 5.14. Compared to the semi-log plot of the data in Figure 0.69b, does the log-log plot look linear? Based on this, is the population data modeled better by an exponential function or a polynomial (power) function?

T 62. Construct a semi-log plot of the data in exercise 59. Compared to the log-log plot already constructed, does this plot look linear? Based on this, is this data better modeled by an exponential or power function?

63. The concentration $[H^+]$ of free hydrogen ions in a chemical solution determines the solution's pH, as defined by $\text{pH} = -\log [H^+]$. Find $[H^+]$ if the pH equals (a) 7, (b) 8, and (c) 9. For each increase in pH of 1, by what factor does $[H^+]$ change?

64. Gastric juice is considered an acid, with a pH of about 2.5. Blood is considered alkaline, with a pH of about 7.5. Compare the concentrations of hydrogen ions in the two substances (see exercise 63).

65. The Richter magnitude M of an earthquake is defined in terms of the energy E in joules released by the earthquake, with $\log_{10} E = 4.4 + 1.5M$. Find the energy for earthquakes with magnitudes (a) 4, (b) 5 and (c) 6. For each increase in M of one, by what factor does E change?

66. It puzzles some people who have not grown up around earthquakes that a magnitude 6 quake is considered much more severe than a magnitude 3 quake. Compare the amount of energy released in the two quakes.

67. The decibel level of a noise is defined in terms of the intensity I of the noise, with $\text{dB} = 10 \log_{10}(I/I_0)$. Here, $I_0 = 10^{-12} \text{ W/m}^2$ is the intensity of a barely audible sound. Compute the intensity levels of sounds with (a) $\text{dB} = 80$, (b) $\text{dB} = 90$ and (c) $\text{dB} = 100$. For each increase in decibels of 10, by what factor does I change?
68. At a basketball game, a courtside decibel meter shows crowd noises ranging from 60 dB to 110 dB. How much louder is the 110-dB noise compared to the 60-dB noise (see exercise 67)?
- T** 69. Use a graphing calculator to graph $y = xe^{-x}$, $y = xe^{-2x}$, $y = xe^{-3x}$ and so on. Estimate the locations of the maximum for each. In general, state a rule for the location of the maximum of $y = xe^{-kx}$.
- T** 70. In golf, the task is to hit a golf ball into a small hole. If the ground near the hole is not flat, the golfer must judge how much the ball will curve. Suppose the golfer is at the point $(-13, 0)$, the hole is at the point $(0, 0)$ and the path of the ball is, for $-13 \leq x \leq 0$, $y = -1.672x + 72 \ln(1 + 0.02x)$. Show that the ball goes in the hole and estimate the point on the y -axis at which the golfer aimed.

Exercises 71–76 refer to the hyperbolic functions.

71. Show that the range of the hyperbolic cosine is $\cosh x \geq 1$ and the range of the hyperbolic sine is the entire real line.
72. Show that $\cosh^2 x - \sinh^2 x = 1$ for all x .
- T** 73. The Saint Louis Gateway Arch is both 630 feet wide and 630 feet tall. (Most people think that it looks taller than it is wide.) One model for the outline of the arch is $y = 757.7 - 127.7 \cosh\left(\frac{x}{127.7}\right)$ for $y \geq 0$. Use a graphing calculator to approximate the x - and y -intercepts and determine if the model has the correct horizontal and vertical measurements.
- T** 74. To model the outline of the arch with a parabola, you can start with $y = -c(x + 315)(x - 315)$ for some constant c . Explain why this gives the correct x -intercepts. Determine the constant c that gives a y -intercept of 630. Graph this parabola and the hyperbolic cosine in exercise 73 on the same axes. Are the graphs nearly identical or very different?
75. Find all solutions of $\sinh(x^2 - 1) = 0$.

76. Find all solutions of $\cosh(3x + 2) = 0$.

77. On a standard piano, the A below middle C produces a sound wave with frequency 220 Hz (cycles per second). The frequency of the A one octave higher is 440 Hz. In general, doubling the frequency produces the same note an octave higher. Find an exponential formula for the frequency f as a function of the number of octaves x above the A below middle C.
78. There are 12 notes in an octave on a standard piano. Middle C is 3 notes above A (see exercise 77). If the notes are tuned equally, this means that middle C is a quarter-octave above A. Use $x = \frac{1}{4}$ in your formula from exercise 77 to estimate the frequency of middle C.

✚ EXPLORATORY EXERCISES

- T** 1. Graph $y = x^2$ and $y = 2^x$ and approximate the two positive solutions of the equation $x^2 = 2^x$. Graph $y = x^3$ and $y = 3^x$ and approximate the two positive solutions of the equation $x^3 = 3^x$. Explain why $x = a$ will always be a solution of $x^a = a^x$, $a > 0$. What is different about the role of $x = 2$ as a solution of $x^2 = 2^x$ compared to the role of $x = 3$ as a solution of $x^3 = 3^x$? To determine the a -value at which the change occurs, graphically solve $x^a = a^x$ for $a = 2.1, 2.2, \dots, 2.9$ and note that $a = 2.7$ and $a = 2.8$ behave differently. Continue to narrow down the interval of change by testing $a = 2.71, 2.72, \dots, 2.79$. Then guess the exact value of a .
- T** 2. Graph $y = \ln x$ and describe the behavior near $x = 0$. Then graph $y = x \ln x$ and describe the behavior near $x = 0$. Repeat this for $y = x^2 \ln x$, $y = x^{1/2} \ln x$, and $y = x^a \ln x$ for a variety of positive constants a . Because the function “blows up” at $x = 0$, we say that $y = \ln x$ has a **singularity** at $x = 0$. The **order** of the singularity at $x = 0$ of a function $f(x)$ is the smallest value of a such that $y = x^a f(x)$ doesn’t have a singularity at $x = 0$. Determine the order of the singularity at $x = 0$ for (a) $f(x) = \frac{1}{x}$, (b) $f(x) = \frac{1}{x^2}$, and (c) $f(x) = \frac{1}{x^3}$. The higher the order of the singularity, the “worse” the singularity is. Based on your work, how bad is the singularity of $y = \ln x$ at $x = 0$?



0.6 TRANSFORMATIONS OF FUNCTIONS

You are now familiar with a long list of functions including polynomials, rational functions, trigonometric functions, exponentials and logarithms. Many students at this stage expect that calculus and higher-level mathematics courses will consist of defining even more functions and solving more equations. Actually, one important goal for this course is to more fully understand the properties of the functions we have already defined. To a large extent, you

will build your understanding by examining a few key properties of functions. To briefly get an idea of where we are headed, recall what the function $f(x) = x^2$ represents. To understand this function, you first had to understand how to multiply two numbers together. From practice multiplying “3 times 3” and “6 times 6” and so on, the idea of multiplying any number by itself became natural. The function x^2 is, in this sense, simply an abstraction. We have put multiplication into the broader context of functions. Now that we have functions, we need to make the manipulations of functions as natural as possible.

The first few function manipulations we define are straightforward.

DEFINITION 6.1

Suppose that $f(x)$ and $g(x)$ are functions with domains D_1 and D_2 , respectively. The functions $f + g$, $f - g$ and $f \cdot g$ are defined by

$$(f + g)(x) = f(x) + g(x),$$

$$(f - g)(x) = f(x) - g(x),$$

and

$$(f \cdot g)(x) = f(x) \cdot g(x),$$

for all x in $D_1 \cap D_2$ (i.e., $x \in D_1$, and $x \in D_2$). The function $\frac{f}{g}$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

for all x in $D_1 \cap D_2$ such that $g(x) \neq 0$.

In example 6.1, we examine various combinations of several simple functions.

EXAMPLE 6.1 Combinations of Functions

If $f(x) = x - 3$ and $g(x) = \sqrt{x - 1}$, determine the functions $f + g$, $3f - g$ and $\frac{f}{g}$, stating the domains of each.

Solution First, note that the domain of f is the entire real line and the domain of g is the set of all $x \geq 1$. Now,

$$(f + g)(x) = x - 3 + \sqrt{x - 1}$$

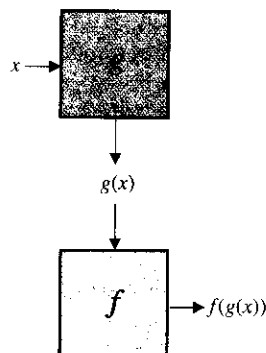
and

$$(3f - g)(x) = 3(x - 3) - \sqrt{x - 1} = 3x - 9 - \sqrt{x - 1}.$$

Notice that the domain of both $(f + g)$ and $(3f - g)$ is $\{x | x \geq 1\}$. For

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x - 3}{\sqrt{x - 1}},$$

the domain is $\{x | x > 1\}$, where we have added the restriction $x \neq 1$ to avoid dividing by 0. ■



$$(f \circ g)(x) = f(g(x)).$$

Definition 6.1 and example 6.1 show us how to do arithmetic with functions. An operation on functions that does not directly correspond to arithmetic is the *composition* of two functions.

DEFINITION 6.2

The **composition** of functions $f(x)$ and $g(x)$, written $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x)),$$

for all x such that x is in the domain of g and $g(x)$ is in the domain of f .

The composition of two functions is a two-step process, as indicated in the margin schematic. Be careful to notice what this definition is saying. In particular, for $f(g(x))$ to be defined, you first need $g(x)$ to be defined, so x must be in the domain of g . Next, f must be defined at the point $g(x)$, so that the number $g(x)$ will need to be in the domain of f .

EXAMPLE 6.2 Finding the Composition of Two Functions

For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 2}$, find the compositions $f \circ g$ and $g \circ f$ and identify the domain of each.

Solution First, we have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(\sqrt{x - 2}) \\ &= (\sqrt{x - 2})^2 + 1 = x - 2 + 1 = x - 1.\end{aligned}$$

It's tempting to write that the domain of $f \circ g$ is the entire real line, but look more carefully. Note that for x to be in the domain of g , we must have $x \geq 2$. The domain of f is the whole real line, so this places no further restrictions on the domain of $f \circ g$. Even though the final expression $x - 1$ is defined for all x , the domain of $(f \circ g)$ is $\{x | x \geq 2\}$. For the second composition,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(x^2 + 1) \\ &= \sqrt{(x^2 + 1) - 2} = \sqrt{x^2 - 1}.\end{aligned}$$

The resulting square root requires $x^2 - 1 \geq 0$ or $|x| \geq 1$. Since the “inside” function f is defined for all x , the domain of $g \circ f$ is $\{x \in \mathbb{R} | |x| \geq 1\}$, which we write in interval notation as $(-\infty, -1] \cup [1, \infty)$. ■

As you progress through the calculus, you will often find yourself needing to recognize that a given function is a composition of simpler functions. For now, it is an important skill to practice.

EXAMPLE 6.3 Identifying Compositions of Functions

Identify functions f and g such that the given function can be written as $(f \circ g)(x)$ for each of (a) $\sqrt{x^2 + 1}$, (b) $(\sqrt{x} + 1)^2$, (c) $\sin x^2$ and (d) $\cos^2 x$. Note that more than one answer is possible for each function.

Solution (a) Notice that $x^2 + 1$ is *inside* the square root. So, one choice is to have $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$.

(b) Here, $\sqrt{x} + 1$ is *inside* the square. So, one choice is $g(x) = \sqrt{x} + 1$ and $f(x) = x^2$.

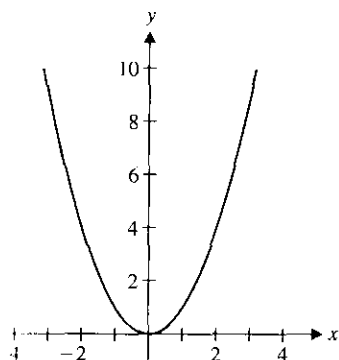


FIGURE 0.70a

$$y = x^2.$$

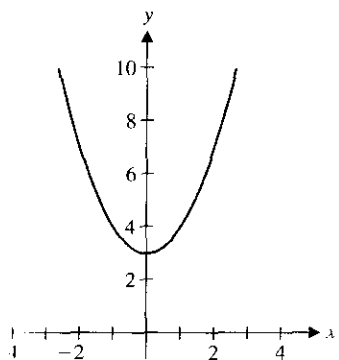


FIGURE 0.70b

$$y = x^2 + 3.$$

(c) The function can be rewritten as $\sin(x^2)$, with x^2 clearly *inside* the sine function. Then, $g(x) = x^2$ and $f(x) = \sin x$ is one choice.

(d) The function as written is shorthand for $(\cos x)^2$. So, one choice is $g(x) = \cos x$ and $f(x) = x^2$. ■

In general, it is quite difficult to take the graphs of $f(x)$ and $g(x)$ and produce the graph of $f(g(x))$. If one of the functions f and g is linear, however, there is a simple graphical procedure for graphing the composition. Such **linear transformations** are explored in the remainder of this section.

The first case is to take the graph of $f(x)$ and produce the graph of $f(x) + c$ for some constant c . You should be able to deduce the general result from the following example.

EXAMPLE 6.4 Vertical Translation of a Graph

Graph $y = x^2$ and $y = x^2 + 3$; compare and contrast the graphs.

Solution You can probably sketch these by hand. You should get something like those in Figures 0.70a and 0.70b. Both figures show parabolas opening up. The main obvious difference is that x^2 has a y -intercept of 0 and $x^2 + 3$ has a y -intercept of 3. Some thought should convince you that this is an important clue. For *any* given value of x , the point on the graph of $y = x^2 + 3$ will be plotted exactly 3 units higher than the corresponding point on the graph of $y = x^2$. This is shown in Figure 0.71a.

In Figure 0.71b, the two graphs are shown on the same set of axes. To many people, it does not look like the top graph is the same as the bottom graph moved up 3 units. This is an unfortunate optical illusion. Humans usually mentally judge distance between curves as the shortest distance between the curves. For these parabolas, the shortest distance is vertical at $x = 0$ but becomes increasingly horizontal as you move away from the y -axis. The distance of 3 between the parabolas is measured vertically.

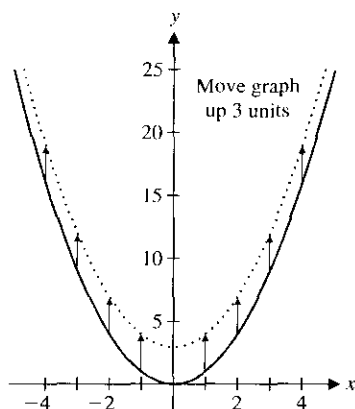


FIGURE 0.71a

Translate graph up.

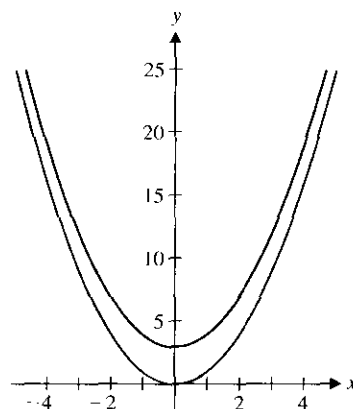


FIGURE 0.71b

$$y = x^2 \text{ and } y = x^2 + 3.$$

In general, the graph of $y = f(x) + c$ is exactly the same as the graph of $y = f(x)$ shifted up (if $c > 0$) or down (if $c < 0$) by $|c|$ units. We usually refer to $f(x) + c$ as a **vertical translation** (up or down, by $|c|$ units).

In example 6.5, we explore what happens if a constant is added to x .

EXAMPLE 6.5 A Horizontal Translation

Compare and contrast the graphs of $y = x^2$ and $y = (x - 1)^2$.

Solution The graphs are shown in Figures 0.72a and 0.72b, respectively.

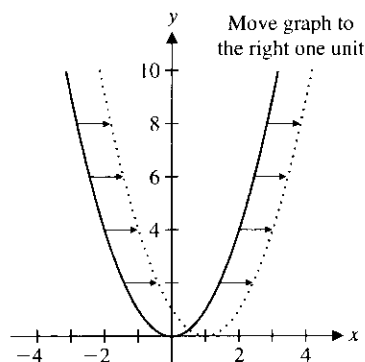


FIGURE 0.73
Translation to the right.

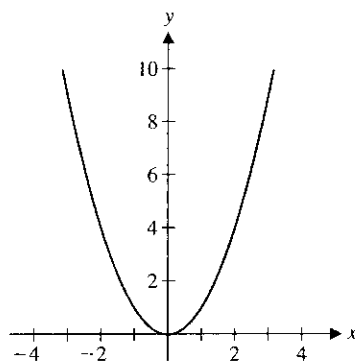


FIGURE 0.72a
 $y = x^2$.

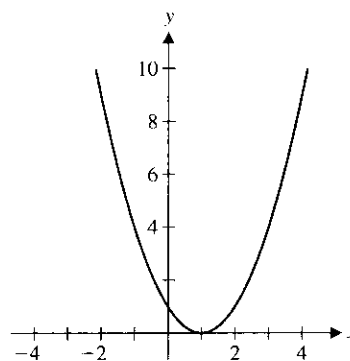


FIGURE 0.72b
 $y = (x - 1)^2$.

Notice that the graph of $y = (x - 1)^2$ appears to be the same as the graph of $y = x^2$, except that it is shifted 1 unit to the right. This should make sense for the following reason. Pick a value of x , say $x = 13$. The value of $(x - 1)^2$ at $x = 13$ is 12^2 , the same as the value of x^2 at $x = 12$, 1 unit to the left. Observe that this same pattern would hold for any x you choose. A simultaneous plot of the two functions (see Figure 0.73) shows this. ■

In general, for $c > 0$, the graph of $y = f(x - c)$ is the same as the graph of $y = f(x)$ shifted c units to the right. Likewise (again, for $c > 0$), you get the graph of $f(x + c)$ by moving the graph of $y = f(x)$ to the left c units. We usually refer to $f(x - c)$ and $f(x + c)$ as **horizontal translations** (to the right and left, respectively, by c units).

To avoid confusion on which way to translate the graph of $y = f(x)$, focus on what makes the argument (the quantity inside the parentheses) zero. For $f(x)$, this is $x = 0$, but for $f(x - c)$ you must have $x = c$ to get $f(0)$ [i.e., the same y -value as $f(x)$ when $x = 0$]. This says that the point on the graph of $y = f(x)$ at $x = 0$ corresponds to the point on the graph of $y = f(x - c)$ at $x = c$.

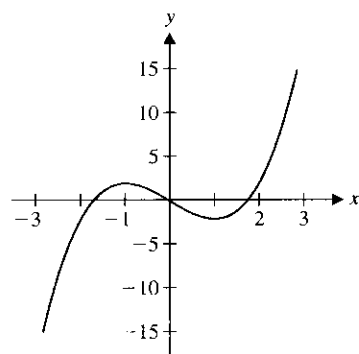


FIGURE 0.74a
 $y = f(x)$.

EXAMPLE 6.6 Comparing Vertical and Horizontal Translations

Given the graph of $y = f(x)$ shown in Figure 0.74a, sketch the graphs of $y = f(x) - 2$ and $y = f(x - 2)$.

Solution To graph $y = f(x) - 2$, simply translate the original graph down 2 units, as shown in Figure 0.74b. To graph $y = f(x - 2)$, simply translate the original graph to the right 2 units (so that the x -intercept at $x = 0$ in the original graph corresponds to an x -intercept at $x = 2$ in the translated graph), as seen in Figure 0.74c.

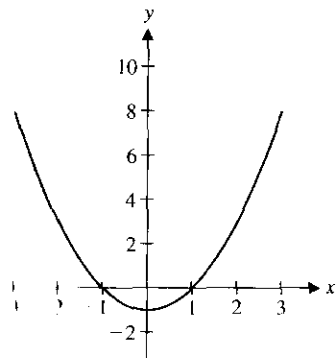


FIGURE 0.75a
 $y = x^2 - 1$.

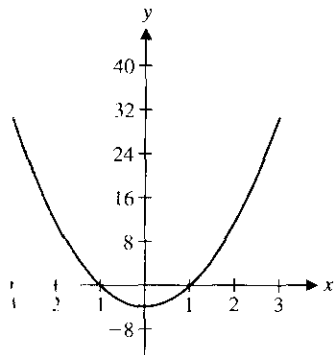


FIGURE 0.75b
 $y = 4(x^2 - 1)$.

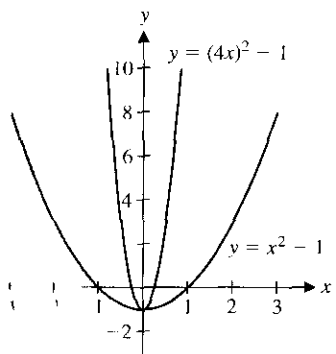


FIGURE 0.75e
 $y = x^2 - 1$ and $y = (4x)^2 - 1$.

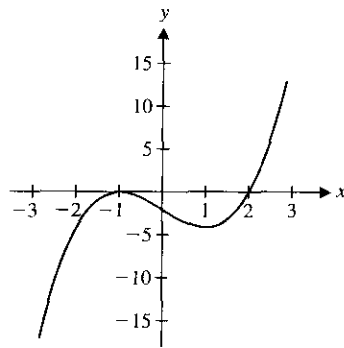


FIGURE 0.74b
 $y = f(x) - 2$.

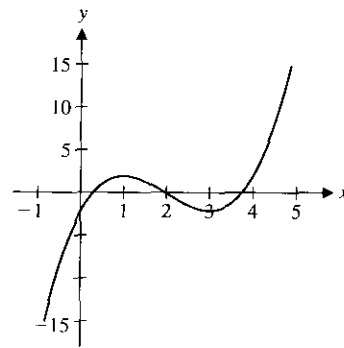


FIGURE 0.74c
 $y = f(x - 2)$.

Example 6.7 explores the effect of multiplying or dividing x or y by a constant.

EXAMPLE 6.7 Comparing Some Related Graphs

Compare and contrast the graphs of $y = x^2 - 1$, $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$.

Solution The first two graphs are shown in Figures 0.75a and 0.75b, respectively.

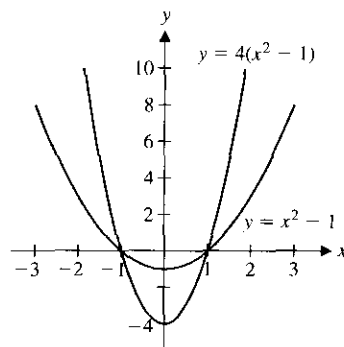


FIGURE 0.75c
 $y = x^2 - 1$ and $y = 4(x^2 - 1)$.

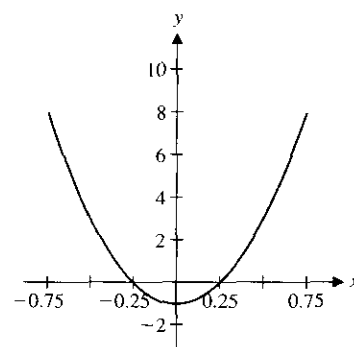


FIGURE 0.75d
 $y = (4x)^2 - 1$.

These graphs look identical until you compare the scales on the y -axes. The values in Figure 0.75b are four times as large, resulting from the multiplication of the original function by 4. The effect looks different when the functions are plotted on the same scale, as in Figure 0.75c. Here, the parabola $y = 4(x^2 - 1)$ looks thinner. Note that the x -intercepts remain the same but the y -intercepts are different. (Why would that be?) The graph of $y = (4x)^2 - 1$ is shown in Figure 0.75d.

Can you spot the difference here? In this case, the x -scale has now changed, by the same factor of 4 as in the function. To see this, note that substituting $x = 1/4$ into $(4x)^2 - 1$ produces $(1)^2 - 1$, exactly the same as substituting $x = 1$ into the original function. When plotted on the same set of axes (as in Figure 0.75e), the parabola $y = (4x)^2 - 1$ looks thinner. Note that here, the x -intercepts are different, but the y -intercepts are the same. (Why would that be?) ■

We can generalize the observations made in example 6.7. Before reading our explanation, try to state a general rule for yourself. How are the graphs of the functions $cf(x)$ and $f(cx)$ related to the graph of $y = f(x)$?

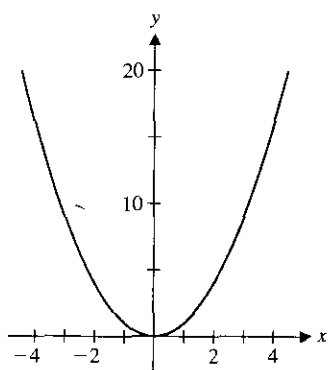


FIGURE 0.76a

$$y = x^2.$$

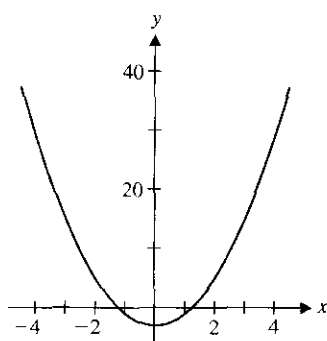


FIGURE 0.76b

$$y = 2x^2 - 3.$$

Based on example 6.7, notice that to obtain a graph of $y = cf(x)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the y -axis by c . To obtain a graph of $y = f(cx)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the x -axis by $1/c$.

These basic rules can be combined to understand more complicated graphs.

EXAMPLE 6.8 A Translation and a Stretching

Describe how to get the graph of $y = 2x^2 - 3$ from the graph of $y = x^2$.

Solution You can get from x^2 to $2x^2 - 3$ by multiplying by 2 and then subtracting 3. In terms of the graph, this has the effect of multiplying the y -scale by 2 and then shifting the graph down by 3 units (see the graphs in Figures 0.76a and 0.76b). ■

EXAMPLE 6.9 A Translation Involving Both x and y Scales

Describe how to get the graph of $y = x^2 + 4x + 3$ from the graph of $y = x^2$.

Solution We can again relate this (and the graph of **every** quadratic) to the graph of $y = x^2$. We must first **complete the square**. Recall that in this process, you take the coefficient of x (4), divide by 2 ($4/2 = 2$) and square the result ($2^2 = 4$). Add and subtract this number and then, rewrite the x terms as a perfect square. We have

$$y = x^2 + 4x + 3 = (x^2 + 4x + 4) - 4 + 3 = (x + 2)^2 - 1.$$

To graph this function, take the parabola $y = x^2$ (see Figure 0.77a) and translate the graph 2 units to the left and 1 unit down (see Figure 0.77b).

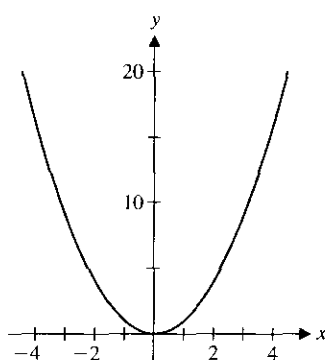


FIGURE 0.77a

$$y = x^2.$$

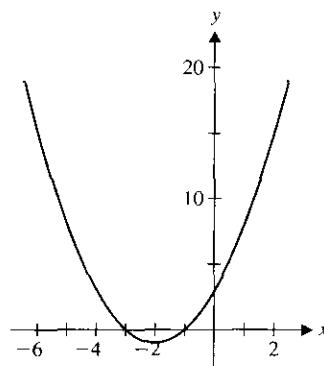


FIGURE 0.77b

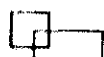
$$y = (x + 2)^2 - 1.$$

The following table summarizes our discoveries in this section.

Transformations of $f(x)$

Transformation	Form	Effect on Graph
Vertical translation	$f(x) + c$	$ c $ units up ($c > 0$) or down ($c < 0$)
Horizontal translation	$f(x + c)$	$ c $ units left ($c > 0$) or right ($c < 0$)
Vertical scale	$cf(x)$ ($c > 0$)	multiply vertical scale by c
Horizontal scale	$f(cx)$ ($c > 0$)	divide horizontal scale by c

You will explore additional transformations in the exercises.



EXERCISES 0.6

WRITING EXERCISES

- The restricted domain of example 6.2 may be puzzling. Consider the following analogy. Suppose you have an airplane flight from New York to Los Angeles with a stop for refueling in Minneapolis. If bad weather has closed the airport in Minneapolis, explain why your flight will be canceled (or at least rerouted) even if the weather is great in New York and Los Angeles.
- Explain why the graphs of $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$ in Figures 0.75c and 0.75e appear "thinner" than the graph of $y = x^2 - 1$.
- As illustrated in example 6.9, completing the square can be used to rewrite any quadratic function in the form $a(x - d)^2 + e$. Using the transformation rules in this section, explain why this means that all parabolas (with $a > 0$) will look essentially the same.
- Explain why the graph of $y = f(x + 4)$ is obtained by moving the graph of $y = f(x)$ four units to the left, instead of to the right.

In exercises 1–6, find the compositions $f \circ g$ and $g \circ f$ and identify their respective domains.

- $f(x) = x + 1$, $g(x) = \sqrt{x - 3}$
- $f(x) = x - 2$, $g(x) = \sqrt{x + 1}$
- $f(x) = e^x$, $g(x) = \ln x$
- $f(x) = \sqrt{1 - x}$, $g(x) = \ln x$
- $f(x) = x^2 + 1$, $g(x) = \sin x$
- $f(x) = \frac{1}{x^2 - 1}$, $g(x) = x^2 - 2$

In exercises 7–16, identify functions $f(x)$ and $g(x)$ such that the given function equals $(f \circ g)(x)$.

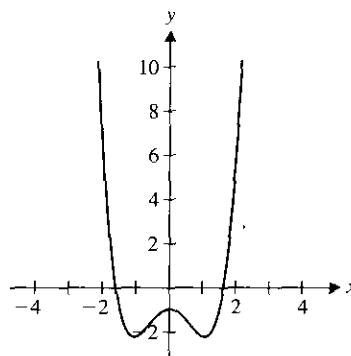
- $\sqrt{x^4 + 1}$
- $\sqrt[3]{x + 3}$
- $\frac{1}{x^2 + 1}$
- $\frac{1}{x^2} + 1$
- $(4x + 1)^2 + 3$
- $4(x + 1)^2 + 3$
- $\sin^3 x$
- $\sin x^3$
- $e^{x^2 + 1}$
- $e^{4x - 2}$

In exercises 17–22, identify functions $f(x)$, $g(x)$ and $h(x)$ such that the given function equals $[f \circ (g \circ h)](x)$.

- $\frac{3}{\sqrt{\sin x + 2}}$
- $\sqrt{e^{4x} + 1}$
- $\cos^3(4x - 2)$
- $\ln \sqrt{x^2 + 1}$
- $4e^{x^2 - 5}$
- $[\tan^{-1}(3x + 1)]^2$

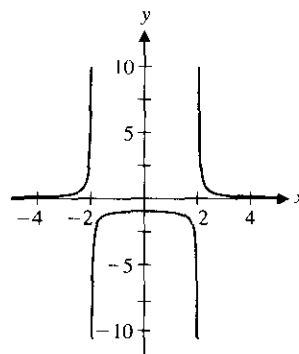
In exercises 23–30, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

- $f(x) - 3$
- $f(x + 2)$
- $f(x - 3)$
- $f(x) + 2$
- $f(2x)$
- $3f(x)$
- $4f(x) - 1$
- $3f(x + 2)$



In exercises 31–38, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

- $f(x - 4)$
- $f(x + 3)$
- $f(2x)$
- $f(2x - 4)$
- $f(3x + 3)$
- $3f(x)$
- $2f(x) - 4$
- $3f(x) + 3$



In exercises 39–44, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

- $f(x) = x^2 + 2x + 1$
- $f(x) = x^2 - 4x + 4$
- $f(x) = x^2 + 2x + 4$
- $f(x) = x^2 - 4x + 2$
- $f(x) = 2x^2 + 4x + 4$
- $f(x) = 3x^2 - 6x + 2$

T In exercises 45–48, graph the given function and compare to the graph of $y = x^2 - 1$.

45. $f(x) = -2(x^2 - 1)$ 46. $f(x) = -3(x^2 - 1)$
 47. $f(x) = -3(x^2 - 1) + 2$ 48. $f(x) = -2(x^2 - 1) - 1$

T In exercises 49–52, graph the given function and compare to the graph of $y = (x - 1)^2 - 1 = x^2 - 2x$.

49. $f(x) = (-x)^2 - 2(-x)$
 50. $f(x) = (-2x)^2 - 2(-2x)$
 51. $f(x) = (-x)^2 - 2(-x) + 1$
 52. $f(x) = (-3x)^2 - 2(-3x) - 3$

53. Based on exercises 45–48, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = cf(x)$ for $c < 0$.

54. Based on exercises 49–52, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = f(cx)$ for $c < 0$.

55. Sketch the graph of $y = |x|^3$. Explain why the graph of $y = |x|^3$ is identical to that of $y = x^3$ to the right of the y -axis. For $y = |x|^3$, describe how the graph to the left of the y -axis compares to the graph to the right of the y -axis. In general, describe how to draw the graph of $y = f(|x|)$ given the graph of $y = f(x)$.

56. For $y = x^3$, describe how the graph to the left of the y -axis compares to the graph to the right of the y -axis. Show that for $f(x) = x^3$, we have $f(-x) = -f(x)$. In general, if you have the graph of $y = f(x)$ to the right of the y -axis and $f(-x) = -f(x)$ for all x , describe how to graph $y = f(x)$ to the left of the y -axis.

T 57. **Iterations** of functions are important in a variety of applications. To iterate $f(x)$, start with an initial value x_0 and compute $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$ and so on. For example, with $f(x) = \cos x$ and $x_0 = 1$, the **iterates** are $x_1 = \cos 1 \approx 0.54$, $x_2 = \cos x_1 \approx \cos 0.54 \approx 0.86$, $x_3 \approx \cos 0.86 \approx 0.65$ and so on. Keep computing iterates and show that they get closer and closer to 0.739085. Then pick your own x_0 (any number you like) and show that the iterates with this new x_0 also converge to 0.739085.

T 58. Referring to exercise 57, show that the iterates of a function can be written as $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, $x_3 = f(f(f(x_0)))$ and so on. Graph $y = \cos(\cos x)$, $y = \cos(\cos(\cos x))$ and $y = \cos(\cos(\cos(\cos x)))$. The graphs should look more and more like a horizontal line. Use the result of exercise 57 to identify the limiting line.

T 59. Compute several iterates of $f(x) = \sin x$ (see exercise 57) with a variety of starting values. What happens to the iterates in the long run?

T 60. Repeat exercise 59 for $f(x) = x^2$.

T 61. In cases where the iterates of a function (see exercise 57) repeat a single number, that number is called a **fixed point**. Explain why any fixed point must be a solution of the equation $f(x) = x$. Find all fixed points of $f(x) = \cos x$ by solving the equation $\cos x = x$. Compare your results to that of exercise 57.

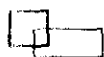
T 62. Find all fixed points of $f(x) = \sin x$ (see exercise 61). Compare your results to those of exercise 59.

✚ EXPLORATORY EXERCISES

1. You have explored how completing the square can transform any quadratic function into the form $y = a(x - d)^2 + e$. We concluded that all parabolas with $a > 0$ look alike. To see that the same statement is not true of cubic polynomials, graph $y = x^3$ and $y = x^3 - 3x$. In this exercise, you will use completing the cube to determine how many different cubic graphs there are. To see what “completing the cube” would look like, first show that $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$. Use this result to transform the graph of $y = x^3$ into the graphs of (a) $y = x^3 - 3x^2 + 3x - 1$ and (b) $y = x^3 - 3x^2 + 3x + 2$. Show that you can’t get a simple transformation to $y = x^3 - 3x^2 + 4x - 2$. However, show that $y = x^3 - 3x^2 + 4x - 2$ can be obtained from $y = x^3 + x$ by basic transformations. Show that the following statement is true: any cubic ($y = ax^3 + bx^2 + cx + d$) can be obtained with basic transformations from $y = ax^3 + kx$ for some constant k .

2. In many applications, it is important to take a section of a graph (e.g., some data) and extend it for predictions or other analysis. For example, suppose you have an electronic signal equal to $f(x) = 2x$ for $0 \leq x \leq 2$. To predict the value of the signal at $x = -1$, you would want to know whether or not the signal was periodic. If the signal is periodic, explain why $f(-1) = 2$ would be a good prediction. In some applications, you would assume that the function is *even*. That is, $f(x) = f(-x)$ for all x . In this case, you want $f(x) = 2(-x) = -2x$ for $-2 \leq x \leq 0$. Graph the *even extension* $f(x) = \begin{cases} -2x & \text{if } -2 \leq x \leq 0 \\ 2x & \text{if } 0 \leq x \leq 2 \end{cases}$. Find the even extension for (a) $f(x) = x^2 + 2x + 1$, $0 \leq x \leq 2$ and (b) $f(x) = e^{-x}$, $0 \leq x \leq 2$.

3. Similar to the even extension discussed in exploratory exercise 2, applications sometimes require a function to be *odd*; that is, $f(-x) = -f(x)$. For $f(x) = x^2$, $0 \leq x \leq 2$, the odd extension requires that for $-2 \leq x \leq 0$, $f(x) = -f(-x) = -(-x)^2 = -x^2$ so that $f(x) = \begin{cases} -x^2 & \text{if } -2 \leq x \leq 0 \\ x^2 & \text{if } 0 \leq x \leq 2 \end{cases}$. Graph $y = f(x)$ and discuss how to graphically rotate the right half of the graph to get the left half of the graph. Find the odd extension for (a) $f(x) = x^2 + 2x$, $0 \leq x \leq 2$ and (b) $f(x) = e^{-x} - 1$, $0 \leq x \leq 2$.



0.7 PARAMETRIC EQUATIONS AND POLAR COORDINATES



RoboCup is an international competition of robot soccer teams. These are not the remote-controlled attack robots you may have seen on television. Instead, the robots are engineered and programmed to automatically respond to the positions of the ball, goal and other players. As programmers of such robots, we would be frustrated by the limitations of the x - y system (called **rectangular coordinates**) we have used in the first six sections of this chapter. First of all, the path followed by a robot is unlikely to pass the vertical line test for a function. Even if it did, the path of the robot is irrelevant unless we know *when* the robot was at each point. Further, what a robot needs to know about a competitor are its distance and direction, not its x - and y -coordinates.

In this section, we introduce two alternatives to functions in rectangular coordinates. Parametric equations will enable us to track the position of an object as a function of time. Polar coordinates enable us to keep track of points in terms of distance and direction. These alternative descriptions give us a great deal of needed flexibility for attacking a variety of problems. Many complicated looking graphs have a very simple description in parametric equations or polar coordinates.

□ Parametric Equations

To describe the position of an object (such as a robot) in terms of time, we need functions $g(t)$ and $h(t)$ that give the object's x - and y -coordinates, respectively, at any given time t . So, from this description, we not only know the path the object follows, but we also know *when* it passes through each point. This is the motivation for the following general definition of parametric equations.

Given any pair of functions $g(t)$ and $h(t)$ defined on the same domain D , the equations

$$x = g(t), \quad y = h(t)$$

are called **parametric equations**. Notice that for each choice of t , the parametric equations specify a point $(x, y) = (g(t), h(t))$ in the xy -plane. The collection of all such points is called the **graph** of the parametric equations. The graph is a curve in the xy -plane, referred to as a **parametric curve**.

The choice of the letter t to denote the independent variable (called the **parameter**) should make you think of *time*. In many situations, this is exactly what the parameter represents. In some applications, the parameter has an interpretation other than time; in others, it has no physical meaning at all. In general, the parameter can be *any* quantity that is convenient for describing the relationship between x and y . In our first example, we can simplify our discussion by eliminating the parameter.

EXAMPLE 7.1 Graphing a Parametric Curve

Sketch the curve defined by the parametric equations $x = 6 - t^2$, $y = t/2$, for $-2 \leq t \leq 4$.

Solution In the table that follows, we list a number of values of the parameter t and the corresponding values of x and y .

t	-2	-1	0	1	2	3	4
x	2	5	6	5	2	-3	-10
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

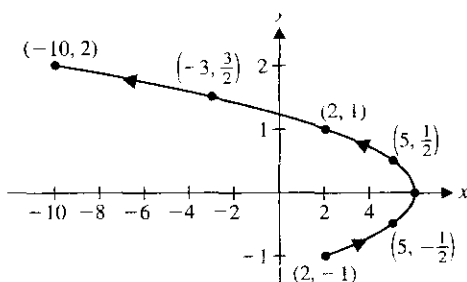
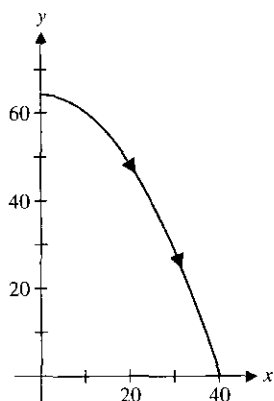


FIGURE 0.78

$$x = 6 - t^2, y = \frac{t}{2}, -2 \leq t \leq 4.$$

We have plotted these points and connected them with a smooth curve in Figure 0.78. You might also notice that we can easily eliminate the parameter here, by solving for t in terms of y . We have $t = 2y$, so that $x = 6 - 4y^2$. The graph of this last equation is a parabola opening to the left. The plane curve we're looking for is the portion of this parabola corresponding to $-2 \leq t \leq 4$. From the table, notice that this corresponds to $-1 \leq y \leq 2$, so that the plane curve is the portion of the parabola indicated in Figure 0.78, where we have also indicated a number of points on the curve. ■

You probably noticed the small arrows drawn on top of the plane curve in Figure 0.78. These indicate the **orientation** of the curve (i.e., the direction of increasing t). In the case where t represents time and the curve represents the path of an object, notice that the orientation indicates the direction followed by the object as it traverses the path, as in the following example.

FIGURE 0.79
Path of projectile.

EXAMPLE 7.2 The Path of a Projectile

The path of a projectile thrown horizontally with initial speed of 20 ft/s from a height of 64 feet, has parametric equations $x = 20t$, $y = 64 - 16t^2$, for $0 \leq t \leq 2$. Sketch the path.

Solution As in example 7.1, we can solve for t , to get $t = x/20$, so that $y = 64 - \frac{x^2}{25}$.

With $0 \leq t \leq 2$, we get $0 \leq x \leq 40$. The portion of the parabola $y = 64 - \frac{x^2}{25}$ with $0 \leq x \leq 40$ is shown in Figure 0.79. Note that in this case, the orientation indicated in the graph gives the direction of motion. Although we used the x - y equation for graphing, observe that the parametric equations also tell us *when* the object is located at a given point and indicates the direction of motion. ■

Graphing calculators and computer algebra systems sketch a parametric graph by plotting points corresponding to a large number of values of the parameter t and then connecting the plotted points with a smooth curve. The appearance of the resulting graph depends greatly on the graphing window used *and also* on the particular choice of t -values. This can be seen in the following example.

EXAMPLE 7.3 Parametric Equations Involving Sines and Cosines

Sketch the curve defined by the parametric equations

$$x = 2 \cos t, y = 2 \sin t, \text{ for (a) } 0 \leq t \leq 2\pi \text{ and (b) } 0 \leq t \leq \pi \quad (7.1)$$

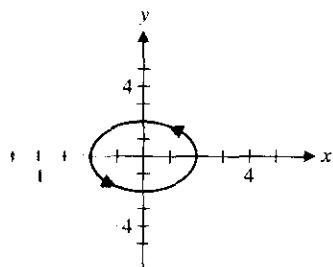


FIGURE 0.80a

$$x = 2 \cos t, y = 2 \sin t.$$

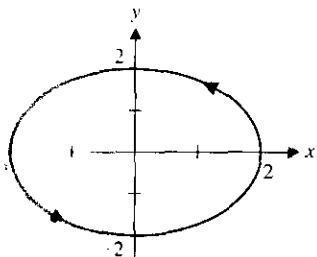


FIGURE 0.80b

$$x = 2 \cos t, y = 2 \sin t.$$

REMARK 7.1

To sketch a parametric graph on a CAS, you may need to write the equations in **vector format**. For instance, in the case of example 7.3, instead of entering $x = 2 \cos t$ and $y = 2 \sin t$, you would enter the ordered pair of functions $(2 \cos t, 2 \sin t)$.

Solution The default graph produced by most graphing calculators looks something like the curve shown in Figure 0.80a (where we have added arrows indicating the orientation). With some thought, we can improve this sketch. First, notice that since $x = 2 \cos t$, x ranges between -2 and 2 . Similarly, y ranges between -2 and 2 . Changing the graphing window to $-2.1 \leq x \leq 2.1$ and $-2.1 \leq y \leq 2.1$ produces the curve shown in Figure 0.80b. The curve still looks like an ellipse, but with some more thought we can identify it as a circle. Rather than eliminate the parameter by solving for t in terms of either x or y , instead notice from (7.1) that

$$x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4(\cos^2 t + \sin^2 t) = 4.$$

So, the plane curve lies on the circle of radius 2 centered at the origin. In fact, it's the whole circle, as we can see by recognizing what the parameter represents in this case. Recall from the definition of sine and cosine that if (x, y) is a point on the unit circle and θ is the angle from the positive x -axis to the line segment joining (x, y) and the origin, then we define $\cos \theta = x$ and $\sin \theta = y$. Since we have $x = 2 \cos t$ and $y = 2 \sin t$, the parameter t corresponds to the angle θ . Further, the curve is the entire circle of radius 2, traced out as the angle t ranges from 0 to 2π . A "square" graphing window is one with the same scale on the x - and y -axes (not necessarily the same x and y ranges, but the same scales!). Some programs generate such a window when you specify an **aspect ratio** of 1. Such a square window gives us the circle seen in Figure 0.80c. Finally, what would change if the domain were limited to $0 \leq t \leq \pi$? Since we've identified t as the angle as measured from the positive x -axis, it should be clear that you will now get the top half of the circle of radius 2, as shown in Figure 0.80d.

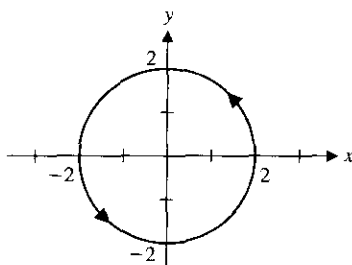


FIGURE 0.80c

A circle.

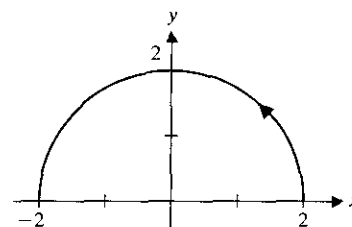


FIGURE 0.80d

Top semicircle.

Simple modifications to the parametric equations in example 7.3 will produce a variety of circles and ellipses. We explore this in example 7.4 and the exercises.

EXAMPLE 7.4 More Circles and Ellipses Defined by Parametric Equations

Identify the plane curves (a) $x = 2 \cos t$, $y = 3 \sin t$, (b) $x = 2 + 4 \cos t$, $y = 3 + 4 \sin t$ and (c) $x = 3 \cos 2t$, $y = 3 \sin 2t$, all for $0 \leq t \leq 2\pi$.

Solution A computer-generated sketch of (a) is shown in Figure 0.81a. It's difficult to determine from the sketch whether the curve is an ellipse or simply a distorted graph of a circle. You should rule out a circle, since the parametric equations produce x -values between -2 and 2 and y -values between -3 and 3 . To verify that this is an ellipse, notice that

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4 \cos^2 t}{4} + \frac{9 \sin^2 t}{9} = \cos^2 t + \sin^2 t = 1.$$

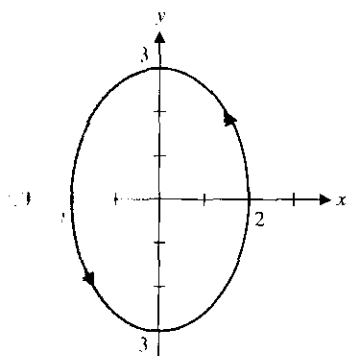


FIGURE 0.81a

$$x = 2 \cos t, y = 3 \sin t.$$

A computer-generated sketch of (b) is shown in Figure 0.81b. You should verify that this is the circle $(x - 2)^2 + (y - 3)^2 = 16$. Finally, a computer sketch of (c) is shown in Figure 0.81c. You should verify that this is the circle $x^2 + y^2 = 9$. So, what is the role of the 2 in the argument of cosine and sine? If you sketched this on a calculator, you may have noticed that the circle was completed long before the calculator finished graphing. Because of the 2, a complete circle corresponds to $0 \leq 2t \leq 2\pi$ or $0 \leq t \leq \pi$. With the domain $0 \leq t \leq 2\pi$, the circle is traced out twice. You might say that the factor of 2 in the argument doubles the speed with which the curve is traced.

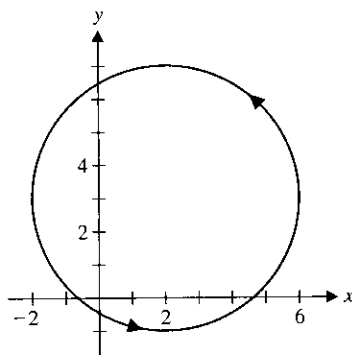


FIGURE 0.81b
 $x = 2 + 4 \cos t$, $y = 3 + 4 \sin t$.

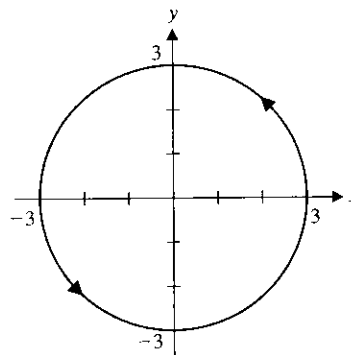


FIGURE 0.81c
 $x = 3 \cos 2t$, $y = 3 \sin 2t$.

REMARK 7.2

Look carefully at the plane curves in examples 7.3 and 7.4 until you can identify the roles of each of the constants in the equations $x = a + b \cos ct$, $y(t) = d + e \sin ct$. These interpretations are important in applications. In particular, (a, d) is the center of an ellipse with axes of length $2b$ and $2e$.

In example 7.5, we see how to find parametric equations for a line segment.

EXAMPLE 7.5 Parametric Equations for a Line Segment

Find parametric equations for the line segment joining the points $(1, 2)$ and $(4, 7)$.

Solution For a line segment, notice that the parametric equations can be linear functions. That is,

$$x = a + bt, \quad y = c + dt,$$

for some constants a, b, c and d . (Eliminate the parameter t to see why this generates a line.) The simplest way to choose these constants is to have $t = 0$ correspond to the starting point $(1, 2)$. Note that if $t = 0$, the equations reduce to $x = a$ and $y = c$. To start our segment at $x = 1$ and $y = 2$, we set $a = 1$ and $c = 2$. Now note that with $t = 1$, the equations are $x = a + b$ and $y = c + d$. To produce the endpoint $(4, 7)$, we must have $a + b = 4$ and $c + d = 7$. With $a = 1$ and $c = 2$, solve to get $b = 3$ and $d = 5$. We now have that

$$x = 1 + 3t, \quad y = 2 + 5t, \quad \text{for } 0 \leq t \leq 1$$

is a pair of parametric equations describing the line segment. ■

In general, for parametric equations of the form $x = a + bt$, $y = c + dt$, notice that you can always choose a and c to be the x - and y -coordinates, respectively, of the starting point (since $x = a$, $y = c$ corresponds to $t = 0$). Then b is the difference in x -coordinates (endpoint minus starting point) and d is the difference in y -coordinates. With these choices, the line segment is always sketched out for $0 \leq t \leq 1$.

REMARK 7.3

Note that in example 7.5, as for all parametric equations, there are an infinite number of choices of parameters that produce a given curve. For instance, you can easily verify that the parametric equations

$$x = -2 + 3t, \quad y = -3 + 5t, \quad \text{for } 1 \leq t \leq 2$$

and

$$x = t, \quad y = \frac{1 + 5t}{3}, \quad \text{for } 1 \leq t \leq 4$$

also produce the line segment of example 7.5. We say that each of these pairs of parametric equations is a different **parameterization** of the curve.

As we illustrate with example 7.6, every equation of the form $y = f(x)$ can be simply written in parametric equations.

EXAMPLE 7.6 Parametric Equations from an x - y Equation

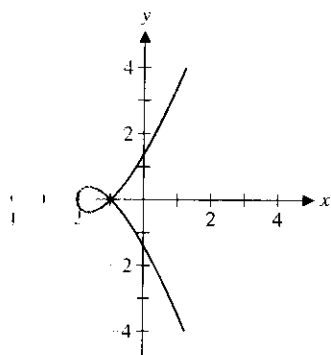
Find parametric equations for the portion of the parabola $y = x^2$ from $(-1, 1)$ to $(3, 9)$.

Solution Any equation of the form $y = f(x)$ can be converted to parametric form by simply letting x equal t . Here, this gives us $y = x^2 = t^2$. Then

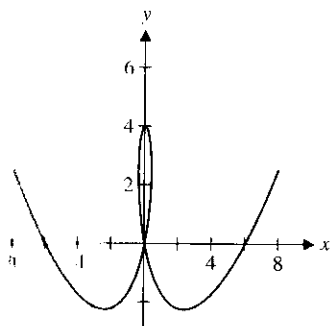
$$x = t, \quad y = t^2, \quad \text{for } -1 \leq t \leq 3,$$

is a parametric representation of the curve. (Of course, you can use the letter x as the parameter instead of the letter t if you prefer.) ■

We can describe many plane curves parametrically that are unlike anything you've seen so far in your studies. Many of these are difficult to draw by hand, but can be easily plotted with a graphing calculator or CAS.

**FIGURE 0.82a**

$$x = t^2 - 2, \quad y = t^3 - t.$$

**FIGURE 0.82b**

$$x = t^3 - t, \quad y = t^4 - 5t^2 + 4.$$

EXAMPLE 7.7 Some Unusual Plane Curves

Sketch the plane curves (a) $x = t^2 - 2$, $y = t^3 - t$ and (b) $x = t^3 - t$, $y = t^4 - 5t^2 + 4$.

Solution A sketch of (a) is shown in Figure 0.82a. This is clearly not the graph of any function (consider the vertical line test to see why), so converting to an x - y equation would not be particularly helpful. You should, however, examine the parametric equations to see if important portions of the graph have been left out (e.g., is there supposed to be anything to the left of $x = -2$?). In this case, $x = t^2 - 2 \geq -2$ for all t and $y = t^3 - t$ has no maximum or minimum (think about why). It seems that most of the graph is indeed shown in Figure 0.82a. A computer sketch of (b) is shown in Figure 0.82b. Again, this is not the graph of a function. To get an idea of the scope of the graph, note that $x = t^3 - t$ has no maximum or minimum. To find the minimum of $y = t^4 - 5t^2 + 4$, note that

$$t^4 - 5t^2 + 4 = (t^2 - 5/2)^2 - \frac{9}{4}.$$

You should conclude that $y \geq -\frac{9}{4}$, as indicated in Figure 0.82b. ■

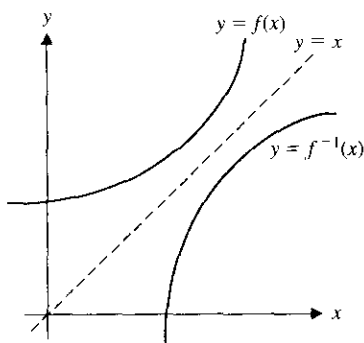


FIGURE 0.83a
 $y = f(x)$ and $y = f^{-1}(x)$.

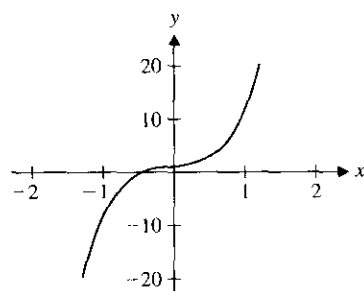


FIGURE 0.83b
 $y = x^5 + 8x^3 + x + 1$.

In section 0.3, we found the relationship between graphs of one-to-one functions and their inverses. Recall that you can reflect the graph of $y = f(x)$ across the line $y = x$ to get the graph of $y = f^{-1}(x)$. (See Figure 0.83a.) Using parametric equations, this can be easily implemented on a graphing calculator.

EXAMPLE 7.8 Drawing the Graph of an Unknown Inverse Function

Draw a graph of the inverse function of $f(x) = x^5 + 8x^3 + x + 1$.

Solution Given the graph of $f(x)$ (see Figure 0.83b), we could certainly draw by hand part of the graph of the inverse. However, recall that if (x, y) is a point on the graph of $f(x)$ [i.e., $y = f(x)$], then (y, x) is a point on the graph of $f^{-1}(x)$ [i.e., $x = f^{-1}(y)$]. The simplest choice of parametric equations for $y = f(x)$ is

$$x = t$$

$$y = f(t).$$

Swapping x and y now gives parametric equations for $f^{-1}(x)$, namely,

$$x = f(t)$$

$$y = t.$$

In this case, we want $x = t^5 + 8t^3 + t + 1$ and $y = t$. The graph of $f^{-1}(x)$ along with the graphs of $f(x)$ and $y = x$ are shown in Figures 0.83c and 0.83d.

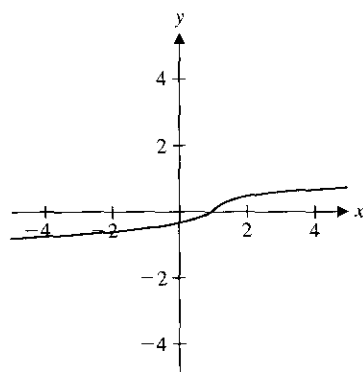


FIGURE 0.83c
 $y = f^{-1}(x)$.

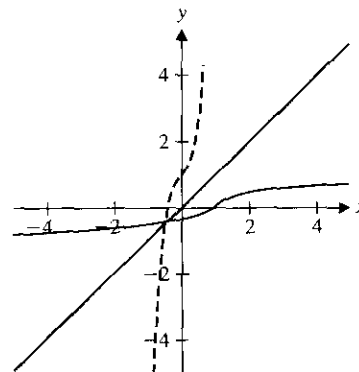


FIGURE 0.83d
 $y = f(x)$ and $y = f^{-1}(x)$.

You should now have some idea of the flexibility of parametric equations. This flexibility alone increases our ability to solve problems and makes the study of parametric equations worthwhile.

□ Polar Coordinates

The search for a more convenient description of a graph leads us to some important ideas. Recall that in our discussion of robot soccer, we observed that in the heat of the action, information about the distance and direction to the goal is more important than the x - and y -coordinates of the goal. This kind of description is the idea behind polar coordinates.

You are familiar with the rectangular coordinates (x, y) of a point, where we identify a point by its horizontal displacement x and vertical displacement y from the origin.

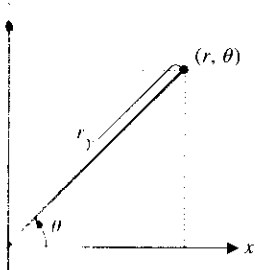


FIGURE 0.84
Polar coordinates.

An alternative description of a point in the xy -plane consists of specifying the distance r from the point to the origin and the angle θ (in radians) measured from the positive x -axis counterclockwise to the ray connecting the point and the origin (see Figure 0.84). We describe the point by the ordered pair (r, θ) and refer to r and θ as **polar coordinates** for the point.

EXAMPLE 7.9 Converting from Polar to Rectangular Coordinates

Plot the points with the indicated polar coordinates and determine the corresponding rectangular coordinates (x, y) , for: (a) $(2, 0)$, (b) $(3, \frac{\pi}{2})$, (c) $(-3, \frac{\pi}{2})$ and (d) $(2, \pi)$.

Solution (a) Notice that the angle $\theta = 0$ locates the point on the positive x -axis. At a distance of $r = 2$ units from the origin, this corresponds to the point $(2, 0)$ in rectangular coordinates (see Figure 0.85a).

(b) The angle $\theta = \frac{\pi}{2}$ locates points on the positive y -axis. At a distance of $r = 3$ units from the origin, this corresponds to the point $(0, 3)$ in rectangular coordinates (see Figure 0.85b).

(c) The angle is the same as in (b), but a negative value of r indicates that the point is located 3 units in the opposite direction, at the point $(0, -3)$ in rectangular coordinates (see Figure 0.85b).

(d) The angle $\theta = \pi$ corresponds to the negative x -axis. The distance of $r = 2$ units from the origin gives us the point $(-2, 0)$ in rectangular coordinates (see Figure 0.85c).

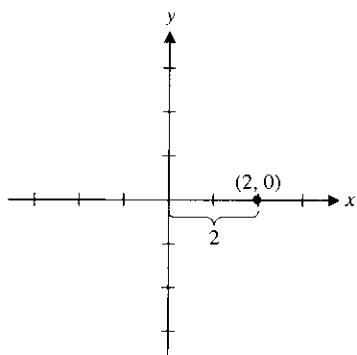


FIGURE 0.85a
The point $(2, 0)$ in polar coordinates.

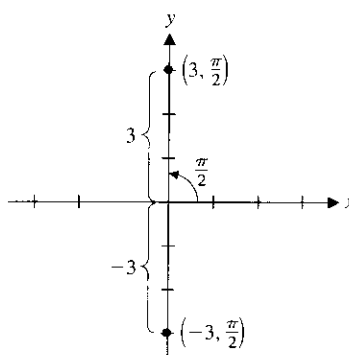


FIGURE 0.85b
The points $(3, \frac{\pi}{2})$ and $(-3, \frac{\pi}{2})$ in polar coordinates.

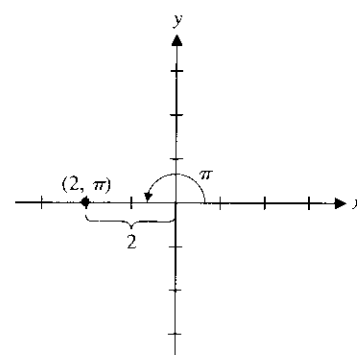


FIGURE 0.85c
The point $(2, \pi)$ in polar coordinates.

EXAMPLE 7.10 Converting from Rectangular to Polar Coordinates

Find all polar coordinate representations of the rectangular point $(1, 1)$.

Solution From Figure 0.86a (on the following page), notice that the point lies on the line $y = x$, which makes an angle of $\frac{\pi}{4}$ with the positive x -axis. From the distance formula, we get that $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. This says that we can write the point as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates. Referring to Figure 0.86b, notice that we can specify the same point by using a negative value of r , $r = -\sqrt{2}$ with the angle $\frac{5\pi}{4}$. (Think about this some.) Notice further, that the angle $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ corresponds to the same ray shown in Figure 0.86a

(see Figure 0.86c). In fact, all of the polar points $(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ and $(-\sqrt{2}, \frac{5\pi}{4} + 2n\pi)$ for any integer n correspond to the same point in the xy -plane.

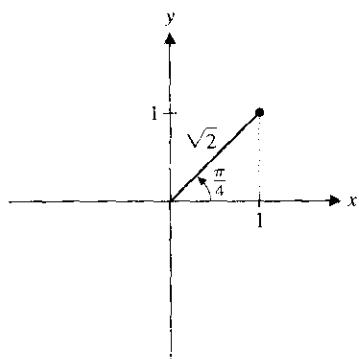


FIGURE 0.86a
Polar coordinates for the point $(1, 1)$.

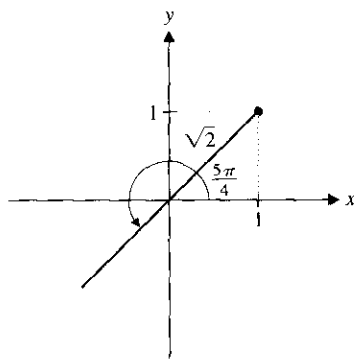


FIGURE 0.86b
An alternative polar representation of $(1, 1)$.

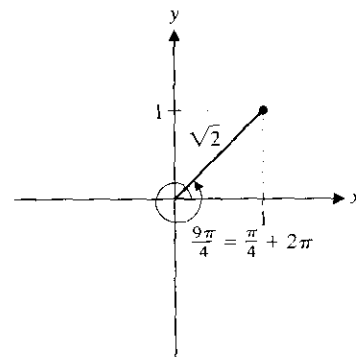


FIGURE 0.86c
Another polar representation of the point $(1, 1)$.

REMARK 7.4

Given a distance r and angle θ , there is exactly one point in the xy -plane with the polar coordinates (r, θ) . However, as we saw in example 7.10, for a given point (x, y) in the plane, there are an infinite number of possible polar coordinate representations. In particular, you can use both positive and negative values of r . Also, for a given angle θ , the angles $\theta \pm 2\pi$, $\theta \pm 4\pi$ and so on, all correspond to the same ray and can also be used. For convenience, we use the notation $\theta + 2n\pi$ (for any integer n) to represent all of these possible angles.

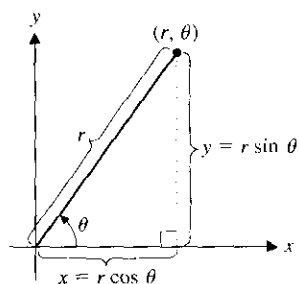


FIGURE 0.87
Converting from polar to rectangular coordinates.

Referring to Figure 0.87, notice that it is a simple matter to find the rectangular coordinates (x, y) of a point specified in polar coordinates as (r, θ) . From the usual definitions for $\sin \theta$ and $\cos \theta$, we get

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta. \quad (7.2)$$

As we've already observed, a given point (x, y) in the plane will have infinitely many polar coordinate representations. From equations (7.2), notice that

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

and for $x \neq 0$,

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

That is, every polar coordinate representation (r, θ) of the point (x, y) , where $x \neq 0$ must satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x}. \quad (7.3)$$



Notice that since there's more than one choice of r and θ , we cannot actually solve equations (7.3) to produce formulas for r and θ . In particular, while you might be tempted to write $\theta = \tan^{-1}(\frac{y}{x})$, this is not the only possible choice. Remember that for (r, θ) to be a polar representation of the point (x, y) , θ can be *any* angle for which $\tan \theta = \frac{y}{x}$, while $\tan^{-1}(\frac{y}{x})$ gives you only one angle θ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Finding polar coordinates for a given point is typically a process involving some graphing and some thought.

EXAMPLE 7.11 Converting from Rectangular to Polar Coordinates

Find all polar coordinate representations for the rectangular points (a) $(2, 3)$ and (b) $(-3, 1)$.

Solution (a) With $x = 2$ and $y = 3$, we have from (7.3) that

$$r^2 = x^2 + y^2 = 2^2 + 3^2 = 13,$$

so that $r = \pm\sqrt{13}$. Also,

$$\tan \theta = \frac{y}{x} = \frac{3}{2}.$$

One solution of this (the most obvious solution) is $\theta = \tan^{-1}(\frac{3}{2}) \approx 0.98$ radians. To determine which choice of r corresponds to this angle, note that $(2, 3)$ is located in the first quadrant (see Figure 0.88a). Since 0.98 radians also puts you in the first quadrant, this angle corresponds to the positive value of r , so that $(\sqrt{13}, \tan^{-1}(\frac{3}{2}))$ is one polar representation of the point. The negative choice of r corresponds to an angle one half-circle (i.e., π radians), away (see Figure 0.88b), so that another representation is $(-\sqrt{13}, \tan^{-1}(\frac{3}{2}) + \pi)$. Every other polar representation is found by adding multiples of 2π to the two angles used above. That is, every polar representation of the point $(2, 3)$ must have the form $(\sqrt{13}, \tan^{-1}(\frac{3}{2}) + 2n\pi)$ or $(-\sqrt{13}, \tan^{-1}(\frac{3}{2}) + 2n\pi)$, for some integer choice of n .

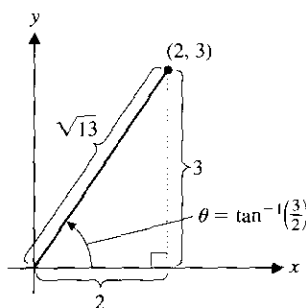


FIGURE 0.88a
The point $(2, 3)$.

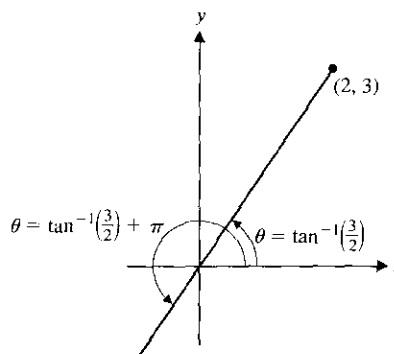


FIGURE 0.88b
Negative value of r .

(b) For the point $(-3, 1)$, we have $x = -3$ and $y = 1$. From (7.3), we have

$$r^2 = x^2 + y^2 = (-3)^2 + 1^2 = 10,$$

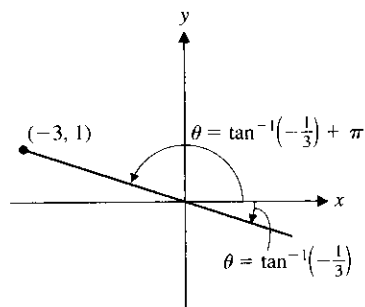


FIGURE 0.89

The point $(-3, 1)$.

so that $r = \pm\sqrt{10}$. Further,

$$\tan \theta = \frac{y}{x} = \frac{1}{-3},$$

so that the most obvious choice for the polar angle is $\theta = \tan^{-1}(-\frac{1}{3}) \approx -0.32$, which lies in the fourth quadrant. Since the point $(-3, 1)$ is in the second quadrant, this choice of the angle corresponds to the negative value of r (see Figure 0.89). The positive value of r then corresponds to the angle $\theta = \tan^{-1}(-\frac{1}{3}) + \pi$. Observe that all polar coordinate representations must then be of the form $(-\sqrt{10}, \tan^{-1}(-\frac{1}{3}) + 2n\pi)$ or $(\sqrt{10}, \tan^{-1}(-\frac{1}{3}) + \pi + 2n\pi)$, for some integer choice of n . ■

Observe that the conversion from polar coordinates to rectangular coordinates is completely straightforward, as in example 7.12.

REMARK 7.5

Notice that for any point (x, y) specified in rectangular coordinates ($x \neq 0$), we can always write the point in polar coordinates using either of the polar angles $\tan^{-1}(\frac{y}{x})$ or $\tan^{-1}(\frac{y}{x}) + \pi$. You can determine which angle corresponds to $r = \sqrt{x^2 + y^2}$ and which corresponds to $r = -\sqrt{x^2 + y^2}$ by looking at the quadrant in which the point lies.

EXAMPLE 7.12 Converting from Polar to Rectangular Coordinates

Find the rectangular coordinates for the polar points (a) $(3, \frac{\pi}{6})$ and (b) $(-2, 3)$.

Solution For (a), we have from (7.2) that

$$x = r \cos \theta = 3 \cos \frac{\pi}{6} = \frac{3\sqrt{3}}{2}$$

and

$$y = r \sin \theta = 3 \sin \frac{\pi}{6} = \frac{3}{2}.$$

The rectangular point is then $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$. For (b), we have

$$x = r \cos \theta = -2 \cos 3 \approx 1.98$$

and

$$y = r \sin \theta = -2 \sin 3 \approx -0.28.$$

The rectangular point is $(-2 \cos 3, -2 \sin 3)$, which is located at approximately $(1.98, -0.28)$. ■

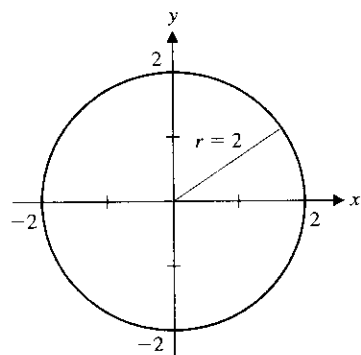


FIGURE 0.90a

The circle $r = 2$.

The **graph** of a polar equation $r = f(\theta)$ is the set of all points (x, y) for which $x = r \cos \theta$, $y = r \sin \theta$ and $r = f(\theta)$. In other words, the graph of a polar equation is a graph in the xy -plane of all those points whose polar coordinates satisfy the given equation. We begin by sketching two very simple (and familiar) graphs. The key to drawing the graph of a polar equation is to always keep in mind what the polar coordinates represent.

EXAMPLE 7.13 Some Simple Graphs in Polar Coordinates

Sketch the graphs of (a) $r = 2$ and (b) $\theta = \pi/3$.

Solution For (a), notice that $2 = r = \sqrt{x^2 + y^2}$, and so, we want all points whose distance from the origin is 2 (with any polar angle θ). Of course, this is the definition of a circle of radius 2 with center at the origin (see Figure 0.90a). For (b), notice that $\theta = \pi/3$ specifies all points with a polar angle of $\pi/3$ from the positive x -axis (at any

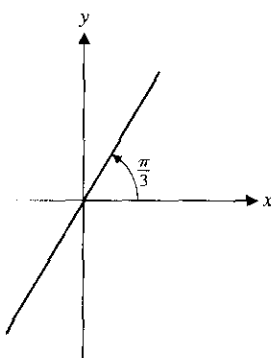


FIGURE 0.90b
The line $\theta = \frac{\pi}{3}$.

distance r from the origin). Including negative values for r , this defines a line with slope $\tan \pi/3 = \sqrt{3}$ (see Figure 0.90b). ■

REMARK 7.6

Calculators sketch polar graphs of $r = f(\theta)$ by computing the value of $f(\theta)$ for numerous values of θ at regularly spaced values of θ and then plotting the resulting points (x, y) . You should be aware that the appearance of a calculator plot depends on the x - y graphing window specified and also on the range of displayed values of θ . When drawing polar graphs, you should identify any values of θ corresponding to $r = 0$ or to where r reaches a maximum or minimum. In addition, you should identify the range of values of θ that produces one copy of the polar curve, when this is appropriate.

It turns out that many familiar curves have simple polar equations.

EXAMPLE 7.14 A Surprisingly Simple Polar Graph

Sketch the graph of the polar equation $r = \sin \theta$.

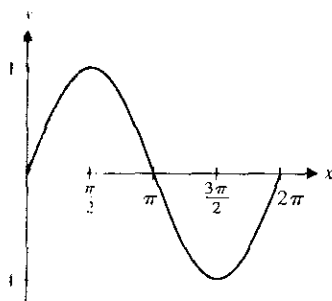


FIGURE 0.91
 $y = \sin \theta$ plotted in rectangular coordinates.

Solution For reference, we first sketch a graph of the sine function in **rectangular** coordinates on the interval $[0, 2\pi]$ (see Figure 0.91). Notice that on the interval $0 \leq \theta \leq \frac{\pi}{2}$, $\sin \theta$ increases from 0 to its maximum value of 1. Then, on the interval $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin \theta$ decreases from 1 to 0. When plotting the polar graph, keep in mind that $r = 0$ corresponds to the origin. Next, on the interval $\pi \leq \theta \leq \frac{3\pi}{2}$, $\sin \theta$ decreases from 0 to its minimum value of -1 . Since the values of r are negative, remember that this means that the points plotted are in the *opposite* quadrant (i.e., the first quadrant). Notice that this traces out the same curve in the first quadrant as we've already drawn for $0 \leq \theta \leq \frac{\pi}{2}$. Likewise, taking θ in the interval $\frac{3\pi}{2} \leq \theta \leq 2\pi$ retraces the portion of the curve in the second quadrant. Since $\sin \theta$ is periodic of period 2π , taking further values of θ simply retraces portions of the curve that we have already traced. A sketch of the polar graph is shown in Figure 0.92. We now verify that this curve is actually a circle. Notice that if we multiply the equation $r = \sin \theta$ through by r , we get

$$r^2 = r \sin \theta.$$

You should immediately recognize from (7.2) and (7.3) that $y = r \sin \theta$ and $r^2 = x^2 + y^2$. This gives us the rectangular equation

$$x^2 + y^2 = y$$

or

$$0 = x^2 + y^2 - y.$$

Completing the square, we get

$$0 = x^2 + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4}$$

and, adding $\frac{1}{4}$ to both sides,

$$\left(\frac{1}{2}\right)^2 = x^2 + \left(y - \frac{1}{2}\right)^2.$$

This is the rectangular equation for the circle of radius $\frac{1}{2}$ centered at the point $(0, \frac{1}{2})$, which is what we see in Figure 0.92. ■

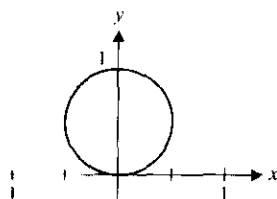


FIGURE 0.92
The circle $r = \sin \theta$.

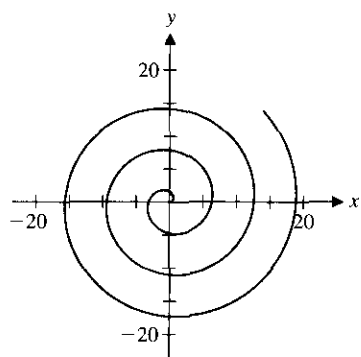


FIGURE 0.93

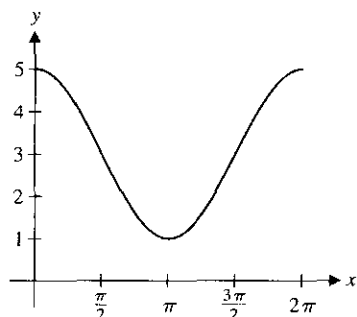
The spiral $r = \theta$, $\theta \geq 0$.

FIGURE 0.94

 $y = 3 + 2 \cos x$ in rectangular coordinates.

The graphs of many polar equations are not the graphs of *any* functions of the form $y = f(x)$, as in example 7.15.

EXAMPLE 7.15 An Archimedean Spiral

Sketch the graph of the polar equation $r = \theta$, for $\theta \geq 0$.

Solution Notice that here, as θ increases, so too does r . That is, as the polar angle increases, the distance from the origin also increases accordingly. This produces the spiral (an example of an **Archimedean spiral**) seen in Figure 0.93. ■

The graph in example 7.16 is in the general class known as **limaçons**. This class of graphs is defined by $r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ for positive constants a and b . Additional graphs of limaçons are shown in Appendix A.

EXAMPLE 7.16 A Limaçon

Sketch the graph of the polar equation $r = 3 + 2 \cos \theta$.

Solution We begin by sketching the graph of $y = 3 + 2 \cos x$ in rectangular coordinates on the interval $[0, 2\pi]$ to use as a reference (see Figure 0.94). Notice that in this case, we have $r = 3 + 2 \cos \theta > 0$ for all values of θ . Further, the maximum value of r is 5 (corresponding to when $\cos \theta = 1$ at $\theta = 0, 2\pi$, etc.) and the minimum value of r is 1 (corresponding to when $\cos \theta = -1$ at $\theta = \pi, 3\pi$, etc.). In this case, the graph is traced out with $0 \leq \theta \leq 2\pi$. We summarize the intervals of increase and decrease for r in the following table.

Interval	$\cos \theta$	$r = 3 + 2 \cos \theta$
$[0, \frac{\pi}{2}]$	Decreases from 1 to 0	Decreases from 5 to 3
$[\frac{\pi}{2}, \pi]$	Decreases from 0 to -1	Decreases from 3 to 1
$[\pi, \frac{3\pi}{2}]$	Increases from -1 to 0	Increases from 1 to 3
$[\frac{3\pi}{2}, 2\pi]$	Increases from 0 to 1	Increases from 3 to 5

In Figures 0.95a–0.95d, we show how the sketch progresses through each interval indicated in the table, with the completed figure (called a *limaçon*) shown in Figure 0.95d.

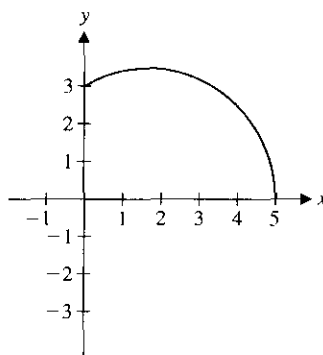


FIGURE 0.95a

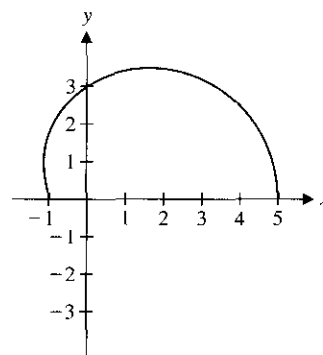
 $0 \leq \theta \leq \frac{\pi}{2}$.

FIGURE 0.95b

 $0 \leq \theta \leq \pi$.

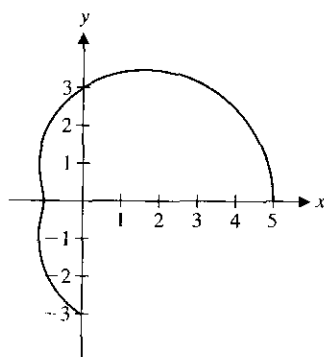


FIGURE 0.95c
 $0 \leq \theta \leq \frac{3\pi}{2}$.

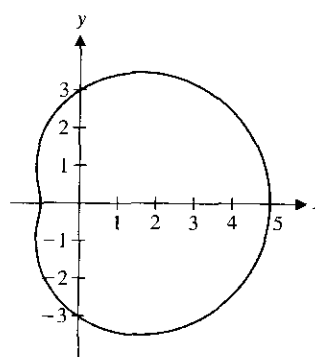


FIGURE 0.95d
 $0 \leq \theta \leq 2\pi$.

EXERCISES 0.7

WRITING EXERCISES

- Interpret in words the roles of each of the constants in the parametric equations $\begin{cases} x = a_1 + b_1 \cos(ct) \\ y = a_2 + b_2 \sin(ct) \end{cases}$.
- As indicated in remark 7.3, a given curve can be described by numerous sets of parametric equations. Explain why several different equations can all be correct. (Hint: Emphasize the fact that t is a dummy variable.)
- Suppose a point has polar representation (r, θ) . Explain why another polar representation of the same point is $(-r, \theta + \pi)$.
- After working with rectangular coordinates for so long, the idea of polar representations may seem slightly awkward. However, polar representations are entirely natural in many settings. For instance, if you were on a ship at sea and another ship was approaching you, explain whether you would use a polar representation (distance and bearing) or a rectangular representation (distance east-west and distance north-south).

In exercises 1–12, sketch the graph defined by the given parametric equations and find a corresponding x - y equation for the curve.

- $\begin{cases} x = 3 \cos t \\ y = 3 \sin t \end{cases}$
- $\begin{cases} x = 2 \cos t \\ y = 3 \sin t \end{cases}$
- $\begin{cases} x = 1 + 2 \cos t \\ y = -2 + 2 \sin t \end{cases}$
- $\begin{cases} x = 2 \sin t \\ y = 3 \cos t \end{cases}$
- $\begin{cases} x = -1 + 2t \\ y = 3t \end{cases}$
- $\begin{cases} x = 4 + 3t \\ y = 2 - 4t \end{cases}$

- $\begin{cases} x = 1 + t \\ y = t^2 + 2 \end{cases}$
- $\begin{cases} x = 2 - t \\ y = t^2 + 1 \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = 2t \end{cases}$
- $\begin{cases} x = t^2 + 1 \\ y = t - 1 \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = t^2 + 1 \end{cases}$
- $\begin{cases} x = \cos t \\ y = 3 \cos t - 1 \end{cases}$

T In exercises 13–20, use your CAS or graphing calculator to sketch the plane curves defined by the given parametric equations.

- $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3 \end{cases}$
- $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3t \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t^2 \end{cases}$
- $\begin{cases} x = \cos t - \frac{1}{2} \cos 3t \\ y = \sin t - \frac{1}{2} \sin 3t \end{cases}$
- $\begin{cases} x = \cos t - \frac{1}{2} \cos 4t \\ y = \sin t - \frac{1}{2} \sin 4t \end{cases}$
- $\begin{cases} x = \cos 2t \\ y = \sin 7t \end{cases}$
- $\begin{cases} x = \cos 2t \\ y = \sin \pi t \end{cases}$

In exercises 21–26, match the parametric equations with the corresponding plane curve displayed below. Give reasons for your choices.

- $\begin{cases} x = t^2 - 1 \\ y = t^4 \end{cases}$
- $\begin{cases} x = t - 1 \\ y = t^3 \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = \sin t \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = \sin 2t \end{cases}$
- $\begin{cases} x = \cos 3t \\ y = \sin 2t \end{cases}$
- $\begin{cases} x = 3 \cos t \\ y = 2 \sin t \end{cases}$

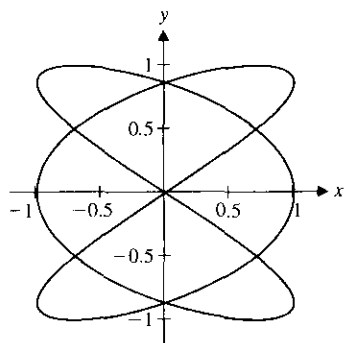


FIGURE A

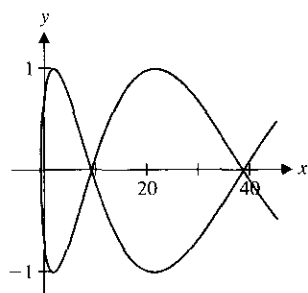


FIGURE B

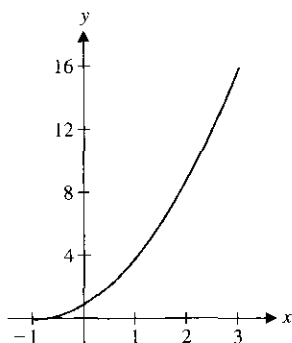


FIGURE C

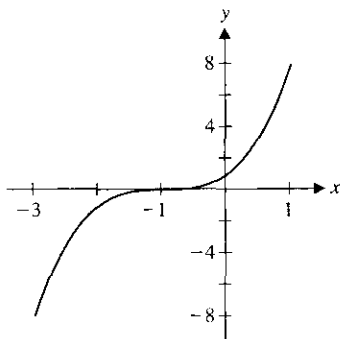


FIGURE D

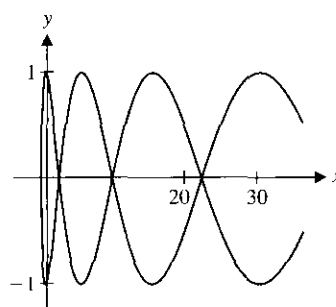


FIGURE E

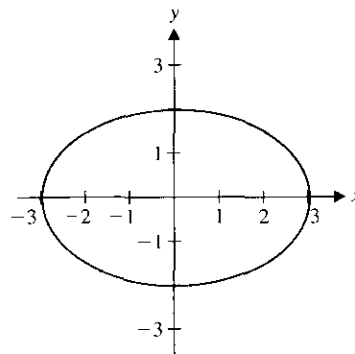


FIGURE F

In exercises 27–32, find parametric equations describing the given curve.

27. (a) The line segment from $(0, 1)$ to $(3, 4)$
(b) The line segment from $(3, 1)$ to $(1, 3)$
28. (a) The portion of the parabola $y = x^2 + 1$ from $(1, 2)$ to $(2, 5)$
(b) The portion of the parabola $y = 2x^2 - 1$ from $(0, -1)$ to $(2, 7)$
29. (a) The circle of radius 3 centered at $(2, 1)$, drawn counter-clockwise.
(b) The circle of radius 5 centered at $(-1, 3)$, drawn counter-clockwise.
30. (a) The ellipse with vertices $(2, 0)$, $(0, 4)$, $(-2, 0)$ and $(0, -4)$
(b) The ellipse with vertices $(0, 3)$, $(2, 4)$, $(4, 3)$ and $(2, 2)$
31. (a) The inverse of $f(x) = x^5 + 2x^3 + 4x - 2$
(b) The inverse of $f(x) = 4x^3 + 2x$
32. (a) The inverse of $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
(b) The inverse of $f(x) = \tan^{-1} x$

In exercises 33–38, plot the given polar points (r, θ) and find their rectangular representation.

- | | | |
|----------------------------|-----------------|---------------------------|
| 33. $(2, 0)$ | 34. $(2, \pi)$ | 35. $(-2, \pi)$ |
| 36. $(-3, \frac{3\pi}{2})$ | 37. $(3, -\pi)$ | 38. $(5, -\frac{\pi}{2})$ |

In exercises 39–44, find all polar coordinates representations of the given rectangular point.

39. $(1, 1)$ 40. $(-1, 1)$
 41. $(1, -1)$ 42. $(3, 4)$
 43. $(-\sqrt{3}, 1)$ 44. $(0, -4)$

In exercises 45–48, find rectangular coordinates for the given polar point.

45. $(1, \pi/4)$ 46. $(0, 3)$
 47. $(1, \pi/2)$ 48. $(-3, 1)$

In exercises 49–56, sketch the graph of the polar equation and find a corresponding x - y equation.

49. $r = 1$ 50. $r = \sqrt{3}$
 51. $\theta = \pi/6$ 52. $\theta = 3\pi/4$
 53. $r = \cos \theta$ 54. $r = 2 \cos \theta$
 55. $r = \sin \theta$ 56. $r = 2 \sin \theta$

In exercises 57–64, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

57. $r = \cos 2\theta$ 58. $r = \cos 3\theta$
 59. $r = 1 + 2 \sin \theta$ 60. $r = 2 - 2 \cos \theta$
 61. $r = 1 - 4 \sin \theta$ 62. $r = 2 + 2 \sin \theta$
 63. $r = e^{i\theta}$ 64. $r = e^{i\theta/4}$

65. Based on your graphs in exercises 53 and 54, conjecture the graph of $r = a \cos \theta$ for any positive constant a .

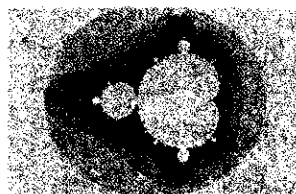
66. Based on the graphs in exercises 57 and 58 and others (try $r = \cos 4\theta$ and $r = \cos 5\theta$), conjecture the graph of $r = \cos n\theta$ for any positive integer n .

In exercises 67–72, find a polar equation corresponding to the given rectangular equation.

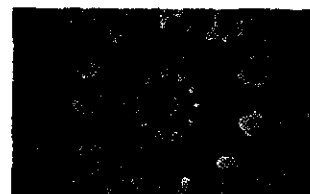
67. $x^2 + y^2 = 4$ 68. $x^2 + y^2 = 9$
 69. $x^2 + y^2 = 16$ 70. $x^2 + y^2 = x$
 71. $y = 3$ 72. $x = 2$

73. Compare the graphs of $\begin{cases} x = \cos 2t \\ y = \sin t \end{cases}$ and $\begin{cases} x = \cos t \\ y = \sin 2t \end{cases}$. Use the identities $\cos 2t = \cos^2 t - \sin^2 t$ and $\sin 2t = 2 \cos t \sin t$ to find x - y equations for each graph.

74. Sketch the graph of $\begin{cases} x = \frac{1}{2} \cos t - \frac{1}{4} \cos 2t \\ y = \frac{1}{2} \sin t - \frac{1}{4} \sin 2t \end{cases}$. This heart-shaped curve is the largest feature of the **Mandelbrot set**, one of the most famous mathematical sets. Portions of the Mandelbrot set have been turned into colorful T-shirts and posters that you may have seen.



Mandelbrot set.



Mandelbrot zoom.

To progress further on a sketch of the Mandelbrot set, add the

circle $\begin{cases} x = -1 + \frac{1}{4} \cos t \\ y = \frac{1}{4} \sin t \end{cases}$ to your initial sketch.

75. One situation where polar coordinates apply directly to sports is in making a golf putt. The two factors that the golfer tries to control are distance (determined by speed) and direction (usually called the “line”). Suppose a putter is d feet from the hole, which has radius $h = \frac{1}{6}$. Show that the path of the ball will intersect the hole if the angle A in the figure satisfies $-\sin^{-1}(h/d) < A < \sin^{-1}(h/d)$.



76. The distance r that the golf ball in exercise 75 travels also needs to be controlled. The ball must reach the front of the hole. In rectangular coordinates, the hole has equation $(x-d)^2 + y^2 = h^2$, so the left side of the hole is $x = d - \sqrt{h^2 - y^2}$. Show that this converts in polar coordinates to $r = d \cos \theta - \sqrt{d^2 \cos^2 \theta - (d^2 - h^2)}$. (Hint: Substitute for x and y , isolate the square root term, square both sides, combine r^2 terms and use the quadratic formula.)

77. The golf putt in exercises 75 and 76 will not go in the hole if it is hit too hard. Suppose that the putt would go $r = d + c$ feet if it did not go in the hole ($c > 0$). For a putt hit toward the center of the hole, define b to be the largest value of c such that the putt goes in (i.e., if the ball is hit more than b feet past the hole, it is hit too hard). Experimental evidence (see Dave Pelz's *Putt Like the Pros*) shows that at other angles A , the distance r

must be less than $d + b \left(1 - \left[\frac{A}{\sin^{-1}(h/d)} \right]^2 \right)$. The results of

exercises 75 and 76 define limits for the angle A and distance r of a successful putt. Identify the functions $r_1(A)$ and $r_2(A)$ such that $r_1(A) < r < r_2(A)$ and constants A_1 and A_2 such that $A_1 < A < A_2$.

- T** 78. Take the general result of exercise 77 and apply it to a putt of $d = 15$ feet with a value of $b = 4$ feet. Visualize this by graphing the region

$$15 \cos \theta - \sqrt{225 \cos^2 \theta - (225 - 1/36)} \\ < r < 15 + 4 \left(1 - \left[\frac{\theta}{\sin^{-1}(1/90)} \right]^2 \right)$$

with $-\sin^{-1}(1/90) < \theta < \sin^{-1}(1/90)$. A good choice of graphing windows is $13.8 \leq x \leq 19$ and $-0.5 \leq y \leq 0.5$.

79. Find parametric equations for the path traced out by a specific point on a circle of radius r rolling left to right. (Hint: First, find parametric equations for the center of the circle, then add to these parametric equations for the point going around the center of the circle.) This curve is called a **cycloid**.
80. Find parametric equations for the path traced out by a specific point inside the circle as the circle rolls from left to right. (Hint: If r is the radius of the circle, let $d < r$ be the distance from the point to the center.) This curve is called a **trochoid**.
81. A microphone is set up at a specific place in a concert hall. Due to reflections from walls and other surfaces, sound bounces to the microphone from a variety of directions and is recorded by the microphone with a variety of amplitudes. If the amplitude r is given as a function of the direction θ , the resulting function is called the **polar pattern** of the microphone. If $r = 2.5 + 2.5 \cos \theta$, in which direction (which value of θ) do you think the microphone is pointed? If you were standing behind the microphone ($\theta = \pi$), would it be safe for you to talk?
82. Compare the properties of the microphone in exercise 81 to a microphone with polar pattern $r = 3.6 + 1.4 \cos \theta$.

✚ EXPLORATORY EXERCISES

- T** 1. The Flying Zucchini Circus Troupe has a human cannonball act, shooting a performer from a cannon into a specially padded seat of a turning Ferris wheel. The Ferris wheel has a radius of 40 feet and rotates counterclockwise at one revolution per minute. The special seat starts at ground level. Carefully explain why parametric equations for the seat are $\begin{cases} x = 40 \cos(\frac{\pi}{30}t - \frac{\pi}{2}) \\ y = 40 + 40 \sin(\frac{\pi}{30}t - \frac{\pi}{2}) \end{cases}$. The cannon is located 200 feet left of the Ferris wheel with the muzzle 10 feet aboveground. The performer is launched 35 seconds after the wheel starts turning with an initial velocity of 100 ft/s at an angle of $\frac{\pi}{5}$ above the horizontal. Carefully explain why parametric equations for the human cannonball are $\begin{cases} x = (100 \cos \frac{\pi}{5})(t - 35) - 200 \\ y = -16(t - 35)^2 + (100 \sin \frac{\pi}{5})(t - 35) + 10 \end{cases} \quad (t \geq 35)$. Determine whether the act is safe or the Flying Zucchini comes down squash.

- T** 2. Rework exploratory exercise 1 with initial velocity 135 ft/s, launch angle 30° and a 27-second delay. How close does the Flying Zucchini get to the special seat? Given that a Ferris wheel seat actually has height, width and depth, do you think that this is close enough? Repeat with (a) initial velocity 75 ft/s, launch angle 47° and 47.25-second delay; (b) initial velocity 118 ft/s, launch angle 35° and 28-second delay. Develop criteria for a safe and exciting human cannonball act. Consider each of the following: Should the launch velocity be large or small? Should the seat be high or low when the cannonball lands? Should the human have a positive or negative vertical velocity at landing? How close (vertically and horizontally) should the human need to get to the center of the seat? Based on your criteria, which of the launches in this exercise is the best? Find an initial velocity, launch angle and launch delay that is better.

◆ CONCEPTS

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

REVIEW EXERCISES



Slope of line	Parallel lines	Perpendicular lines
Domain	Intercepts	Zeros of function
Graphing window	Local maximum	Vertical asymptote
Inverse function	One-to-one function	Periodic function
Sine function	Cosine function	Arcsine function
e	Exponential function	Logarithm
Composition	Parametric equations	Polar coordinates



REVIEW EXERCISES

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. For a graph, you can compute the slope using any two points and get the same value.
2. All graphs must pass the vertical line test.
3. A cubic function has a graph with one local maximum and one local minimum.
4. If a function has no local maximum or minimum, then it is one-to-one.
5. The graph of the inverse of f can be obtained by reflecting the graph of f across the diagonal $y = x$.
6. If f is a trigonometric function, then the solution of the equation $f(x) = 1$ is $f^{-1}(1)$.
7. Exponential and logarithmic functions are inverses of each other.
8. All quadratic functions have graphs that look like the parabola $y = x^2$.
9. Polar coordinates are a specific example of parametric equations.
10. Every curve has an infinite number of parametric representations.

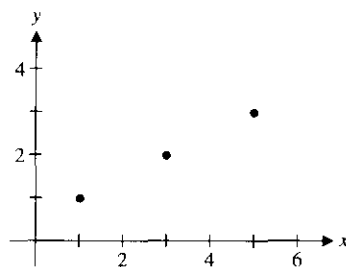
In exercises 1 and 2, find the slope of the line through the given points.

1. $(2, 3), (0, 7)$
2. $(1, 4), (3, 1)$

In exercises 3 and 4, determine if the lines are parallel, perpendicular or neither.

3. $y = 3x + 1$ and $y = 3(x - 2) + 4$
4. $y = 2(x + 1) - 1$ and $y = \frac{1}{2}x + 2$
5. Determine if the points $(1, 2)$, $(2, 4)$ and $(0, 6)$ form the vertices of a right triangle.
6. The data represents populations at various times. Plot the points, discuss any patterns and predict the population at the next time: $(0, 2100)$, $(1, 3050)$, $(2, 4100)$ and $(3, 5050)$.

7. Find an equation of the line through the given points and compute the y -coordinate corresponding to $x = 4$.



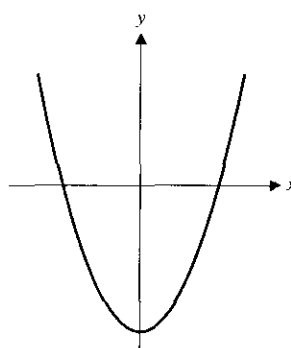
8. For $f(x) = x^2 - 3x - 4$, compute $f(0)$, $f(2)$ and $f(4)$.

In exercises 9 and 10, find an equation of the line with given slope and point.

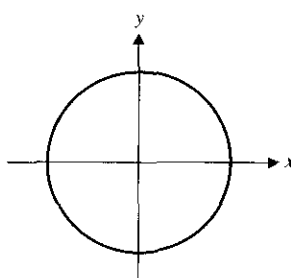
9. $m = -\frac{1}{3}, (-1, -1)$
10. $m = \frac{1}{4}, (0, 2)$

In exercises 11 and 12, use the vertical line test to determine if the curve is the graph of a function.

11.



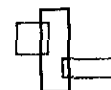
12.



In exercises 13 and 14, find the domain of the given function.

13. $f(x) = \sqrt{4 - x^2}$
14. $f(x) = \frac{x - 2}{x^2 - 2}$

REVIEW EXERCISES



In exercises 15–28, sketch a graph of the function showing extrema, intercepts and asymptotes.

15. $f(x) = x^2 + 2x - 8$

16. $f(x) = x^3 - 6x + 1$

17. $f(x) = x^4 - 2x^2 + 1$

18. $f(x) = x^5 - 4x^3 + x - 1$

19. $f(x) = \frac{4x}{x+2}$

21. $f(x) = \sin 3x$

23. $f(x) = \sin x + 2 \cos x$

25. $f(x) = 4e^{2x}$

27. $f(x) = \ln 3x$

20. $f(x) = \frac{x-2}{x^2-x-2}$

22. $f(x) = \tan 4x$

24. $f(x) = \sec 2x$

26. $f(x) = 3e^{-4x}$

28. $f(x) = e^{\ln 2x}$

29. Determine all intercepts of $y = x^2 + 2x - 8$ (see exercise 15).30. Determine all intercepts of $y = x^4 - 2x^2 + 1$ (see exercise 17).31. Find all vertical asymptotes of $y = \frac{4x}{x+2}$.32. Find all vertical asymptotes of $y = \frac{x-2}{x^2-x-2}$.

In exercises 33–36, find or estimate all zeros of the given function.

33. $f(x) = x^2 - 3x - 10$

34. $f(x) = x^3 + 4x^2 + 3x$

35. $f(x) = x^3 - 3x^2 + 2$

36. $f(x) = x^4 - 3x - 2$

In exercises 37 and 38, determine the number of solutions.

37. $\sin x = x^3$

38. $\sqrt{x^2+1} = x^2 - 1$

39. A surveyor stands 50 feet from a telephone pole and measures an angle of 34° to the top. How tall is the pole?40. Find $\sin \theta$ given that $0 < \theta < \frac{\pi}{2}$ and $\cos \theta = \frac{1}{5}$.41. Convert to fractional or root form: (a) $5^{-1/2}$ (b) 3^{-2} .42. Convert to exponential form: (a) $\frac{2}{\sqrt{x}}$ (b) $\frac{3}{x^2}$.43. Rewrite $\ln 8 - 2 \ln 2$ as a single logarithm.44. Solve the equation for x : $e^{\ln 4x} = 8$.

In exercises 45 and 46, solve the equation for x .

45. $3e^{2x} = 8$

46. $2 \ln 3x = 5$.

In exercises 47 and 48, find $f \circ g$ and $g \circ f$ and identify their respective domains.

47. $f(x) = x^2, g(x) = \sqrt{x-1}$

48. $f(x) = x^2, g(x) = \frac{1}{x^2-1}$

In exercises 49 and 50, identify functions $f(x)$ and $g(x)$ such that $(f \circ g)(x)$ equals the given function.

49. e^{3x^2+2}

50. $\sqrt{\sin x + 2}$

In exercises 51 and 52, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

51. $f(x) = x^2 - 4x + 1$

52. $f(x) = x^2 + 4x + 6$

In exercises 53–56, determine if the function is one-to-one. If so, find its inverse.

53. $x^3 - 1$

54. e^{-4x}

55. e^{2x^2}

56. $x^3 - 2x + 1$

In exercises 57–60, graph the inverse without solving for the inverse.

57. $x^5 + 2x^3 - 1$

58. $x^3 + 5x + 2$

59. $\sqrt{x^3+4x}$

60. e^{x^3+2x}

In exercises 61–64, evaluate the quantity using the unit circle.

61. $\sin^{-1} 1$

62. $\cos^{-1}(-\frac{1}{2})$

63. $\tan^{-1}(-1)$

64. $\csc^{-1}(-2)$

In exercises 65–68, simplify the expression using a triangle.

65. $\sin(\sec^{-1} 2)$

66. $\tan(\cos^{-1}(4/5))$

67. $\sin^{-1}(\sin(3\pi/4))$

68. $\cos^{-1}(\sin(-\pi/4))$

In exercises 69 and 70, find all solutions of the equation.

69. $\sin 2x = 1$

70. $\cos 3x = \frac{1}{2}$

In exercises 71–74, sketch the plane curve defined by the parametric equations and find a corresponding x - y equation for the curve.

71. $\begin{cases} x = -1 + 3 \cos t \\ y = 2 + 3 \sin t \end{cases}$

72. $\begin{cases} x = 2 - t \\ y = 1 + 3t \end{cases}$

REVIEW EXERCISES

$$73. \begin{cases} x = t^2 + 1 \\ y = t^2 - 1 \end{cases}$$

$$74. \begin{cases} x = \cos t \\ y = \cos^2 t - 1 \end{cases}$$

In exercises 75–78, sketch the plane curves defined by the parametric equations.

$$75. \begin{cases} x = \cos 2t \\ y = \sin 6t \end{cases}$$

$$76. \begin{cases} x = \cos 6t \\ y = \sin 2t \end{cases}$$

$$77. \begin{cases} x = \cos 2t \cos t \\ y = \cos 2t \sin t \end{cases}$$

$$78. \begin{cases} x = \cos 2t \cos 3t \\ y = \cos 2t \sin 3t \end{cases}$$

In exercises 79–82, match the parametric equations with the corresponding plane curve.

$$79. \begin{cases} x = t^3 \\ y = t^2 - 1 \end{cases}$$

$$80. \begin{cases} x = t^3 \\ y = t^2 - 1 \end{cases}$$

$$81. \begin{cases} x = \cos 2t \cos t \\ y = \cos 2t \sin t \end{cases}$$

$$82. \begin{cases} x = \cos(t + \cos t) \\ y = \cos(t + \sin t) \end{cases}$$

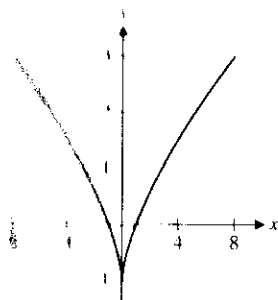


FIGURE A

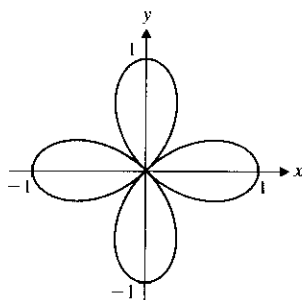


FIGURE B

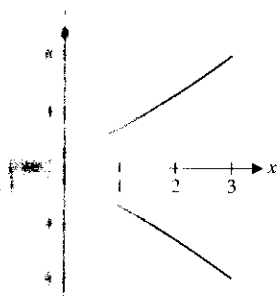


FIGURE C

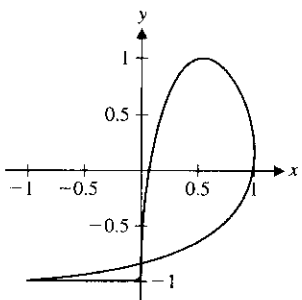


FIGURE D

In exercises 83 and 84, find parametric equations for the given curve.

83. The line segment from (2, 1) to (4, 7)

84. The portion of the parabola $y = x^2 + 1$ from (1, 2) to (3, 10)

In exercises 85 and 86, sketch the graph of the polar equation and find a corresponding x - y equation.

$$85. r = 3 \cos \theta$$

$$86. r = 2 \sec \theta$$

In exercises 87–94, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

$$87. r = 2 \sin \theta$$

$$88. r = 2 - 2 \sin \theta$$

$$89. r = 2 - 3 \sin \theta$$

$$90. r = \cos 3\theta + \sin 2\theta$$

$$91. r^2 = 4 \sin 2\theta$$

$$92. r = e^{\cos \theta} - 2 \cos 4\theta$$

$$93. r = \frac{2}{1 + 2 \sin \theta}$$

$$94. r = \frac{2}{1 + 2 \cos \theta}$$

In exercises 95 and 96, find a polar equation corresponding to the rectangular equation.

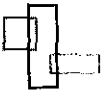
$$95. x^2 + y^2 = 9$$

$$96. (x - 3)^2 + y^2 = 9$$

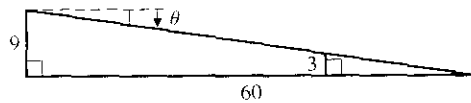
CONNECTIONS

- Sketch a graph of any function $y = f(x)$ that has an inverse. (Your choice.) Sketch the graph of the inverse function $y = f^{-1}(x)$. Then sketch the graph of $y = g(x) = f(x + 2)$. Sketch the graph of $y = g^{-1}(x)$ and use the graph to determine a formula for $g^{-1}(x)$ in terms of $f^{-1}(x)$. Repeat this for $h(x) = f(x) + 3$ and $k(x) = f(x - 4) + 5$.
- In tennis, a serve must clear the net and then land inside of a box drawn on the other side of the net. In this exercise, you will explore the margin of error for successfully serving. First, consider a straight serve (this essentially means a serve hit infinitely hard) struck 9 feet above the ground. Call the starting point (0, 9). The back of the service box is 60 feet away, at (60, 0). The top of the net is 3 feet above the ground and 39 feet from the server, at (39, 3). Find the service angle θ (i.e., the angle as measured from the horizontal) for the triangle formed by the points (0, 9), (0, 0) and (60, 0). Of course, most serves curve down due to gravity. Ignoring air resistance, the path of the ball struck at angle θ and initial speed v ft/s is $y = -\frac{16}{(v \cos \theta)^2} x^2 - (\tan \theta)x + 9$. To hit the back of the service line, you need $y = 0$ when $x = 60$. Substitute in these values along with $v = 120$. Multiply by $\cos^2 \theta$ and replace $\sin \theta$ with $\sqrt{1 - \cos^2 \theta}$. Replacing $\cos \theta$ with z gives you an algebraic equation in z . Numerically estimate z . Similarly, substitute $x = 39$ and $y = 3$ and find an equation for $w = \cos \theta$.

REVIEW EXERCISES



Numerically estimate w . The margin of error for the serve is given by $\cos^{-1} z < \theta < \cos^{-1} w$.



3. Baseball players often say that an unusually fast pitch rises or even hops up as it reaches the plate. One explanation of this illusion involves the players' inability to track the ball all the way to the plate. The player must compensate by predicting where

the ball will be when it reaches the plate. Suppose the height of a pitch when it reaches home plate is $h = -(240/v)^2 + 6$ feet for a pitch with velocity v ft/s. (This equation takes into consideration gravity but not air resistance.) Halfway to the plate, the height would be $h = -(120/v)^2 + 6$ feet. Compare the halfway heights for pitches with $v = 132$ and $v = 139$ (about 90 and 95 mph, respectively). Would a batter be able to tell much difference between them? Now compare the heights at the plate. Why might the batter think that the faster pitch hopped up right at the plate? How many inches did the faster pitch hop?

LIMITS AND CONTINUITY

In the extreme sport of speed skydiving, contestants start at an altitude of about 4000 meters and accelerate to almost 500 km/h (over 300 mph!) in a vertical head-down position. At an altitude of 1500 meters, a diver switches from the vertical position to a horizontal position, to provide maximum braking. This change of position slows down the diver to a speed of about 200 km/h, at which point the diver's parachute is released. This large variation of possible speeds helps explain how teams of skydivers can catch up to each other and fall in unison. In order to carefully control their speed, skydivers need a working understanding of the physics involved. For us, one immediate question is how we might mathematically describe the speed of a skydiver or any other falling object.

We know some properties of falling objects from experience. First, the (downward) speed of a dropped object will increase as it continues to fall. This is why a book falling from the top shelf of a bookcase will cause more damage than a book falling from the bottom shelf. We also know that there is a maximum value beyond which the speed never increases. For instance, while raindrops fall from great heights, they never fall faster than about 15 mph. (Fortunately; imagine getting pelted by a 300 mph raindrop!)

To translate the preceding observations into mathematical terms, let t represent time and $s = f(t)$ represent the speed of a falling object at time t . From what we've just observed, the function f has the following properties:

1. $f(0) = 0$.
2. Larger values of t give larger values of $f(t)$.
3. As t increases, $f(t)$ increases toward some maximum value.

The possible graph of such a function is shown in Figure 1.1.

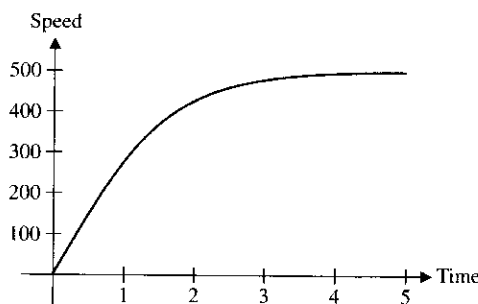
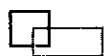


FIGURE 1.1
Speed of a falling skydiver.

CHAPTER

1

Finding a function with these properties can be a challenge. (Try it!) You are familiar with several functions with properties 1 and 2. For instance, $f(t) = 2t$ satisfies $f(0) = 0$ (just substitute in $t = 0$ to check this) and the positive slope corresponds to property 2. However, property 3 involves the concept of **limit**. Understanding the computational and visual aspects of limits such as this is the main goal of this chapter.



1.1 A BRIEF PREVIEW OF CALCULUS: TANGENT LINES AND THE LENGTH OF A CURVE

In this section, we approach the boundary between precalculus mathematics and the calculus by investigating several important problems requiring the use of calculus. We do not present here an exhaustive list, but rather, a small sampling of representative problems.

Recall that we think of the slope of a straight line as the change in y divided by the change in x . This fraction is the same regardless of which two points on the line you use to compute the slope. For example, the points $(0, 1)$, $(1, 4)$ and $(3, 10)$ all lie on the line $y = 3x + 1$. The slope of 3 can be obtained from any two of the points. For instance,

$$m = \frac{4 - 1}{1 - 0} = 3$$

or

$$m = \frac{10 - 1}{3 - 0} = 3.$$

In the calculus, we generalize this problem to find the slope of a *curve* at a point. For instance, suppose we wanted to find the slope of the curve $y = x^2 + 1$ at the point $(1, 2)$. You might think of picking a second point on the parabola, say $(2, 5)$. The slope of the line through these two points (called a **secant line**; see Figure 1.2a) is easy enough to compute. We have

$$m_{\text{sec}} = \frac{5 - 2}{2 - 1} = 3.$$

However, if you use the point $(0, 1)$ instead of $(2, 5)$, you get a different slope (see Figure 1.2b):

$$m_{\text{sec}} = \frac{2 - 1}{1 - 0} = 1.$$

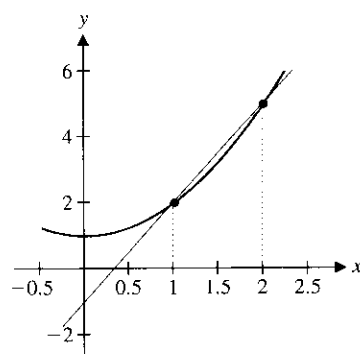


FIGURE 1.2a
Secant line, slope = 3.

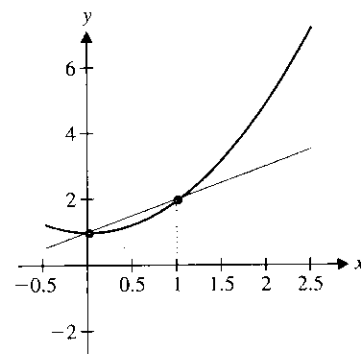


FIGURE 1.2b
Secant line, slope = 1.

For curves other than straight lines, the slopes of secant lines joining different points are generally *not* the same. Refer to Figures 1.2a and 1.2b for a graphical representation of this.

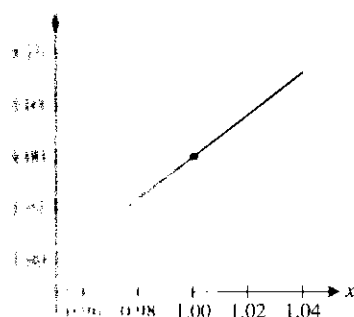


FIGURE 1.1

If you get different slopes using different pairs of points, then what exactly does it mean for a curve to have a slope at a point? The answer can be visualized by graphically zooming in on the specified point. Take the graph of $y = x^2 + 1$ and zoom in tightly on the point $(1, 2)$. You should get a graph something like the one in Figure 1.3. The graph looks very much like a straight line. In fact, the more you zoom in, the straighter the curve appears to be. Further, the closer the curve is to being straight, the less it matters which two points are used to compute a slope. So, here's the strategy: pick several points on the parabola, each closer to the point $(1, 2)$ than the previous one. Compute the slopes of the lines through $(1, 2)$ and each of the points. The closer the second point gets to $(1, 2)$, the closer the computed slope is to the answer you seek.

For example, the point $(1.5, 3.25)$ is on the parabola fairly close to $(1, 2)$. The slope of the line joining these points is

$$m_{\text{sec}} = \frac{3.25 - 2}{1.5 - 1} = 2.5.$$

The point $(1.1, 2.21)$ is even closer to $(1, 2)$. The slope of the secant line joining these two points is

$$m_{\text{sec}} = \frac{2.21 - 2}{1.1 - 1} = 2.1.$$

Continuing in this way, the point $(1.01, 2.0201)$ is closer yet to the point $(1, 2)$. The slope of the secant lines through these points is

$$m_{\text{sec}} = \frac{2.0201 - 2}{1.01 - 1} = 2.01.$$

The slopes of the secant lines (2.5, 2.1, 2.01) are getting closer and closer to the slope of the parabola at the point $(1, 2)$. So what could we mean by the slope of the curve? Based on these calculations, 2 would seem like a reasonable answer.

Example 1.1 takes our introductory example just a bit further.

EXAMPLE 1.1 Estimating the Slope of a Curve

Estimate the slope of $y = x^2 + 1$ at $x = 1$.

Solution We focus on the point whose coordinates are $x = 1$ and $y = 1^2 + 1 = 2$. To estimate the slope, choose a sequence of points near $(1, 2)$ and compute the slopes of the secant lines joining those points with $(1, 2)$. (We showed sample secant lines in Figures 1.2a and 1.2b.) Choosing points with $x > 1$ (x -values of 2, 1.1 and 1.01) and points with $x < 1$ (x -values of 0, 0.9 and 0.99), we compute the corresponding y -values using $y = x^2 + 1$ and get the slopes shown in the following table.

Second Point	m_{sec}
(2, 5)	$\frac{5 - 2}{2 - 1} = 3$
(1.1, 2.21)	$\frac{2.21 - 2}{1.1 - 1} = 2.1$
(1.01, 2.0201)	$\frac{2.0201 - 2}{1.01 - 1} = 2.01$

Second Point	m_{sec}
(0, 1)	$\frac{1 - 2}{0 - 1} = 1$
(0.9, 1.81)	$\frac{1.81 - 2}{0.9 - 1} = 1.9$
(0.99, 1.9801)	$\frac{1.9801 - 2}{0.99 - 1} = 1.99$

Observe that in both columns, as the second point gets closer to $(1, 2)$, the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the curve at the point $(1, 2)$ is then 2. ■

In the differential calculus, we will develop a technique for computing such slopes exactly (and easily). It turns out that this process generalizes to a powerful tool (the **derivative**), which we bring to bear in later chapters on a wide range of significant problems in areas such as physics, biology, and economics. For now, note what distinguishes the calculus problem from the corresponding algebra problem. The calculus problem involves a process of finding something we call a **limit**. For the moment, the slope of the curve cannot be computed exactly, but is estimated using a sequence of approximations. The limit enables us to extend the concept of slope from simple straight lines to more general curves.

EXAMPLE 1.2 Estimating the Slope of a Curve

Estimate the slope of $y = \sin x$ at $x = 0$.

Solution This turns out to be a very important problem, one to which we will return later. For now, choose a sequence of points near $(0, 0)$ and compute the slopes of the secant lines joining those points with $(0, 0)$. The following table shows one set of choices.

Second Point	m_{sec}
$(1, \sin 1)$	0.8415
$(0.1, \sin 0.1)$	0.9983
$(0.01, \sin 0.01)$	0.9999

Second Point	m_{sec}
$(-1, \sin(-1))$	0.8415
$(-0.1, \sin(-0.1))$	0.9983
$(-0.01, \sin(-0.01))$	0.9999

Note that as the second point gets closer and closer to $(0, 0)$, the slope of the secant line (m_{sec}) appears to get closer and closer to 1. A good estimate of the slope of the curve at the point $(0, 0)$ would then appear to be 1. Although at this stage, we have no way of computing the slope exactly, this is consistent with the graph of $y = \sin x$ shown in Figure 1.4. Note that near $(0, 0)$, the graph resembles that of $y = x$, a straight line of slope 1. ■

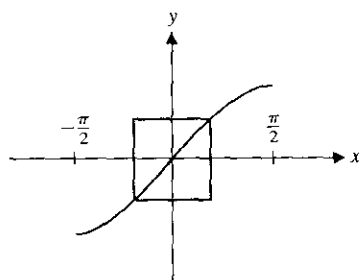


FIGURE 1.4
 $y = \sin x$.

A second problem requiring the power of calculus is that of computing distance along a curved path. While this problem is of less significance than our first example (both historically and in the development of the calculus), it provides a good indication of the need for mathematics beyond simple algebra. You should pay special attention to the similarities between the development of this problem and our earlier work with slope.

Recall that the (straight-line) distance between the two points (x_1, y_1) and (x_2, y_2) is given by

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For instance, the distance between the points $(0, 1)$ and $(3, 4)$ is

$$d\{(0, 1), (3, 4)\} = \sqrt{(3 - 0)^2 + (4 - 1)^2} = 3\sqrt{2} \approx 4.24264.$$

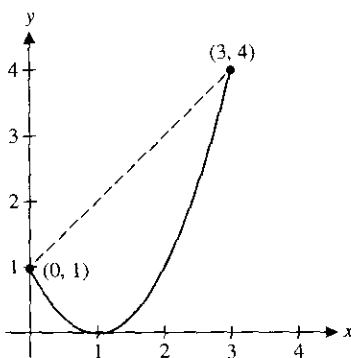


FIGURE 1.5a
 $y = (x - 1)^2$.

However, this is not the only way we might want to compute the distance between these two points. For example, suppose that you needed to drive a car from $(0, 1)$ to $(3, 4)$ along the path defined by the curve $y = (x - 1)^2$ (see Figure 1.5a). In this case, you don't care about the straight-line distance connecting the two points, but only about how far you must drive along the curve (the **length** of the curve or **arc length**).

Notice that the distance along the curve must be greater than $3\sqrt{2}$ since $3\sqrt{2}$ is the straight-line distance. But how much greater? Taking a cue from the slope problem, you might want to formulate a strategy for obtaining a sequence of increasingly accurate approximations. Instead of using just one line segment to get the approximation of $3\sqrt{2}$, you could use two line segments, as shown in Figure 1.5b. Notice that the sum of the lengths of

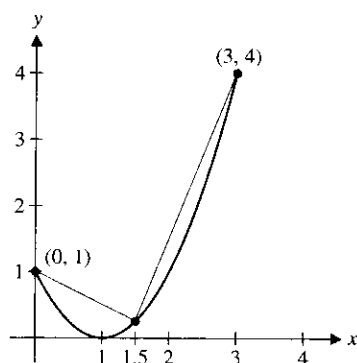


FIGURE 1.5b
Two line segments.

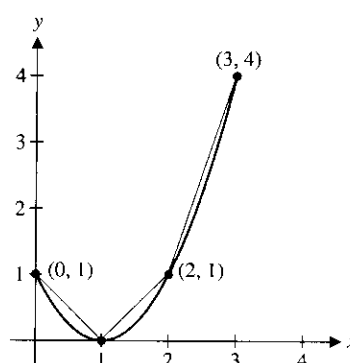


FIGURE 1.5c
Three line segments.

the two line segments appears to be a much better approximation to the actual length of the curve than the straight-line distance used previously. This distance is

$$\begin{aligned} d_2 &= d\{(0, 1), (1.5, 0.25)\} + d\{(1.5, 0.25), (3, 4)\} \\ &= \sqrt{(1.5 - 0)^2 + (0.25 - 1)^2} + \sqrt{(3 - 1.5)^2 + (4 - 0.25)^2} \approx 5.71593. \end{aligned}$$

You're probably way ahead of us by now. If approximating the length of the curve with two line segments gives an improved approximation, why not use three or four or more? Using the three line segments indicated in Figure 1.5c, we get the further improved approximation

$$\begin{aligned} d_3 &= d\{(0, 1), (1, 0)\} + d\{(1, 0), (2, 1)\} + d\{(2, 1), (3, 4)\} \\ &= \sqrt{(1 - 0)^2 + (0 - 1)^2} + \sqrt{(2 - 1)^2 + (1 - 0)^2} + \sqrt{(3 - 2)^2 + (4 - 1)^2} \\ &= 2\sqrt{2} + \sqrt{10} \approx 5.99070. \end{aligned}$$

No. of Segments	Distance
1	4.24264
2	5.71593
3	5.99070
4	6.03562
5	6.06906
6	6.08713
7	6.09711

Note from Figure 1.5c that the arc length of the curve is greater than d_3 . Also, note that the more line segments we use, the better the approximation appears to be. While this process may be tedious, it will become much less so with the development of the definite integral in Chapter 4. For the moment, we list a number of these successively better approximations (produced using points on the curve with evenly spaced x -coordinates) in the table found in the margin. The table suggests that the length of the curve is approximately 6.1 (quite far from the straight-line distance of about 4.2). If you continued this rather tedious process using more and more line segments, the sum of their lengths would approach the actual length of the curve (about 6.126). This is the idea behind using the definite integral (which we develop in Chapter 4) to calculate such quantities. As with the problem of computing the slope of a curve, this also involves the notion of limit.

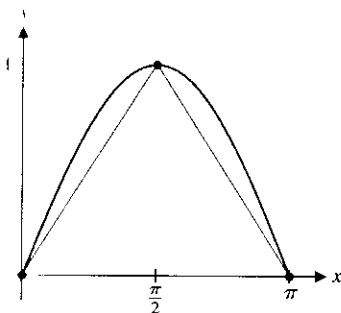


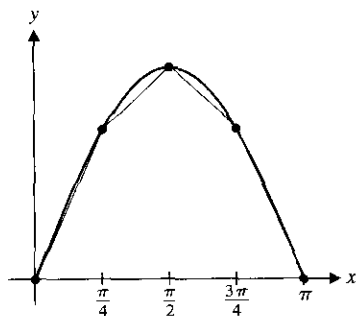
FIGURE 1.6a
Approximating the curve with two line segments.

EXAMPLE 1.3 Estimating the Length of a Curve

Estimate the length of the curve $y = \sin x$ for $0 \leq x \leq \pi$.

Solution The endpoints of the curve on this interval are $(0, 0)$ and $(\pi, 0)$. The distance between these points is $d_1 = \pi$. The point on the graph of $y = \sin x$ corresponding to the midpoint of the interval $[0, \pi]$ is $(\pi/2, 1)$. The distance from $(0, 0)$ to $(\pi/2, 1)$ plus the distance from $(\pi/2, 1)$ to $(\pi, 0)$ (illustrated in Figure 1.6a) is

$$d_2 = \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} + \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \approx 3.7242.$$

**FIGURE 1.6b**

Approximating the curve with four line segments.

Number of Line Segments	Sum of Lengths
8	3.8125
16	3.8183
32	3.8197
64	3.8201

Using the five points $(0, 0)$, $(\pi/4, 1/\sqrt{2})$, $(\pi/2, 1)$, $(3\pi/4, 1/\sqrt{2})$ and $(\pi, 0)$ (i.e., four line segments, as indicated in Figure 1.6b), the sum of the lengths of these line segments is

$$d_4 = 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}} + 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} \approx 3.7901.$$

Using nine points (i.e., eight line segments), you need a good calculator and some patience to compute the distance of 3.8125. A table showing further approximations is given in the margin. At this stage, it would be reasonable to estimate the length of the sine curve on the interval $[0, \pi]$ as slightly more than 3.8. ■

You may be relieved to know that we will develop some techniques to make such calculations much easier. For now, don't let the somewhat tedious arithmetic get in the way of your understanding the basic ideas behind the calculus. Notice that we extended the notion of distance to length of a curve by making some reasonably obvious (straight-line) approximations and then systematically improving on those approximations. As with our earlier problem on computing the slope of a curve, the essence of this is the concept of *limit*. In this case, the larger the number of line segments we used, the closer the approximate length got to the exact length.

It is the notion of limit that separates precalculus mathematics from the calculus. We develop this idea carefully in the rest of the chapter and use limits extensively throughout the book.

In the sections that follow, we carefully develop the notion of limit and then subsequently use limits to produce an assortment of powerful mathematical tools known collectively as the calculus.



EXERCISES 1.1



WRITING EXERCISES

1. Explain why each approximation of length in example 1.3 is less than the actual length.
2. To estimate the slope of $f(x) = x^2 + 1$ at $x = 1$, you would compute the slopes of various secant lines. Note that $y = x^2 + 1$ curves up. Explain why the secant line connecting $(1, 2)$ and $(1.1, 2.21)$ will have slope greater than the slope of the tangent line. Discuss how the slope of the secant line between $(1, 2)$ and $(0.9, 1.81)$ compares to the slope of the tangent line.

T In exercises 1–12, estimate the slope (as in example 1.1) of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 + 1, a = 1$
2. $f(x) = x^2 + 1, a = 2$
3. $f(x) = \cos x, a = 0$
4. $f(x) = \cos x, a = \pi/2$
5. $f(x) = x^3 + 2, a = 0$
6. $f(x) = x^3 + 2, a = 1$
7. $f(x) = \sqrt{x+1}, a = 0$
8. $f(x) = \sqrt{x+1}, a = 3$

9. $f(x) = e^x, a = 0$

10. $f(x) = e^x, a = 1$

11. $f(x) = \ln x, a = 1$

12. $f(x) = \ln x, a = 2$

T In exercises 13–20, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n 's and conjecture the actual length of the curve.

13. $f(x) = x^2 + 1, 0 \leq x \leq 2$

14. $f(x) = x^3 + 2, 0 \leq x \leq 1$

15. $f(x) = \cos x, 0 \leq x \leq \pi/2$

16. $f(x) = \sin x, 0 \leq x \leq \pi/2$

17. $f(x) = \sqrt{x+1}, 0 \leq x \leq 3$

18. $f(x) = 1/x, 1 \leq x \leq 2$

19. $f(x) = x^2 + 1, -2 \leq x \leq 2$

20. $f(x) = x^3 + 2, -1 \leq x \leq 1$

✚ EXPLORATORY EXERCISE

- Several central concepts of calculus have been introduced in this section. An important aspect of our future development of calculus is to derive simple techniques for computing quantities such as slope and length. In this exercise, you will learn how to directly compute the slope of a curve at a point. Suppose you want the slope of $y = x^2$ at $x = 1$. You could start by computing slopes of secant lines connecting the point $(1, 1)$ with nearby points. Suppose the nearby point has x -coordinate

$1 + h$, where h is a small (positive or negative) number. Explain why the corresponding y -coordinate is $(1 + h)^2$. Show that the slope of the secant line is $\frac{(1 + h)^2 - 1}{1 + h - 1} = 2 + h$. As h gets closer and closer to 0, this slope better approximates the slope of the tangent line. By setting h equal to 0, show that the slope of the tangent line equals 2. In a similar way, show that the slope of $y = x^2$ at $x = 2$ is 4 and find the slope of $y = x^2$ at $x = 3$. Based on your answers, conjecture a formula for the slope of $y = x^2$ at $x = a$, for any unspecified value of a .

1.2 THE CONCEPT OF LIMIT

In this section, we develop the notion of limit using some common language and illustrate the idea with some simple examples. The notion turns out to be a rather subtle one, easy to think of intuitively, but a bit harder to pin down in precise terms. We present the precise definition of limit in Appendix B. There, we carefully define limits in considerable detail. The more informal notion of limit that we introduce and work with in this chapter is adequate for most purposes.

As a start, consider the functions

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5}{x - 2}.$$

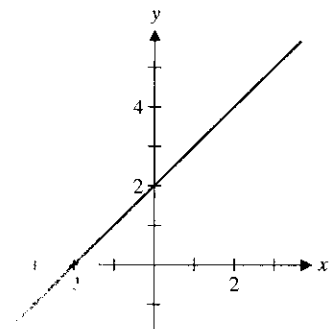


FIGURE 1.7a

calculator graph of $y = \frac{x^2 - 4}{x - 2}$.

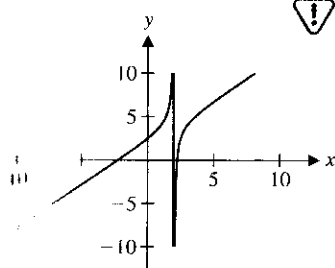


FIGURE 1.7b

calculator graph of $y = \frac{x^2 - 5}{x - 2}$.

What's the first thing that you notice about these functions? You probably observed that both functions are undefined at $x = 2$. So, what does this mean, beyond saying that you cannot substitute 2 for x ? We often find important clues about the behavior of a function from a calculator or computer-generated graph (see Figures 1.7a and 1.7b). Before beginning a careful analysis of each function though, we need to briefly comment on some shortcomings of these graphs.

The computer-generated graph in Figure 1.7a appears to be a straight line with no unusual behavior noticeable near $x = 2$. Actually, there is a hole in the true graph of $y = g(x)$ at $x = 2$, but it's not visible in Figure 1.7a. This is another limitation of your calculator or computer software. Computers and calculators typically won't show a single point removed from a graph. We indicate such a point with an open circle, as shown in Figure 1.8a (on the following page).

Many calculator- and computer-generated graphs will include the vertical segment displayed in Figure 1.7b, although this is *not* actually part of the true graph. Rather, it results from your calculator or computer connecting two points, one slightly to the left of $x = 2$ and the other slightly to the right of $x = 2$, where one function value is large and positive and the other is large in absolute value, but negative. The vertical line segment plotted suggests the vertical asymptote indicated correctly in Figure 1.8b (on the following page) as the dashed vertical line $x = 2$. When you see such vertical segments, you should suspect that they represent vertical asymptotes. We discuss asymptotes in more detail in section 1.5.

Notice that the graphs of these two functions look quite different in the vicinity of $x = 2$. Although we can't say anything about the value of these functions at $x = 2$ (since this is outside the domain of both functions), we can examine their behavior in the vicinity of

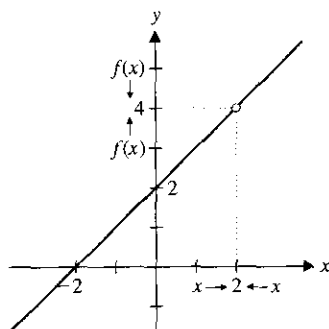


FIGURE 1.8a

$$y = \frac{x^2 - 4}{x - 2}$$

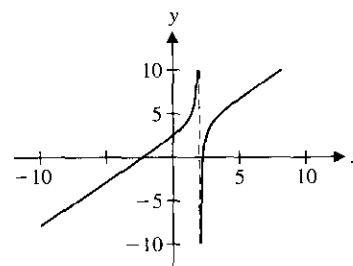


FIGURE 1.8b

$$y = \frac{x^2 - 5}{x - 2}$$

x	$f(x) = \frac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.99999	3.99999
1.999999	3.999999

this point. We consider these functions one at a time. First, for $f(x) = \frac{x^2 - 4}{x - 2}$, we compute some values of the function for x close to 2, as in the table shown in the margin.

Notice that as you move down the first column of the table, the x -values get closer to 2, but are all less than 2. We use the notation $x \rightarrow 2^-$ to indicate that x approaches 2 from the left side. Notice that the table and the graph both suggest that as x gets closer and closer to 2 (with $x < 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the left** is 4, written

$$\lim_{x \rightarrow 2^-} f(x) = 4.$$

Likewise, we need to consider what happens to the function values for x close to 2 but larger than 2. Here, we use the notation $x \rightarrow 2^+$ to indicate that x approaches 2 from the right side. We compute some of these values in the table shown in the margin.

Again, the table and graph both suggest that as x gets closer and closer to 2 (with $x > 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the right** is 4, written

$$\lim_{x \rightarrow 2^+} f(x) = 4.$$

We call $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$ **one-sided limits**. Since the two one-sided limits of $f(x)$ are the same, we summarize our results by saying that the **limit of $f(x)$ as x approaches 2** is 4, written

$$\lim_{x \rightarrow 2} f(x) = 4.$$

The notion of limit as we have described it here is intended to communicate the behavior of a function *near* some point of interest, but not actually *at* that point. We finally observe that we can also determine this limit algebraically, as follows. Notice that since the expression in the numerator of $f(x) = \frac{x^2 - 4}{x - 2}$ factors, we can write

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} && \text{Cancel two factors of } (x - 2). \\ &= \lim_{x \rightarrow 2} (x + 2) = 4, && \text{As } x \text{ approaches 2, } (x + 2) \text{ approaches 4.} \end{aligned}$$

x	$g(x) = \frac{x^2 - 5}{x - 2}$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	10003.9999
1.99999	100003.99999
1.999999	1000003.999999

where we can cancel the factors of $(x - 2)$ since in the limit as $x \rightarrow 2$, x is *close* to 2, but $x \neq 2$, so that $x - 2 \neq 0$.

Similarly, we consider one-sided limits for $g(x) = \frac{x^2 - 5}{x - 2}$, as $x \rightarrow 2$. Based on the graph in Figure 1.8b and the table of approximate function values shown in the margin, observe that as x gets closer and closer to 2 (with $x < 2$), $g(x)$ increases without bound. Since there is no number that $g(x)$ is approaching, we say that the **limit of $g(x)$ as x approaches 2 from the left does not exist**, written

$$\lim_{x \rightarrow 2^-} g(x) \text{ does not exist.}$$

Similarly, the graph and the table of function values for $x > 2$ (shown in the margin) suggest that $g(x)$ decreases without bound as x approaches 2 from the right. Since there is no number that $g(x)$ is approaching, we say that

$$\lim_{x \rightarrow 2^+} g(x) \text{ does not exist.}$$

Finally, since there is no common value for the one-sided limits of $g(x)$ (in fact, neither limit exists), we say that the **limit of $g(x)$ as x approaches 2 does not exist**, written

$$\lim_{x \rightarrow 2} g(x) \text{ does not exist.}$$

x	$g(x) = \frac{x^2 - 5}{x - 2}$
2.1	-5.9
2.01	-95.99
2.001	-995.999
2.0001	-9995.9999
2.00001	-99995.99999
2.000001	-999995.999999

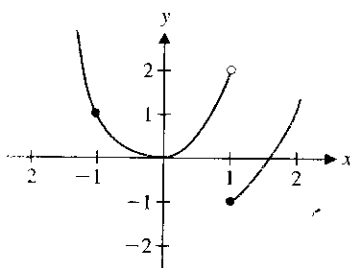


FIGURE 1.9
 $y = f(x)$.

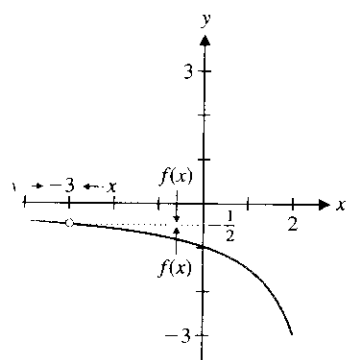


FIGURE 1.10
 $\lim_{x \rightarrow -3} \frac{3x + 9}{x^2 - 9} = -\frac{1}{2}$.

Before moving on, we should summarize what we have said about limits. **A limit exists if and only if both one-sided limits exist and are equal.** That is,

$$\lim_{x \rightarrow a} f(x) = L, \text{ for some number } L, \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L.$$

In other words, we say that $\lim_{x \rightarrow a} f(x) = L$ if we can make $f(x)$ as close as we might like to L , by making x sufficiently close to a (on either side of a), but not equal to a .

Note that we can think about limits from a purely graphical viewpoint, as in example 2.1.

EXAMPLE 2.1 Determining Limits Graphically

Use the graph in Figure 1.9 to determine $\lim_{x \rightarrow 1^-} f(x)$, $\lim_{x \rightarrow 1^+} f(x)$, $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow -1} f(x)$.

Solution For $\lim_{x \rightarrow 1^-} f(x)$, we consider the y -values as x gets closer and closer to 1, with $x < 1$. That is, we follow the graph toward $x = 1$ *from the left* ($x < 1$). Observe that the graph dead-ends into the open circle at the point $(1, 2)$. Therefore, we say that $\lim_{x \rightarrow 1^-} f(x) = 2$. For $\lim_{x \rightarrow 1^+} f(x)$, we follow the graph toward $x = 1$ *from the right* ($x > 1$). In this case, the graph dead-ends into the solid circle located at the point $(1, -1)$. For this reason, we say that $\lim_{x \rightarrow 1^+} f(x) = -1$. Because $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$, we say that $\lim_{x \rightarrow 1} f(x)$ does not exist. Finally, we have that $\lim_{x \rightarrow -1} f(x) = 1$, since the graph approaches a y -value of 1, as x approaches -1 both from the left and from the right. ■

EXAMPLE 2.2 A Limit Where Two Factors Cancel

Evaluate $\lim_{x \rightarrow -3} \frac{3x + 9}{x^2 - 9}$.

Solution Here, the domain of the function includes all values of x except $x = \pm 3$. Again, we examine a graph (see Figure 1.10) and compute some function values for x

x	$\frac{3x+9}{x^2-9}$
-3.1	-0.491803
-3.01	-0.499168
-3.001	-0.499917
-3.0001	-0.499992
-3.00001	-0.499999

x	$\frac{3x+9}{x^2-9}$
-2.9	-0.508475
-2.99	-0.500835
-2.999	-0.500083
-2.9999	-0.500008
-2.99999	-0.500001

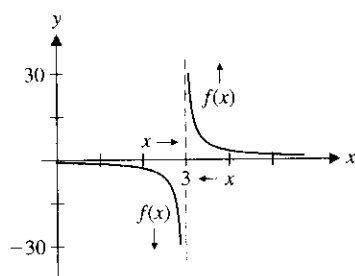


FIGURE 1.11

$$y = \frac{3x+9}{x^2-9}$$

x	$\frac{3x+9}{x^2-9}$
3.1	30
3.01	300
3.001	3000
3.0001	30,000
3.00001	300,000

x	$\frac{3x+9}{x^2-9}$
2.9	-30
2.99	-300
2.999	-3000
2.9999	-30,000
2.99999	-300,000

near -3 . Based on this numerical and graphical evidence, it's reasonable to conjecture that

$$\lim_{x \rightarrow -3^+} \frac{3x+9}{x^2-9} = \lim_{x \rightarrow -3^-} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

Further, note that

$$\begin{aligned} \lim_{x \rightarrow -3} \frac{3x+9}{x^2-9} &= \lim_{x \rightarrow -3} \frac{3(x+3)}{(x+3)(x-3)} \quad \text{Cancel two factors of } (x+3). \\ &= \lim_{x \rightarrow -3} \frac{3}{x-3} = -\frac{1}{2}, \end{aligned}$$

since $(x-3) \rightarrow -6$ as $x \rightarrow -3$. Again, the cancellation of the two factors of $(x+3)$ is valid since in the limit as $x \rightarrow -3$, x is *close* to -3 , but $x \neq -3$, so that $x+3 \neq 0$. Likewise,

$$\lim_{x \rightarrow -3^+} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

Finally, since the function approaches the *same* value as $x \rightarrow -3$ both from the right and from the left (i.e., the one-sided limits are equal), we write

$$\lim_{x \rightarrow -3} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

In example 2.2, the limit exists because both one-sided limits exist and are equal. In example 2.3, neither one-sided limit exists.

EXAMPLE 2.3 A Limit That Does Not Exist

Determine whether or not $\lim_{x \rightarrow 3} \frac{3x+9}{x^2-9}$ exists.

Solution We first draw a graph (see Figure 1.11) and compute some function values for x close to 3.

Based on this numerical and graphical evidence, it appears that, as $x \rightarrow 3^+$, $\frac{3x+9}{x^2-9}$ is increasing without bound. Thus,

$$\lim_{x \rightarrow 3^+} \frac{3x+9}{x^2-9} \text{ does not exist.}$$

Similarly, from the graph and the table of values for $x < 3$, we can say that

$$\lim_{x \rightarrow 3^-} \frac{3x+9}{x^2-9} \text{ does not exist.}$$

Since neither one-sided limit exists, we say

$$\lim_{x \rightarrow 3} \frac{3x+9}{x^2-9} \text{ does not exist.}$$

Here, we considered both one-sided limits for the sake of completeness. Of course, you should keep in mind that if *either* one-sided limit fails to exist, then the limit does not exist. ■

Many limits cannot be resolved using algebraic methods. In these cases, we can approximate the limit using graphical and numerical evidence, as we see in example 2.4.

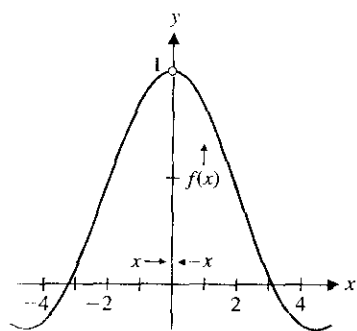


FIGURE 1.12

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

EXAMPLE 2.4 Approximating the Value of a Limit

Evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

Solution Unlike some of the limits considered previously, there is no algebra that will simplify this expression. However, we can still draw a graph (see Figure 1.12) and use a calculator to compute some approximate function values.

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.99999983
0.0001	0.9999999983
0.00001	0.999999999983

x	$\frac{\sin x}{x}$
-0.1	0.998334
-0.01	0.999983
-0.001	0.99999983
-0.0001	0.9999999983
-0.00001	0.999999999983

The graph and the tables of values lead us to the conjectures:

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1,$$

from which we conjecture that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

In Chapter 2, we see that this limit has special significance. ■

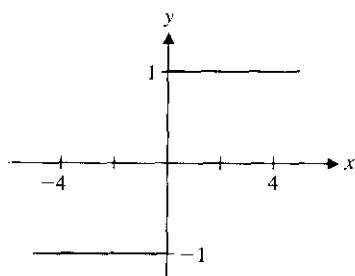


FIGURE 1.13a

$$f(x) = \frac{x}{|x|}.$$

**REMARK 2.1**

Computer or calculator computation of limits is unreliable. We use graphs and tables of values only as (strong) evidence pointing to what a plausible answer might be. To be certain, we need to obtain careful verification of our conjectures. We see how to do this in the remaining sections of this chapter.

Many functions have graphs with jumps, where the y -values take a sudden jump from one value to another. We analyze such behavior with one-sided limits, as in example 2.5.

EXAMPLE 2.5 A Case Where One-Sided Limits Are Needed

Evaluate $\lim_{x \rightarrow 0} \frac{x}{|x|}$.

Solution The computer-generated graph shown in Figure 1.13a is revealing. As we have previously observed, your calculator or computer may draw a vertical segment connecting the two horizontal segments. Again, such a vertical segment is not actually a part of the graph, but results from the computer or calculator connecting two points on either side of the x -axis. In this case, though, there is no vertical asymptote at $x = 0$. The graph in Figure 1.13b correctly shows open circles at the intersections of the two halves of the graph with the y -axis. We also have

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{x}{|x|} &= \lim_{x \rightarrow 0^+} \frac{x}{x} && \text{Since } |x| = x, \text{ when } x > 0. \\ &= \lim_{x \rightarrow 0^+} 1 \\ &= 1 \end{aligned}$$

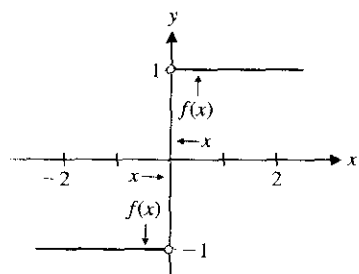


FIGURE 1.13b

$\lim_{x \rightarrow 0} \frac{x}{|x|}$ does not exist.

and

$$\begin{aligned}\lim_{x \rightarrow 0^-} \frac{x}{|x|} &= \lim_{x \rightarrow 0^-} \frac{x}{-x} && \text{Since } |x| = -x, \text{ when } x < 0. \\ &= \lim_{x \rightarrow 0^-} -1 \\ &= -1.\end{aligned}$$

It now follows that

$$\lim_{x \rightarrow 0} \frac{x}{|x|} \text{ does not exist,}$$

since the one-sided limits are not the same. You should also keep in mind that this observation is entirely consistent with what we see in the graph. Also, note that the function can be rewritten as

$$\frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$$

Whenever you encounter a piecewise-defined function such as this, you should compute one-sided limits as we did here. ■



Regulation baseball, showing stitching.

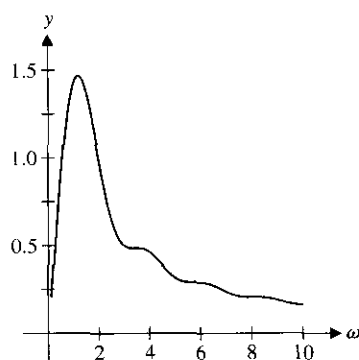


FIGURE 1.14

$$y = \frac{1.7}{\omega} - \frac{5}{8\omega^2} \sin(2.72\omega).$$

EXAMPLE 2.6 A Limit Describing the Movement of a Baseball Pitch

The knuckleball is one of the most exotic pitches in baseball. Batters describe the ball as unpredictably moving left, right, up and down. The motion of the knuckleball is mostly due to the raised stitches on the baseball, which act like an airplane wing to give a “lift force” to the ball. The motion is erratic because as the ball moves, the orientation of the stitches shifts and hence the lift force on the baseball changes.

For a typical knuckleball speed of 60 mph, the left/right position of the ball (in feet) as it crosses the plate is given by:

$$f(\omega) = \frac{1.7}{\omega} - \frac{5}{8\omega^2} \sin(2.72\omega)$$

(derived from experimental data in Watts and Bahill’s excellent book *Keeping Your Eye on the Ball*), where ω is the rotational speed of the ball in radians per second and where $f(\omega) = 0$ corresponds to the middle of home plate. Folk wisdom among baseball pitchers has it that the less spin on the ball, the better the pitch. To investigate this theory, we consider the limit of $f(\omega)$ as $\omega \rightarrow 0$. As always, we look at a graph (see Figure 1.14) and generate a table of function values. The graphical and numerical evidence suggests that $\lim_{\omega \rightarrow 0} f(\omega) = 0$.

ω	$f(\omega)$
10	0.1645
1	1.4442
0.1	0.2088
0.01	0.021
0.001	0.0021
0.0001	0.0002

The limit indicates that a knuckleball with absolutely no spin doesn’t move at all (and therefore would be easy to hit). According to Watts and Bahill, a very slow rotation

rate of about 1 to 3 radians per second produces the best pitch (i.e., the most movement). Take another look at Figure 1.14 to convince yourself that this makes sense. ■

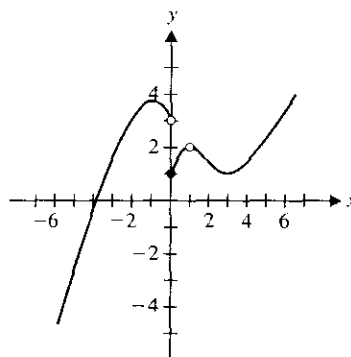
EXERCISES 1.2

WRITING EXERCISES

- Suppose your professor says, "You can think of the limit of $f(x)$ as x approaches a as *what $f(a)$ should be*." Critique this statement. What does it mean? Does it provide important insight? Is there anything misleading about it? Replace the phrase in italics with your own best description of what the limit is.
- Your friend's professor says, "The limit is a *prediction* of what $f(a)$ will be." Compare and contrast this statement to the one in exercise 1. Does the inclusion of the word **prediction** make the limit idea seem more useful and important?
- We have emphasized that $\lim_{x \rightarrow a} f(x)$ does not depend on the actual value of $f(a)$, or even on whether $f(a)$ exists. In principle, functions such as $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 13 & \text{if } x = 2 \end{cases}$ are as "normal" as functions such as $g(x) = x^2$. With this in mind, explain why it is important that the limit concept is independent of how (or whether) $f(a)$ is defined.
- The most common limit encountered in everyday life is the **speed limit**. Describe how this type of limit is very different from the limits discussed in this section.

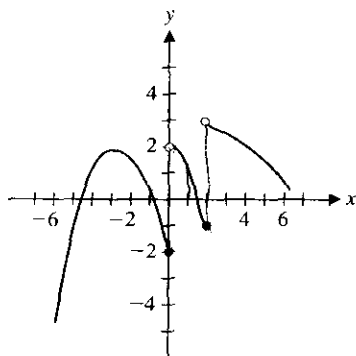
- For the function graphed below, identify each limit or state that it does not exist.

- | | |
|-------------------------------------|-------------------------------------|
| (a) $\lim_{x \rightarrow 0^-} f(x)$ | (b) $\lim_{x \rightarrow 0^+} f(x)$ |
| (c) $\lim_{x \rightarrow 0} f(x)$ | (d) $\lim_{x \rightarrow 2^-} f(x)$ |
| (e) $\lim_{x \rightarrow -2} f(x)$ | (f) $\lim_{x \rightarrow 1^-} f(x)$ |
| (g) $\lim_{x \rightarrow 1^+} f(x)$ | (h) $\lim_{x \rightarrow 1} f(x)$ |
| (i) $\lim_{x \rightarrow -1} f(x)$ | (j) $\lim_{x \rightarrow 3} f(x)$ |



- For the function graphed below, identify each limit or state that it does not exist.

- | | |
|-------------------------------------|-------------------------------------|
| (a) $\lim_{x \rightarrow 0^-} f(x)$ | (b) $\lim_{x \rightarrow 0^+} f(x)$ |
| (c) $\lim_{x \rightarrow 0} f(x)$ | (d) $\lim_{x \rightarrow 1^-} f(x)$ |
| (e) $\lim_{x \rightarrow -1} f(x)$ | (f) $\lim_{x \rightarrow 2^-} f(x)$ |
| (g) $\lim_{x \rightarrow 2^+} f(x)$ | (h) $\lim_{x \rightarrow 2} f(x)$ |
| (i) $\lim_{x \rightarrow -2} f(x)$ | (j) $\lim_{x \rightarrow 3} f(x)$ |



- Sketch the graph of $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ and identify each limit.

- | | |
|-------------------------------------|-------------------------------------|
| (a) $\lim_{x \rightarrow 2^-} f(x)$ | (b) $\lim_{x \rightarrow 2^+} f(x)$ |
| (c) $\lim_{x \rightarrow 2} f(x)$ | (d) $\lim_{x \rightarrow 1} f(x)$ |

- Sketch the graph of $f(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x+1} - 2 & \text{if } x > 0 \end{cases}$ and identify each limit.

- | | | |
|-------------------------------------|-------------------------------------|-----------------------------------|
| (a) $\lim_{x \rightarrow 0^-} f(x)$ | (b) $\lim_{x \rightarrow 0^+} f(x)$ | (c) $\lim_{x \rightarrow 0} f(x)$ |
| (d) $\lim_{x \rightarrow -1} f(x)$ | (e) $\lim_{x \rightarrow 3} f(x)$ | |

- Sketch the graph of $f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \geq -1 \end{cases}$ and identify each limit.

- | | |
|--------------------------------------|--------------------------------------|
| (a) $\lim_{x \rightarrow -1^-} f(x)$ | (b) $\lim_{x \rightarrow -1^+} f(x)$ |
| (c) $\lim_{x \rightarrow -1} f(x)$ | (d) $\lim_{x \rightarrow 0} f(x)$ |

6. Sketch the graph of $f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ 3 & \text{if } -1 \leq x < 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$ and identify each limit.

(a) $\lim_{x \rightarrow -1^-} f(x)$ (b) $\lim_{x \rightarrow -1^+} f(x)$ (c) $\lim_{x \rightarrow -1} f(x)$
 (d) $\lim_{x \rightarrow 1} f(x)$ (e) $\lim_{x \rightarrow 0} f(x)$

- T** 7. Evaluate $f(1.5)$, $f(1.1)$, $f(1.01)$ and $f(1.001)$ and conjecture a value for $\lim_{x \rightarrow 1^+} f(x)$ for $f(x) = \frac{x-1}{\sqrt{x}-1}$. Evaluate $f(0.5)$, $f(0.9)$, $f(0.99)$ and $f(0.999)$ and conjecture a value for $\lim_{x \rightarrow 1^-} f(x)$ for $f(x) = \frac{x-1}{\sqrt{x}-1}$. Does $\lim_{x \rightarrow 1} f(x)$ exist?

- T** 8. Evaluate $f(-1.5)$, $f(-1.1)$, $f(-1.01)$ and $f(-1.001)$ and conjecture a value for $\lim_{x \rightarrow -1^-} f(x)$ for $f(x) = \frac{x+1}{x^2-1}$. Evaluate $f(-0.5)$, $f(-0.9)$, $f(-0.99)$ and $f(-0.999)$ and conjecture a value for $\lim_{x \rightarrow -1^+} f(x)$ for $f(x) = \frac{x+1}{x^2-1}$. Does $\lim_{x \rightarrow -1} f(x)$ exist?

- T** In exercises 9–14, use numerical and graphical evidence to conjecture values for each limit.

9. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 2}$

10. $\lim_{x \rightarrow -1} \frac{x^2 + x}{x^2 - x - 2}$

11. $\lim_{x \rightarrow 0} \frac{x^2 + x}{\sin x}$

12. $\lim_{x \rightarrow \pi} x \sin x$

13. $\lim_{x \rightarrow 0} \frac{\tan x}{\sin x}$

14. $\lim_{x \rightarrow 2} \sqrt{x^2 - 2x + 2}$

- T** In exercises 15–26, use numerical and graphical evidence to conjecture whether or not the limit at $x = a$ exists. If not, describe what is happening at $x = a$ graphically.

15. $\lim_{x \rightarrow 2} \frac{x - 2}{|x - 2|}$

16. $\lim_{x \rightarrow -1} \frac{|x + 1|}{x^2 - 1}$

17. $\lim_{x \rightarrow -1} \frac{x^2 - 1}{x^2 + 2x - 3}$

18. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

19. $\lim_{x \rightarrow 1} \frac{x^2 + 1}{x - 1}$

20. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 + 2x - 3}$

21. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 2x + 1}$

22. $\lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1}$

23. $\lim_{x \rightarrow 0} \ln x$

24. $\lim_{x \rightarrow 0} x \ln(x^2)$

25. $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

26. $\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x}$

27. Use your results from exercises 17–22 to investigate the following. Suppose that $f(x)$ and $g(x)$ are polynomials with $g(a) = 0$ and $f(a) \neq 0$. What can you conjecture about $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$?

28. Use your results from exercises 17–22 to investigate the following. Suppose that $f(x)$ and $g(x)$ are polynomials with $f(a) = 0$ and $g(a) \neq 0$. What can you conjecture about $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$?

In exercises 29–32, sketch a graph of a function with the given properties.

29. $f(-1) = 2$, $f(0) = -1$, $f(1) = 3$ and $\lim_{x \rightarrow 1} f(x)$ does not exist.

30. $f(x) = 1$ for $-2 \leq x \leq 1$, $\lim_{x \rightarrow 1^+} f(x) = 3$ and $\lim_{x \rightarrow -2} f(x) = 1$.

31. $f(0) = 1$, $\lim_{x \rightarrow 0^-} f(x) = 2$ and $\lim_{x \rightarrow 0^+} f(x) = 3$.

32. $\lim_{x \rightarrow 0} f(x) = -2$, $f(0) = 1$, $f(2) = 3$ and $\lim_{x \rightarrow 2} f(x)$ does not exist.

33. As we see in Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is given by $m = \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h}$. Estimate the slope m . Graph $y = \sqrt{x}$ and the line with slope m through the point $(1, 1)$.

34. As we see in Chapter 2, the velocity of an object that has traveled \sqrt{x} miles in x hours at the $x = 1$ hour mark is given by $v = \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1}$. Estimate this limit.

35. Consider the following arguments concerning $\lim_{x \rightarrow 0^+} \sin \frac{\pi}{x}$.

First, as $x > 0$ approaches 0, $\frac{\pi}{x}$ increases without bound, since $\sin t$ oscillates for increasing t , the limit does not exist. Second: taking $x = 1, 0.1, 0.01$ and so on, we compute $\sin \pi = \sin 10\pi = \sin 100\pi = \dots = 0$; therefore the limit equals 0. Which argument sounds better to you? Explain. Explore the limit and determine which answer is correct.

36. Consider the following argument concerning $\lim_{x \rightarrow 0} e^{-1/x}$. As x approaches 0, $\frac{1}{x}$ increases without bound and $\frac{-1}{x}$ decreases without bound. Since e^t approaches 0 as t decreases without bound, the limit equals 0. Discuss all the errors made in this argument.

- T** 37. Numerically estimate the one-sided limits $\lim_{x \rightarrow 0^+} (1+x)^{1/x}$ and $\lim_{x \rightarrow 0^-} (1+x)^{1/x}$. Note that the function values for $x > 0$ increase as x decreases, while for $x < 0$ the function values decrease as x increases. Explain why this indicates that the limit $\lim_{x \rightarrow 0} (1+x)^{1/x}$ is between function values for positive and negative x 's. Approximate this limit correct to eight digits.

38. Explain what is wrong with the following logic (you may use exercise 37 to convince yourself that the answer is wrong, but discuss the logic without referring to exercise 37): as $x \rightarrow 0$, it is clear that $(1+x) \rightarrow 1$. Since 1 raised to any power is 1, $\lim_{x \rightarrow 0} (1+x)^{1/x} = \lim_{x \rightarrow 0} 1 = 1$.

19 Numerically estimate $\lim_{x \rightarrow 0^+} x^{\sec x}$. Try to numerically estimate $\lim_{x \rightarrow 0^-} x^{\sec x}$. If your computer has difficulty evaluating the function for negative x 's, explain why.

20 Explain what is wrong with the following logic (note from exercise 39 that the answer is accidentally correct): since 0 to any power is 0, $\lim_{x \rightarrow 0} x^{\sec x} = \lim_{x \rightarrow 0} 0^{\sec x} = 0$.

21 Give an example of a function $f(x)$ such that $\lim_{x \rightarrow 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function $g(x)$ such that $g(0)$ exists but $\lim_{x \rightarrow 0} g(x)$ does not exist.

22 Give an example of a function $f(x)$ such that $\lim_{x \rightarrow 0} f(x)$ exists and $f(0)$ exists, but $\lim_{x \rightarrow 0} f(x) \neq f(0)$.

23 In the text, we described $\lim_{x \rightarrow a} f(x) = L$ as meaning "as x gets closer and closer to a , $f(x)$ is getting closer and closer to L ." As x gets closer and closer to 0, it is true that x^2 gets closer and closer to 0, but it is certainly not true that $\lim_{x \rightarrow 0} x^2 = -0.01$. Try to modify the description of limit to make it clear that $\lim_{x \rightarrow 0} x^2 \neq -0.01$. We explore a very precise definition of limit in Appendix B.

24 In Figure 1.14, the final position of the knuckleball at time $t = 0.68$ is shown as a function of the rotation rate ω . The batter must decide at time $t = 0.4$ whether or not to swing at the pitch. At $t = 0.4$, the left/right position of the ball is given by $h(\omega) = \frac{1}{\omega} - \frac{5}{8\omega^2} \sin(1.6\omega)$. Graph $h(\omega)$ and compare to Figure 1.14. Conjecture the limit of $h(\omega)$ as $\omega \rightarrow 0$. For $\omega = 0$, is there any difference in ball position between what the batter sees at $t = 0.4$ and what he tries to hit at $t = 0.68$?

25 A parking lot charges \$2 for each hour or portion of an hour, with a maximum charge of \$12 for all day. If $f(t)$ equals the total parking bill for t hours, sketch a graph of $y = f(t)$ for $0 \leq t \leq 24$. Determine the limits $\lim_{t \rightarrow 3.5} f(t)$ and $\lim_{t \rightarrow 4} f(t)$, if they exist.

26 For the parking lot in exercise 25, determine all values of a with $0 \leq a \leq 24$ such that $\lim_{t \rightarrow a} f(t)$ does not exist. Briefly discuss the effect this has on your parking strategy (e.g., are there times where you would be in a hurry to move your car or times where it doesn't matter whether you move your car?).

EXPLORATORY EXERCISES

1. In a situation similar to that of example 2.6, the left/right position of a knuckleball pitch in baseball can be modeled by $P = \frac{5}{8\omega^2}(1 - \cos 4\omega t)$, where t is time measured in seconds ($0 \leq t \leq 0.68$) and ω is the rotation rate of the ball measured in radians per second. In example 2.6, we chose a specific t -value and evaluated the limit as $\omega \rightarrow 0$. While this gives us some information about which rotation rates produce hard-to-hit pitches, a clearer picture emerges if we look at $f(t)$ over its entire domain. Set $\omega = 10$ and graph the resulting function $\frac{1}{160}(1 - \cos 40t)$ for $0 \leq t \leq 0.68$. Imagine looking at a pitcher from above and try to visualize a baseball starting at the pitcher's hand at $t = 0$ and finally reaching the batter, at $t = 0.68$. Repeat this with $\omega = 5$, $\omega = 1$, $\omega = 0.1$ and whatever values of ω you think would be interesting. Which values of ω produce hard-to-hit pitches?

2. In this exercise, the results you get will depend on the accuracy of your computer or calculator. Work this exercise and compare your results with your classmates' results. We will investigate $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2}$. Start by verifying the calculations presented in the table:

x	$f(x)$
0.1	-0.499583...
0.01	-0.4999583...
0.001	-0.49999583...

Describe as precisely as possible the pattern shown here. What would you predict for $f(0.0001)$? $f(0.00001)$? Does your computer or calculator give you this answer? If you continue trying powers of 0.1 (0.000001, 0.0000001, etc.) you should eventually be given a displayed result of -0.5. Do you think this is exactly correct or has the answer just been rounded off? Why is rounding off inescapable? It turns out that -0.5 is the exact value for the limit, so the round-off here is somewhat helpful. However, if you keep evaluating the function at smaller and smaller values of x , you will eventually see a reported function value of 0. This round-off error is not so benign; we discuss this error in section 1.6. For now, evaluate $\cos x$ at the current value of x and try to explain where the 0 came from.

1.3 COMPUTATION OF LIMITS

Now that you have an idea of what a limit is, we need to develop some means of calculating limits of simple functions. In this section, we present some basic rules for dealing with common limit problems. We begin with a simple limit.

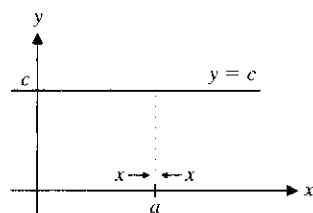


FIGURE 1.15

$$\lim_{x \rightarrow a} c = c.$$

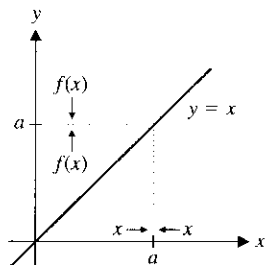


FIGURE 1.16

$$\lim_{x \rightarrow a} x = a.$$

For any constant c and any real number a ,

$$\lim_{x \rightarrow a} c = c. \quad (3.1)$$

In other words, the limit of a constant is that constant. This certainly comes as no surprise, since the function $f(x) = c$ does not depend on x and so, stays the same as $x \rightarrow a$ (see Figure 1.15). Another simple limit is the following.

For any real number a ,

$$\lim_{x \rightarrow a} x = a. \quad (3.2)$$

Again, this is not a surprise, since as $x \rightarrow a$, x will approach a (see Figure 1.16). Be sure that you are comfortable enough with the limit notation to recognize how obvious the limits in (3.1) and (3.2) are. As simple as they are, we use them repeatedly in finding more complex limits. We also need the basic rules contained in Theorem 3.1.

THEOREM 3.1

Suppose that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist and let c be any constant. The following then apply:

- (i) $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x),$
- (ii) $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x),$
- (iii) $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right]$ and
- (iv) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \left(\text{if } \lim_{x \rightarrow a} g(x) \neq 0 \right).$

The proof of Theorem 3.1 requires the formal definition of limit found in Appendix F. You should think of these rules as sensible results that you would certainly expect to be true, given your intuitive understanding of what a limit is. Read them in plain English. For instance, part (ii) says that the limit of a sum (or a difference) equals the sum (or difference) of the limits, *provided the limits exist*. Think of this as follows. If as x approaches a , $f(x)$ approaches L and $g(x)$ approaches M , then $f(x) + g(x)$ should approach $L + M$.

Observe that by applying part (iii) of Theorem 3.1 with $g(x) = f(x)$, we get that whenever $\lim_{x \rightarrow a} f(x)$ exists,

$$\begin{aligned} \lim_{x \rightarrow a} [f(x)]^2 &= \lim_{x \rightarrow a} [f(x) \cdot f(x)] \\ &= \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} f(x) \right] = \left[\lim_{x \rightarrow a} f(x) \right]^2. \end{aligned} \quad (3.3)$$

Likewise, for any positive integer, n , we can apply part (iii) of Theorem 3.1 repeatedly, to yield

$$\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad (3.4)$$

(see exercises 63 and 64).

Notice that taking $f(x) = x$ in (3.4) gives us that for any integer $n > 0$ and any real number a ,

$$\lim_{x \rightarrow a} x^n = a^n. \quad (3.5)$$

That is, to compute the limit of any positive power of x , you simply substitute in the value of x being approached.

EXAMPLE 3.1 Finding the Limit of a Power Function

Evaluate $\lim_{x \rightarrow -2} x^5$.

Solution From equation (3.5), we have

$$\lim_{x \rightarrow -2} x^5 = (-2)^5 = -32. \quad \blacksquare$$

EXAMPLE 3.2 Finding the Limit of a Polynomial

Apply the rules of limits to evaluate $\lim_{x \rightarrow 2} (3x^2 - 5x + 4)$.

Solution We have

$$\begin{aligned} \lim_{x \rightarrow 2} (3x^2 - 5x + 4) &= \lim_{x \rightarrow 2} (3x^2) - \lim_{x \rightarrow 2} (5x) + \lim_{x \rightarrow 2} 4 && \text{By Theorem 3.1 (ii).} \\ &= 3 \lim_{x \rightarrow 2} x^2 - 5 \lim_{x \rightarrow 2} x + 4 && \text{By Theorem 3.1 (i).} \\ &= 3 \cdot (2)^2 - 5 \cdot 2 + 4 = 6. && \text{From equation (3.5). } \blacksquare \end{aligned}$$



HISTORICAL NOTES

Augustin Louis Cauchy (1789–1857) A French mathematician who developed the precise definition of limit (see Appendix B). Cauchy was one of the most prolific mathematicians in history, making important contributions to number theory, linear algebra, differential equations, astronomy, optics and complex variables. A difficult man to get along with, a colleague wrote, “Cauchy is mad and there is nothing that can be done about him, although right now, he is the only one who knows how mathematics should be done.”

EXAMPLE 3.3 Finding the Limit of a Rational Function

Apply the rules of limits to evaluate $\lim_{x \rightarrow 3} \frac{x^3 - 5x + 4}{x^2 - 2}$.

Solution We get

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^3 - 5x + 4}{x^2 - 2} &= \frac{\lim_{x \rightarrow 3} (x^3 - 5x + 4)}{\lim_{x \rightarrow 3} (x^2 - 2)} && \text{By Theorem 3.1 (iv).} \\ &= \frac{\lim_{x \rightarrow 3} x^3 - 5 \lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 4}{\lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 2} && \text{By Theorem 3.1 (i) and (ii).} \\ &= \frac{3^3 - 5 \cdot 3 + 4}{3^2 - 2} = \frac{16}{7}. && \text{From equation (3.5). } \blacksquare \end{aligned}$$

You may have noticed that in the last two examples, we simply ended up substituting in the limiting value of x . In example 3.4, it's not quite so simple.

EXAMPLE 3.4 Finding a Limit by Factoring

Evaluate $\lim_{x \rightarrow 1} \frac{x^2 - 1}{1 - x}$.

Solution Notice right away that

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{1 - x} \neq \frac{\lim_{x \rightarrow 1} (x^2 - 1)}{\lim_{x \rightarrow 1} (1 - x)},$$

since the limit in the denominator is zero. (Recall that the limit of a quotient is the quotient of the limits *only* when both limits exist *and* the limit in the denominator is *not* zero.) However, we can resolve this problem by observing that

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{1 - x} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{-(x - 1)} && \text{Factoring the numerator and} \\ &&& \text{factoring } -1 \text{ from denominator.} \\ &= \lim_{x \rightarrow 1} \frac{(x + 1)}{-1} = -2, \end{aligned}$$

where the cancellation of the two factors of $(x - 1)$ is valid because in the limit as $x \rightarrow 1$, x is close to 1, but $x \neq 1$, so that $x - 1 \neq 0$. ■



In general, in any case where the limits of both the numerator and the denominator are 0, you should try to algebraically simplify the expression, to get a cancellation, as we did in example 3.4.

EXAMPLE 3.5 Finding a Limit By Rationalizing

Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$.

Solution First, notice that both the numerator $(\sqrt{x+2} - \sqrt{2})$ and the denominator (x) approach 0, as x approaches 0. Unlike example 3.4, we can't factor the numerator. However, we can rationalize the numerator, as follows:

$$\begin{aligned} \frac{\sqrt{x+2} - \sqrt{2}}{x} &= \frac{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}{x(\sqrt{x+2} + \sqrt{2})} = \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})} \\ &= \frac{x}{x(\sqrt{x+2} + \sqrt{2})} = \frac{1}{\sqrt{x+2} + \sqrt{2}}, \end{aligned}$$

where the last equality holds if $x \neq 0$ (which is the case in the limit as $x \rightarrow 0$). So, we have

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}. \quad \blacksquare$$

In Theorem 3.2, we show that the limit of a polynomial is simply the value of the polynomial at that point; that is, to find the limit of a polynomial, we simply substitute the value that x is approaching.

THEOREM 3.2

For any polynomial $p(x)$ and any real number a ,

$$\lim_{x \rightarrow a} p(x) = p(a).$$

PROOF

Suppose that $p(x)$ is a polynomial of degree $n \geq 0$,

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0.$$

Then, from Theorem 3.1 and equation (3.5),

$$\begin{aligned} \lim_{x \rightarrow a} p(x) &= \lim_{x \rightarrow a} (c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0) \\ &= c_n \lim_{x \rightarrow a} x^n + c_{n-1} \lim_{x \rightarrow a} x^{n-1} + \cdots + c_1 \lim_{x \rightarrow a} x + \lim_{x \rightarrow a} c_0 \\ &= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 = p(a). \quad \blacksquare \end{aligned}$$

Evaluating the limit of a polynomial is now easy. Many other limits are evaluated just as easily.

THEOREM 3.3

Suppose that $\lim_{x \rightarrow a} f(x) = L$ and n is any positive integer. Then,

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L},$$

where for n even, we assume that $L > 0$.

Notice that this result says that we may (under the conditions outlined in the hypotheses) bring limits “inside” n th roots. We can then use our existing rules for computing the limit inside.

EXAMPLE 3.6 Evaluating the Limit of an n th Root of a Polynomial

Evaluate $\lim_{x \rightarrow 2} \sqrt[3]{3x^2 - 2x}$.

Solution By Theorems 3.2 and 3.3, we have

$$\lim_{x \rightarrow 2} \sqrt[3]{3x^2 - 2x} = \sqrt[3]{\lim_{x \rightarrow 2} (3x^2 - 2x)} = \sqrt[3]{8}. \quad \blacksquare$$

So that we are not restricted to discussing only the algebraic functions (i.e., those that can be constructed by using addition, subtraction, multiplication, division, exponentiation and by taking n th roots), we state the following result now, without proof. For each part, you should sketch a graph to convince yourself that the result makes sense.

THEOREM 3.4

For any real number a , we have

- | | |
|---|--|
| (i) $\lim_{x \rightarrow a} \sin x = \sin a$, | (v) $\lim_{x \rightarrow a} \sin^{-1} x = \sin^{-1} a$, for $-1 \leq a \leq 1$, |
| (ii) $\lim_{x \rightarrow a} \cos x = \cos a$, | (vi) $\lim_{x \rightarrow a} \cos^{-1} x = \cos^{-1} a$, for $-1 \leq a \leq 1$ and |
| (iii) $\lim_{x \rightarrow a} e^x = e^a$, | (vii) $\lim_{x \rightarrow a} \tan^{-1} x = \tan^{-1} a$. |
| (iv) $\lim_{x \rightarrow a} \ln x = \ln a$, for $a > 0$, | |

Notice that Theorem 3.4 says that limits of the sine, cosine, exponential, natural logarithm, inverse sine, inverse cosine and inverse tangent functions are found simply by substitution. A more thorough discussion of functions with this property (called continuity) is found in section 1.4.

EXAMPLE 3.7 Evaluating a Limit of an Inverse Trigonometric Function

Evaluate $\lim_{x \rightarrow 0} \sin^{-1} \left(\frac{x+1}{2} \right)$.

Solution By Theorem 3.4, we have

$$\lim_{x \rightarrow 0} \sin^{-1} \left(\frac{x+1}{2} \right) = \sin^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{6}.$$

So much for limits that we can compute using elementary rules. Certainly, many of the limits that we need to find in practice are of the type discussed in example 3.7. Just as many, however, cannot be determined using these elementary rules and can only be found using more careful analysis, often by an indirect approach. For instance, consider the following problem.

EXAMPLE 3.8 A Limit of a Product That Is Not the Product of the Limits

Evaluate $\lim_{x \rightarrow 0} (x \cot x)$.

Solution Your first reaction might be to say that this is a limit of a product and so, must be the product of the limits:

$$\begin{aligned} \lim_{x \rightarrow 0} (x \cot x) &= \left(\lim_{x \rightarrow 0} x \right) \left(\lim_{x \rightarrow 0} \cot x \right) \quad \text{This is incorrect!} \\ &= 0 \cdot ? = 0, \end{aligned} \quad (3.6)$$

where we've written a "?" since you probably don't know what to do with $\lim_{x \rightarrow 0} \cot x$. Since the first limit is 0, do we really need to worry about the second limit? The problem here is that we are attempting to use a rule in a case where it does not apply. Specifically, Theorem 3.1 says that the limit of a product is the product of the respective limits *when all of the limits exist*. So, what can be said of $\lim_{x \rightarrow 0} \cot x$? Consider the graph of the function found in Figure 1.17. The graph suggests that $\lim_{x \rightarrow 0} \cot x$ does not exist. You should compute some function values, as well, to convince yourself that this is in fact the case. Equation (3.6) then does not hold and we're back to square one. Since none of our rules seem to apply here, the most reasonable step is to draw a graph and compute some function values in an effort to see what is going on. The graph of $y = x \cot x$ appears in Figure 1.18. If we compute some function values (see the table in the margin), we observe that the function values appear to be approaching 1, as $x \rightarrow 0$.

Based on the graph and the table of function values, we make the conjecture

$$\lim_{x \rightarrow 0} (x \cot x) = 1.$$

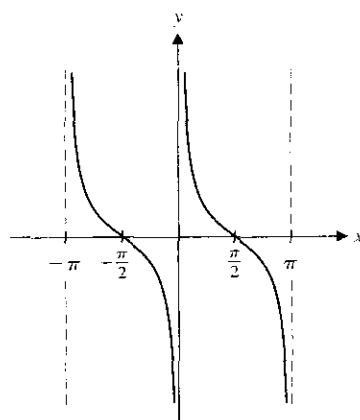


FIGURE 1.17

$y = \cot x$.

x	$x \cot x$
± 0.1	0.9967
± 0.01	0.999967
± 0.001	0.9999967
± 0.0001	0.99999967
± 0.00001	0.999999967

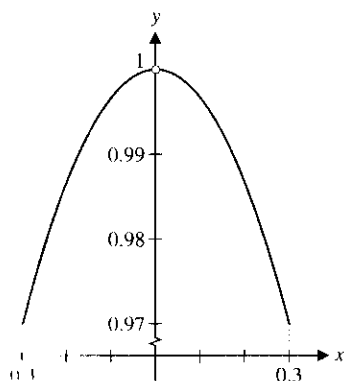


FIGURE 1.18
 $y = x \cot x$.

which is definitely not 0, as you might have initially suspected. You can also think about this limit as follows:

$$\begin{aligned}\lim_{x \rightarrow 0} (x \cot x) &= \lim_{x \rightarrow 0} \left(x \frac{\cos x}{\sin x} \right) = \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \cos x \right) \\ &= \left(\lim_{x \rightarrow 0} \frac{x}{\sin x} \right) \left(\lim_{x \rightarrow 0} \cos x \right) \\ &= \frac{\lim_{x \rightarrow 0} \cos x}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} = \frac{1}{1} = 1,\end{aligned}$$

since $\lim_{x \rightarrow 0} \cos x = 1$ and where we have used the conjecture we made in example 2.4 that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. (It is possible to verify this last conjecture, using the Squeeze Theorem, which follows.) ■

At this point, we introduce a tool that will help us determine a number of important limits. Notice that we said *tool* and not *rule*. You should recognize that most mathematical problems are solved by careful analysis and not by simply applying a series of rules.

THEOREM 3.5 (Squeeze Theorem)

Suppose that

$$f(x) \leq g(x) \leq h(x)$$

for all x in some interval (c, d) , except possibly at the point $a \in (c, d)$ and that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L,$$

for some number L . Then, it follows that

$$\lim_{x \rightarrow a} g(x) = L, \text{ also.}$$

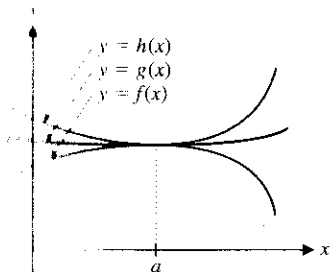


FIGURE 1.19
The Squeeze Theorem.

REMARK 3.1

The Squeeze Theorem also applies to one-sided limits.

Referring to Figure 1.19, you should clearly see that if $g(x)$ lies between $f(x)$ and $h(x)$, except possibly at a itself and both $f(x)$ and $h(x)$ have the same limit as $x \rightarrow a$, then $g(x)$ gets *squeezed* between $f(x)$ and $h(x)$ and therefore should also have a limit of L . The challenge in using the Squeeze Theorem is in finding appropriate functions f and h that bound a given function g from below and above, respectively, and that have the same limit as $x \rightarrow a$.

EXAMPLE 3.9 Using the Squeeze Theorem to Verify the Value of a Limit

Determine the value of $\lim_{x \rightarrow 0} \left[x^2 \cos \left(\frac{1}{x} \right) \right]$.

Solution Your first reaction might be to observe that this is a limit of a product and so, might be the product of the limits:

$$\lim_{x \rightarrow 0} \left[x^2 \cos \left(\frac{1}{x} \right) \right] \stackrel{?}{=} \left(\lim_{x \rightarrow 0} x^2 \right) \left[\lim_{x \rightarrow 0} \cos \left(\frac{1}{x} \right) \right], \quad \text{This is incorrect!} \quad (3.7)$$

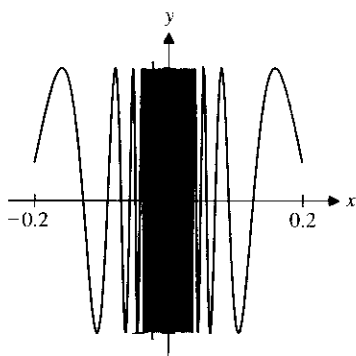


FIGURE 1.20
 $y = \cos\left(\frac{1}{x}\right)$.

where we've written a "?" since you probably don't know what to do with $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$. Consider the graph of $y = \cos\left(\frac{1}{x}\right)$ found in Figure 1.20. The graph suggests that $\cos\left(\frac{1}{x}\right)$ oscillates back and forth between -1 and 1 . Further, the closer x gets to 0 , the more rapid the oscillations become. You should compute some function values, as well, to convince yourself that $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$ does not exist. Equation (3.7) then does not hold. Since none of our rules seem to apply here, we draw a graph and compute some function values to see what is going on. The graph of $y = x^2 \cos\left(\frac{1}{x}\right)$ appears in Figure 1.21a. In the accompanying table of function values, we observe the same oscillation visible in the graph.

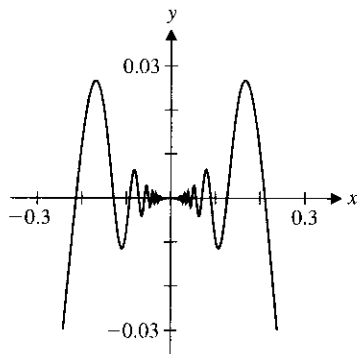


FIGURE 1.21a
 $y = x^2 \cos\left(\frac{1}{x}\right)$.

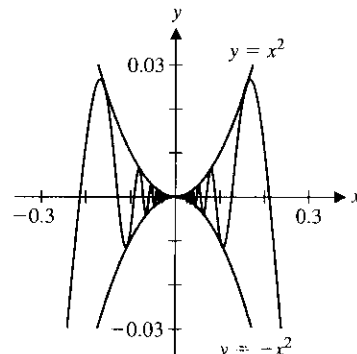


FIGURE 1.21b
 $y = x^2 \cos\left(\frac{1}{x}\right)$, $y = x^2$ and
 $y = -x^2$.

x	$x^2 \cos(1/x)$
± 0.1	-0.008
± 0.01	8.6×10^{-5}
± 0.001	5.6×10^{-7}
± 0.0001	-9.5×10^{-9}
± 0.00001	-9.99×10^{-11}

Despite the oscillation seen in the graph and in the table of function values, we can make the conjecture

$$\lim_{x \rightarrow 0} \left[x^2 \cos\left(\frac{1}{x}\right) \right] = 0.$$

We can prove that this conjecture is correct using the Squeeze Theorem, but we will first need to find functions f and h such that

$$f(x) \leq x^2 \cos\left(\frac{1}{x}\right) \leq h(x),$$

for all $x \neq 0$ and where $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$. How can you be expected to do that?

One difficulty is that there's actually very little that you know about $\cos\left(\frac{1}{x}\right)$. So, ask yourself what you **do** know. Probably the only inequality you can recall concerning this function is

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1, \quad (3.8)$$

for all $x \neq 0$. If we multiply (3.8) through by x^2 (notice that since $x^2 \geq 0$, this multiplication preserves the inequalities), we get

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2,$$

for all $x \neq 0$. We illustrate this inequality in Figure 1.21b. Further,

$$\lim_{x \rightarrow 0} (-x^2) = 0 = \lim_{x \rightarrow 0} x^2.$$

So, from the Squeeze Theorem, it now follows that

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0,$$

also, as we had conjectured. ■

REMARK 3.2

To resolve the limit in example 3.9, we could not apply the rules for limits contained in Theorem 3.1. So, we resorted to an indirect method of finding the limit. This four-pronged force of graphics plus calculation followed by analysis is sometimes referred to as the **Rule of Three**. (The Rule of Three presents a general strategy for attacking new problems. The basic idea is to look at problems graphically, numerically and analytically.) In the case of example 3.9, the first two elements of this “rule” (the graphics in Figure 1.21 and the following table of function values) suggest a plausible conjecture, while the third element provides us with a careful mathematical verification of the conjecture.

Functions are often defined by different expressions on different intervals. Such **piecewise-defined** functions are important and we illustrate such a function in example 3.10.

EXAMPLE 3.10 A Limit for a Piecewise-Defined Function

Evaluate $\lim_{x \rightarrow 0} f(x)$, where f is defined by

$$f(x) = \begin{cases} x^2 + 2 \cos x + 1, & \text{for } x < 0 \\ e^x - 4, & \text{for } x \geq 0 \end{cases}$$

Solution Since f is defined by different expressions for $x < 0$ and for $x \geq 0$, we must consider one-sided limits. We have

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x^2 + 2 \cos x + 1) = 2 \cos 0 + 1 = 3,$$

by Theorem 3.4. Alternatively, we have

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (e^x - 4) = e^0 - 4 = 1 - 4 = -3.$$

Since the one-sided limits are different, we have that $\lim_{x \rightarrow 0} f(x)$ does not exist. ■

We end the section with an example of the use of limits in computing velocity. In section 2.1, we see that for an object moving in a straight line, whose position at time t is given by the function $f(t)$, the instantaneous velocity of that object at time $t = a$ (i.e., the velocity at the instant $t = a$, as opposed to the average velocity over some period of time) is given by the limit

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

EXAMPLE 3.11 Evaluating a Limit Describing Velocity

Suppose that the position function for an object is given by

$$f(t) = t^2 + 2 \text{ (feet)},$$

where t is measured in seconds. Find the instantaneous velocity of the object at time $t = 1$.

Solution Given what we have just learned about limits, this is now an easy problem to solve. We have

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[(1+h)^2 + 2] - 3}{h}.$$

While we can't simply substitute $h = 0$ (Why not?), we can write

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{[(1+h)^2 + 2] - 3}{h} &= \lim_{h \rightarrow 0} \frac{(1+2h+h^2) - 1}{h} && \text{Expanding the squared term.} \\ &= \lim_{h \rightarrow 0} \frac{2h+h^2}{h} = \lim_{h \rightarrow 0} \frac{h(2+h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2+h}{1} = 2. && \text{Canceling factors of } h.\end{aligned}$$

So, the instantaneous velocity of this object at time $t = 1$ is 2 feet per second. ■

EXERCISES 1.3

WRITING EXERCISES

- Given your knowledge of the graphs of polynomials, explain why equations (3.1) and (3.2) and Theorem 3.2 are obvious. Name five nonpolynomial functions for which limits can be evaluated by substitution.
- Suppose that you can draw the graph of $y = f(x)$ without lifting your pencil from your paper. Explain why $\lim_{x \rightarrow a} f(x) = f(a)$, for every value of a .
- In one or two sentences, explain the Squeeze Theorem. Use a real-world analogy (e.g., having the functions represent the locations of three people as they walk) to indicate why it is true.
- Given the graph in Figure 1.21 and the calculations that follow, it may be unclear why we insist on using the Squeeze Theorem before concluding that $\lim_{x \rightarrow 0} [x^2 \cos(1/x)]$ is indeed 0. Review section 1.2 to explain why we are being so fussy.

In exercises 1–34, evaluate the indicated limit, if it exists. Assume that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

- $\lim_{x \rightarrow 0} (x^2 - 3x + 1)$
- $\lim_{x \rightarrow 0} \cos(x^2 + \pi)$
- $\lim_{x \rightarrow 1} \sqrt{x^2 + 2x + 4}$
- $\lim_{x \rightarrow 2} \sqrt[3]{2x + 1}$
- $\lim_{x \rightarrow 2} \frac{x-5}{x^2 + 4}$
- $\lim_{x \rightarrow 0} \cos^{-1}(x^2)$
- $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$
- $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}$
- $\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 4}$
- $\lim_{x \rightarrow 0} \frac{x^2 + x}{x^2 - 3x}$
- $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 + 2x - 3}$
- $\lim_{x \rightarrow 3} \frac{x - 1}{x^2 + 2x - 3}$
- $\lim_{x \rightarrow 0} \frac{\sin x}{\tan x}$
- $\lim_{x \rightarrow 0} \frac{\tan x}{x}$
- $\lim_{x \rightarrow 0} \frac{xe^{-2x+1}}{x^2 + x}$
- $\lim_{x \rightarrow 0^+} x^2 \csc^2 x$

- $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x}$
- $\lim_{x \rightarrow 0} \frac{2x}{3 - \sqrt{x+9}}$
- $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1}$
- $\lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4}$
- $\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right)$
- $\lim_{x \rightarrow 0} \left(\frac{2}{x} - \frac{2}{|x|} \right)$
- $\lim_{x \rightarrow 2} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$
- $\lim_{x \rightarrow -1} f(x)$, where $f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \geq -1 \end{cases}$
- $\lim_{x \rightarrow 0} f(x)$, where $f(x) = \begin{cases} x^2 + 2 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \geq -1 \end{cases}$
- $\lim_{x \rightarrow 1} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$
- $\lim_{x \rightarrow -1} f(x)$, where $f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$
- $\lim_{x \rightarrow 1} f(x)$, where $f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$
- $\lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h}$
- $\lim_{h \rightarrow 0} \frac{(1+h)^3 - 1}{h}$
- $\lim_{h \rightarrow 0} \frac{h^2}{\sqrt{h^2 + h + 3} - \sqrt{h + 3}}$
- $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + x + 4} - 2}{x^2 + x}$
- $\lim_{t \rightarrow -2} \frac{\frac{1}{2} + \frac{1}{t}}{2 + t}$
- $\lim_{x \rightarrow 0} \frac{\tan 2x}{5x}$
- Use numerical and graphical evidence to conjecture the value of $\lim_{x \rightarrow 0} x^2 \sin(1/x)$. Use the Squeeze Theorem to prove that you are correct: identify the functions $f(x)$ and $h(x)$, show graphically that $f(x) \leq x^2 \sin(1/x) \leq h(x)$, and justify $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x)$.
- Why can't you use the Squeeze Theorem as in exercise 35 to prove that $\lim_{x \rightarrow 0} x^2 \sec(1/x) = 0$? Explore this limit graphically.

37. Use the Squeeze Theorem to prove that $\lim_{x \rightarrow 0^+} [\sqrt{x} \cos^2(1/x)] = 0$. Identify the functions $f(x)$ and $h(x)$, show graphically that $f(x) \leq \sqrt{x} \cos^2(1/x) \leq h(x)$ for all $x > 0$, and justify $\lim_{x \rightarrow 0^+} f(x) = 0$ and $\lim_{x \rightarrow 0^+} h(x) = 0$.
38. Suppose that $f(x)$ is bounded: that is, there exist constants m and n such that $m \leq f(x) \leq n$ for all x . Use the Squeeze Theorem to prove that $\lim_{x \rightarrow 0} x^2 f(x) = 0$.

In exercises 39–48, either find the limit or explain why it does not exist.

39. $\lim_{x \rightarrow 4^+} \sqrt{16 - x^2}$ 40. $\lim_{x \rightarrow 4^-} \sqrt{16 - x^2}$
41. $\lim_{x \rightarrow -2^+} \sqrt{x^2 + 3x + 2}$ 42. $\lim_{x \rightarrow -2^-} \sqrt{x^2 + 3x + 2}$
43. Given that $\lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2} = \frac{1}{2}$, quickly evaluate $\lim_{x \rightarrow 0^+} \frac{\sqrt{1 - \cos x}}{x}$.
44. Given that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, quickly evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2}$.
45. Suppose $f(x) = \begin{cases} g(x) & \text{if } x < a \\ h(x) & \text{if } x > a \end{cases}$ for polynomials $g(x)$ and $h(x)$. Explain why $\lim_{x \rightarrow a} f(x) = g(a)$ and determine $\lim_{x \rightarrow a^+} f(x)$.
46. Explain how to determine $\lim_{x \rightarrow a} f(x)$ if g and h are polynomials and $f(x) = \begin{cases} g(x) & \text{if } x < a \\ c & \text{if } x = a \\ h(x) & \text{if } x > a \end{cases}$.
47. Evaluate each limit and justify each step by citing the appropriate theorem or equation.
- (a) $\lim_{x \rightarrow 2} (x^2 - 3x + 1)$ (b) $\lim_{x \rightarrow 0} \frac{x - 2}{x^2 + 1}$
48. Evaluate each limit and justify each step by citing the appropriate theorem or equation.
- (a) $\lim_{x \rightarrow -1} [(x + 1) \sin x]$ (b) $\lim_{x \rightarrow 1} \frac{xe^x}{\tan x}$

In exercises 49–52, use the given position function $f(t)$ to find the velocity at time $t = a$.

49. $f(t) = t^2 + 2, a = 2$ 50. $f(t) = t^2 + 2, a = 0$
51. $f(t) = t^3, a = 0$ 52. $f(t) = t^3, a = 1$
53. In Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is defined by $m = \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h}$. Compute the slope m . Graph $y = \sqrt{x}$ and the line with slope m through the point $(1, 1)$.
54. In Chapter 2, an alternative form for the limit in exercise 53 is given by $\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1}$. Compute this limit.

T In exercises 55–62, use numerical evidence to conjecture the value of the limit if it exists. Check your answer with your CAS. If you disagree, which one of you is correct?

55. $\lim_{x \rightarrow 0^+} (1 + x)^{1/x}$ 56. $\lim_{x \rightarrow 0^+} e^{1/x}$ 57. $\lim_{x \rightarrow 0^+} x^{x^2}$
58. $\lim_{x \rightarrow 0^+} x^{\ln x}$ 59. $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ 60. $\lim_{x \rightarrow 0} e^{1/x}$
61. $\lim_{x \rightarrow 0} \tan^{-1} \frac{1}{x}$ 62. $\lim_{x \rightarrow 0} \ln \left| \frac{1}{x} \right|$
63. Assume that $\lim_{x \rightarrow a} f(x) = L$. Use Theorem 3.1 to prove that $\lim_{x \rightarrow a} [f(x)]^3 = L^3$. Also, show that $\lim_{x \rightarrow a} [f(x)]^4 = L^4$.
64. How did you work exercise 63? You probably used Theorem 3.1 to work from $\lim_{x \rightarrow a} [f(x)]^2 = L^2$ to $\lim_{x \rightarrow a} [f(x)]^3 = L^3$, and then used $\lim_{x \rightarrow a} [f(x)]^3 = L^3$ to get $\lim_{x \rightarrow a} [f(x)]^4 = L^4$. Going one step at a time, we should be able to reach $\lim_{x \rightarrow a} [f(x)]^n = L^n$ for any positive integer n . This is the idea of **mathematical induction**. Formally, we need to show the result is true for a specific value of $n = n_0$ [Equation (3.3) does that for $n_0 = 2$], then assume the result is true for a general $n = k \geq n_0$. If we show that we can get from the result being true for $n = k$ to the result being true for $n = k + 1$, we have proved that the result is true for any positive integer n . In one sentence, explain why this is true. Use this technique to prove that $\lim_{x \rightarrow a} [f(x)]^n = L^n$ for any positive integer n .
65. Find all the errors in the following incorrect string of equalities:
- $$\lim_{x \rightarrow 0} \frac{1}{x} = \lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} x \lim_{x \rightarrow 0} \frac{1}{x^2} = 0 \cdot ? = 0.$$
66. Find all the errors in the following incorrect string of equalities:
- $$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \frac{0}{0} = 1.$$
67. Give an example of functions $f(x)$ and $g(x)$ such that $\lim_{x \rightarrow 0} [f(x) + g(x)]$ exists but $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist.
68. Give an example of functions $f(x)$ and $g(x)$ such that $\lim_{x \rightarrow 0} [f(x) \cdot g(x)]$ exists but at least one of $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ does not exist.
69. If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} g(x)$ does not exist, is it always true that $\lim_{x \rightarrow a} [f(x) + g(x)]$ does not exist? Explain.
70. Is the following true or false? If $\lim_{x \rightarrow 0} f(x)$ does not exist, then $\lim_{x \rightarrow 0} \frac{1}{f(x)}$ does not exist. Explain.
71. Suppose a state's income tax code states the tax liability on x dollars of taxable income is given by
- $$T(x) = \begin{cases} 0.14x & \text{if } 0 \leq x < 10,000 \\ 1500 + 0.21x & \text{if } 10,000 \leq x \end{cases}$$
- Compute $\lim_{x \rightarrow 0^+} T(x)$; why is this good? Compute $\lim_{x \rightarrow 10,000} T(x)$; why is this bad?

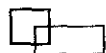
72. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function $T(x) = \begin{cases} a + 0.12x & \text{if } x \leq 20,000 \\ b + 0.16(x - 20,000) & \text{if } x > 20,000 \end{cases}$ such that $\lim_{x \rightarrow 0^+} T(x) = 0$ and $\lim_{x \rightarrow 20,000} T(x)$ exists. Why is it important for these limits to exist?
73. The **greatest integer function** is denoted by $f(x) = [x]$ and equals the greatest integer which is less than or equal to x . Thus, $[2.3] = 2$, $[-1.2] = -2$ and $[3] = 3$. In spite of this last fact, show that $\lim_{x \rightarrow 3} [x]$ does not exist.
74. Investigate the existence of (a) $\lim_{x \rightarrow 1} [x]$, (b) $\lim_{x \rightarrow 1.5} [x]$, (c) $\lim_{x \rightarrow 1.5} [2x]$, (d) $\lim_{x \rightarrow 1} (x - [x])$.

EXPLORATORY EXERCISES

1. The value $x = 0$ is called a **zero of multiplicity n** ($n \geq 1$) for the function $f(x)$ if $\lim_{x \rightarrow 0} \frac{f(x)}{x^n}$ exists and is nonzero but

$\lim_{x \rightarrow 0} \frac{f(x)}{x^{n-1}} = 0$. Show that $x = 0$ is a zero of multiplicity 2 for x^2 , $x = 0$ is a zero of multiplicity 3 for x^3 and $x = 0$ is a zero of multiplicity 4 for x^4 . For polynomials, what does multiplicity describe? The reason the definition is not as straightforward as we might like is so that it can apply to non-polynomial functions, as well. Find the multiplicity of $x = 0$ for $f(x) = \sin x$; $f(x) = x \sin x$; $f(x) = \sin x^2$. If you know that $x = 0$ is a zero of multiplicity m for $f(x)$ and multiplicity n for $g(x)$, what can you say about the multiplicity of $x = 0$ for $f(x) + g(x)$? $f(x) \cdot g(x)$? $f(g(x))$?

2. We have conjectured that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Using graphical and numerical evidence, conjecture the value of $\lim_{x \rightarrow 0} \frac{\sin 2x}{x}$, $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$, $\lim_{x \rightarrow 0} \frac{\sin \pi x}{x}$ and $\lim_{x \rightarrow 0} \frac{\sin x/2}{x}$. In general, conjecture the value of $\lim_{x \rightarrow 0} \frac{\sin cx}{x}$ for any constant c . Given that $\lim_{x \rightarrow 0} \frac{\sin cx}{cx} = 1$ for any constant $c \neq 0$, prove that your conjecture is correct.



1.4 CONTINUITY AND ITS CONSEQUENCES

When you describe something as *continuous*, just what do you have in mind? For example, if you say that a machine has been in *continuous* operation for the past 60 hours, what is it that you are trying to communicate? Most of us would interpret this to mean that the machine has been in operation *all* of that time, without any interruption at all, even for a moment. Mathematicians mean very much the same thing when we say that a function is continuous. A function is said to be **continuous** on an interval if its graph on that interval can be drawn without interruption, that is, without lifting your pencil from the paper.

It is helpful for us to first try to see what it is about the functions whose graphs are shown in Figures 1.22–1.25 that makes them **discontinuous** (i.e., not continuous) at the point $x = a$.

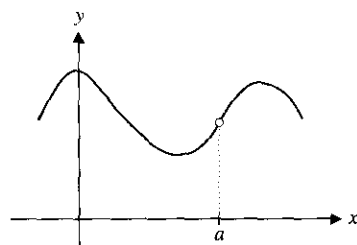


FIGURE 1.22
 $f(a)$ is not defined (the graph has a hole at $x = a$).

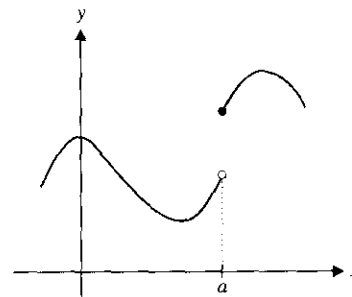


FIGURE 1.23
 $f(a)$ is defined, but $\lim_{x \rightarrow a} f(x)$ does not exist (the graph has a jump at $x = a$).

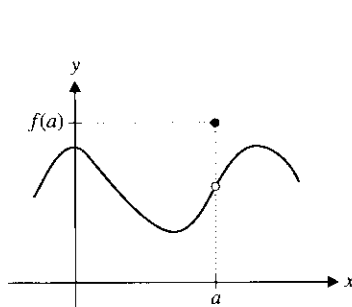


FIGURE 1.24

$\lim_{x \rightarrow a} f(x)$ exists and $f(a)$ is defined, but $\lim_{x \rightarrow a} f(x) \neq f(a)$ (the graph has a hole at $x = a$).

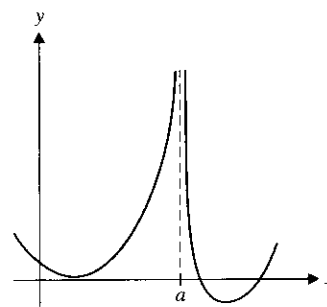


FIGURE 1.25

$\lim_{x \rightarrow a} f(x)$ does not exist (the function “blows up” at $x = a$).

NOTES

The definition of continuity all boils down to the one condition in (iii), since conditions (i) and (ii) must hold whenever (iii) is met.

Further, this says that a function is continuous at a point exactly when you can compute its limit at that point by simply substituting in.

This suggests the following definition of continuity at a point.

DEFINITION 4.1

A function f is *continuous* at $x = a$ when

- (i) $f(a)$ is defined, (ii) $\lim_{x \rightarrow a} f(x)$ exists and (iii) $\lim_{x \rightarrow a} f(x) = f(a)$.

Otherwise, f is said to be *discontinuous* at $x = a$.

REMARK 4.1

For most purposes, it is best for you to think of the intuitive notion of continuity that we've outlined above. Definition 4.1 should then simply follow from your intuitive understanding of the concept.

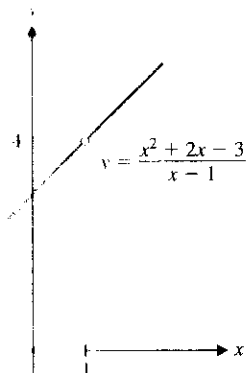


FIGURE 1.26

$$y = \frac{x^2 + 2x - 3}{x - 1}$$

REMARK 4.2

You should be careful not to confuse the continuity of a function at a point with its simply being defined there. A function can be defined at a point without being continuous there. (Look back at Figures 1.23 and 1.24.)

EXAMPLE 4.1 Finding Where a Rational Function Is Continuous

Determine where $f(x) = \frac{x^2 + 2x - 3}{x - 1}$ is continuous.

Solution Note that

$$f(x) = \frac{x^2 + 2x - 3}{x - 1} = \frac{(x - 1)(x + 3)}{(x - 1)} \quad \text{Factoring the numerator.}$$

$$= x + 3, \text{ for } x \neq 1. \quad \text{Canceling common factors.}$$

This says that the graph of f is a straight line, but with a hole in it at $x = 1$, as indicated in Figure 1.26. So, f is continuous for $x < 1$ and $x > 1$ (but not at $x = 1$). We can also say that f is continuous for x in $(-\infty, 1)$ and $(1, \infty)$. (Except at $x = 1$, the graph is a straight line and so, can be drawn without lifting pencil from paper.) ■

EXAMPLE 4.2 Removing a Discontinuity

Make the function from example 4.1 continuous everywhere by redefining it at a single point.

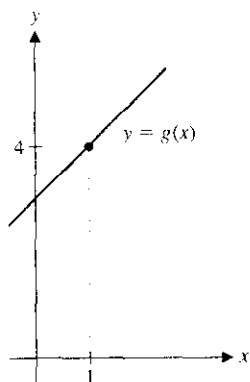


FIGURE 1.27
 $y = g(x)$.

Solution In example 4.1, the function was discontinuous at $x = 1$ since it was undefined there. So, suppose we just go ahead and define it, as follows. Let

$$g(x) = \begin{cases} \frac{x^2 + 2x - 3}{x - 1}, & \text{if } x \neq 1 \\ a, & \text{if } x = 1, \end{cases}$$

for some real number a .

Notice that $g(x)$ is defined for all x and equals $f(x)$ for all $x \neq 1$. Here, we have

$$\begin{aligned} \lim_{x \rightarrow 1} g(x) &= \lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 3) = 4. \end{aligned}$$

Observe that if we choose $a = 4$, we now have that

$$\lim_{x \rightarrow 1} g(x) = 4 = g(1)$$

and so, g is continuous at $x = 1$.

Note that the graph of g is the same as the graph of f seen in Figure 1.26, except that we now include the point $(1, 4)$ (see Figure 1.27). Also, note that there's actually a very simple way to write $g(x)$. (Think about this.) ■

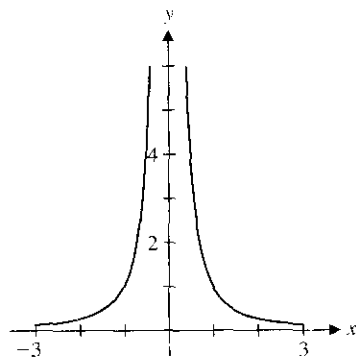


FIGURE 1.28a
 $y = \frac{1}{x^2}$.

When we can remove a discontinuity by redefining the function at that point, we call the discontinuity **removable**. Not all discontinuities are removable, however. Carefully examine Figures 1.22–1.25 and convince yourself that the discontinuities in Figures 1.22 and 1.24 are removable, while those in Figures 1.23 and 1.25 are nonremovable. Briefly, a function f has a removable discontinuity at $x = a$ if $\lim_{x \rightarrow a} f(x)$ exists and either $f(a)$ is undefined or $\lim_{x \rightarrow a} f(x) \neq f(a)$.

EXAMPLE 4.3 Nonremovable Discontinuities

Determine if there are any nonremovable discontinuities of $f(x) = \frac{1}{x^2}$ and $g(x) = \cos\left(\frac{1}{x}\right)$.

Solution You should observe from Figure 1.28a (also, construct a table of function values) that

$$\lim_{x \rightarrow 0} \frac{1}{x^2} \text{ does not exist.}$$

Hence, f is discontinuous at $x = 0$.

Similarly, observe that $\lim_{x \rightarrow 0} \cos(1/x)$ does not exist, due to the endless oscillation of $\cos(1/x)$ as x approaches 0 (see Figure 1.28b and example 3.9).

In both cases, notice that since the limits do not exist, there is no way to redefine either function at $x = 0$ to make it continuous there. ■

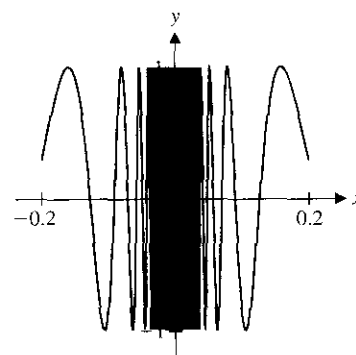


FIGURE 1.28b
 $y = \cos(1/x)$.

From your experience with the graphs of some common functions, the result in Theorem 4.1 should come as no surprise.

THEOREM 4.1

All polynomials are continuous everywhere. Additionally, $\sin x$, $\cos x$, $\tan^{-1} x$ and e^x are continuous everywhere, $\ln x$ is continuous for $x > 0$ and $\sin^{-1} x$ and $\cos^{-1} x$ are continuous for $-1 < x < 1$.

PROOF

We have already established (in Theorem 3.2) that for any polynomial $p(x)$ and any real number a ,

$$\lim_{x \rightarrow a} p(x) = p(a),$$

from which it follows that p is continuous at $x = a$. The rest of the theorem follows from Theorem 3.4 in a similar way. ■

From these very basic continuous functions, we can build a large collection of continuous functions, using the results in Theorem 4.2.

THEOREM 4.2

Suppose that f and g are continuous at $x = a$. Then all of the following are true:

- (i) $(f \pm g)$ is continuous at $x = a$,
- (ii) $(f \cdot g)$ is continuous at $x = a$ and
- (iii) (f/g) is continuous at $x = a$ if $g(a) \neq 0$ and discontinuous at $x = a$ if $g(a) = 0$.

Simply put, Theorem 4.2 says that a sum, difference or product of continuous functions is continuous. Further, a quotient of two continuous functions is continuous at any point at which the denominator is nonzero.

PROOF

(i) If f and g are continuous at $x = a$, then

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) \pm g(x)] &= \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) && \text{From Theorem 3.1.} \\ &= f(a) \pm g(a) && \text{Since } f \text{ and } g \text{ are continuous at } a, \\ &= (f \pm g)(a), \end{aligned}$$

by the usual rules of limits. Thus, $(f \pm g)$ is also continuous at $x = a$.

Parts (ii) and (iii) are proved in a similar way and are left as exercises. ■

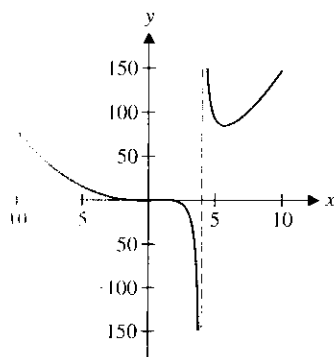


FIGURE 1.29

$$f(x) = \frac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$$

EXAMPLE 4.4 Continuity for a Rational Function

Determine where f is continuous, for $f(x) = \frac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$.

Solution Here, f is a quotient of two polynomial (hence continuous) functions. The graph of the function indicated in Figure 1.29 indicates a vertical asymptote at around $x = 4$, but doesn't indicate any other discontinuity. From Theorem 4.2, f will be continuous at all x where the denominator

$$x^2 - 3x - 4 = (x + 1)(x - 4) \neq 0.$$

Thus, f is continuous for $x \neq -1, 4$. (Think about why you didn't see anything peculiar about the graph at $x = -1$.) ■

With the addition of the result in Theorem 4.3, we will have all the basic tools needed to establish the continuity of most elementary functions.

THEOREM 4.3

Suppose that $\lim_{x \rightarrow a} g(x) = L$ and f is continuous at L . Then,

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(L).$$

A proof of Theorem 4.3 is given in Appendix F.

Notice that this says that if f is continuous then we can bring the limit “inside.” This should make sense, since as $x \rightarrow a$, $g(x) \rightarrow L$ and so, $f(g(x)) \rightarrow f(L)$, since f is continuous at L . The next result follows directly from Theorem 4.3. (Such a result, which follows immediately from another result, is called a **corollary**.)

COROLLARY 4.1

Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a .

We leave the proof of Corollary 4.1 as an exercise.

EXAMPLE 4.5 Continuity for a Composite Function

Determine where $h(x) = \cos(x^2 - 5x + 2)$ is continuous.

Solution Note that

$$h(x) = f(g(x)),$$

where $g(x) = x^2 - 5x + 2$ and $f(x) = \cos x$. Since both f and g are continuous for all x , h is continuous for all x , by Corollary 4.1. ■

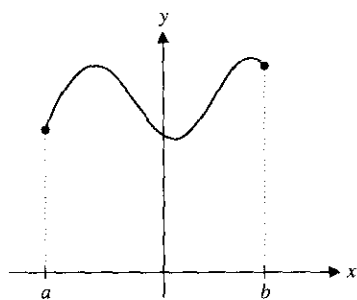


FIGURE 1.30
 f continuous on $[a, b]$.

DEFINITION 4.2

If f is continuous at every point on an open interval (a, b) , we say that f is **continuous on (a, b)** . Following Figure 1.30, we say that f is **continuous on the closed interval $[a, b]$** , if f is continuous on the open interval (a, b) and

$$\lim_{x \rightarrow a^+} f(x) = f(a) \text{ and } \lim_{x \rightarrow b^-} f(x) = f(b).$$

Finally, if f is continuous on all of $(-\infty, \infty)$, we simply say that f is **continuous** (i.e., when we don't specify an interval, we mean continuous everywhere).

For many functions, it's a simple matter to determine the intervals on which the function is continuous. We illustrate this in example 4.6.

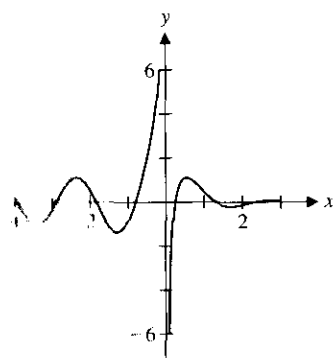


FIGURE 1.31

$$f(x) = \frac{\sin(3x - 7)}{e^x - 1}$$

EXAMPLE 4.6 Intervals Where a Function Is Continuous

Determine the interval(s) where f is continuous, for $f(x) = \frac{\sin(3x - 7)}{e^x - 1}$.

Solution Note that f is the quotient of two continuous functions and hence, is continuous for all x for which the denominator is nonzero. We show a graph of the function in Figure 1.31. Since

$$e^x - 1 = 0$$

only for $x = 0$, we have that f is continuous for all $x \neq 0$. Thus, f is continuous on both of the intervals $(-\infty, 0)$ and $(0, \infty)$. ■

EXAMPLE 4.7 Interval of Continuity for a Logarithm

Determine the interval(s) where $f(x) = \ln(x - 3)$ is continuous.

Solution It follows from Theorem 4.1 and Corollary 4.1 that f is continuous whenever $(x - 3) > 0$ (i.e., $x > 3$). Thus, f is continuous on the interval $(3, \infty)$. ■

The Internal Revenue Service presides over some of the most despised functions in existence. Look up the current Tax Rate Schedules. In 2002, the first few lines (for single taxpayers) looked like:

<i>For taxable income over</i>	<i>but not over</i>	<i>your tax liability is</i>	<i>minus</i>
\$0	6,000	10%	\$0
\$6,000	27,950	15%	\$300
\$27,950	67,700	27%	\$3654

Where do the numbers \$300 and \$3654 come from? If we write the tax liability $T(x)$ as a function of the taxable income x (assuming that x can be any real value and not just a whole dollar amount), we have

$$T(x) = \begin{cases} 0.10x & \text{if } 0 < x \leq 6000 \\ 0.15x - 300 & \text{if } 6000 < x \leq 27,950 \\ 0.27x - 3654 & \text{if } 27,950 < x \leq 67,700. \end{cases}$$

Be sure you understand our translation so far. Note that it is important that this be a continuous function: think of the fairness issues that would arise if it were not!

EXAMPLE 4.8 Continuity of Federal Tax Tables

Verify that the federal tax rate function $T(x)$ is continuous at the “joint” $x = 27,950$. Then, find a to complete the table. (You will find b and c as exercises.)

<i>For taxable income over</i>	<i>but not over</i>	<i>your tax liability is</i>	<i>minus</i>
\$67,700	\$141,250	30%	a
\$141,250	\$307,050	35%	b
\$307,050	—	38.6%	c

Solution For $T(x)$ to be continuous at $x = 27,950$, we must have

$$\lim_{x \rightarrow 27,950^-} T(x) = \lim_{x \rightarrow 27,950^+} T(x).$$

Since both functions $0.15x - 300$ and $0.27x - 3654$ are continuous, we can compute the one-sided limits by substituting $x = 27,950$. Thus,

$$\lim_{x \rightarrow 27,950^-} T(x) = 0.15(27,950) - 300 = 3892.50$$

and

$$\lim_{x \rightarrow 27,950^+} T(x) = 0.27(27,950) - 3654 = 3892.50.$$

Since the one-sided limits agree and equal the value of the function at that point, $T(x)$ is continuous at $x = 27,950$. We leave it as an exercise to establish that $T(x)$ is also continuous at $x = 6000$. (It's worth noting that the function could be written with equal signs on all of the inequalities; this would be incorrect if the function was discontinuous.) To complete the table, we choose a to get the one-sided limits at $x = 67,700$ to match. We have

$$\lim_{x \rightarrow 67,700^-} T(x) = 0.27(67,700) - 3654 = 14,625,$$

while

$$\lim_{x \rightarrow 67,700^+} T(x) = 0.30(67,700) - a = 20,310 - a.$$

So, we set the one-sided limits equal, to obtain

$$14,625 = 20,310 - a$$

or

$$a = 20,310 - 14,625 = 5685. \blacksquare$$



HISTORICAL NOTES

Karl Weierstrass (1815–1897)

A German mathematician who proved the Intermediate Value Theorem and several other fundamental results of the calculus. Weierstrass was known as an excellent teacher whose students circulated his lecture notes throughout Europe, because of their clarity and originality. Also known as a superb fencer, Weierstrass was one of the founders of modern mathematical analysis.

The following theorem should seem an obvious consequence of our intuitive definition of continuity.

THEOREM 4.4 (Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$ and W is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a, b]$ for which $f(c) = W$.

REMARK 4.3

Theorem 4.4 says that if f is continuous on $[a, b]$, then f must take on *every* value between $f(a)$ and $f(b)$ at least once. That is, a continuous function cannot skip over any numbers between its values at the two endpoints. To do so, the graph would need to leap across the horizontal line $y = W$, something that continuous functions cannot do (see Figure 1.32a). Of course, a function may take on a given value W more than once (see Figure 1.32b).

In Corollary 4.2, we see an immediate and useful application of the Intermediate Value Theorem.

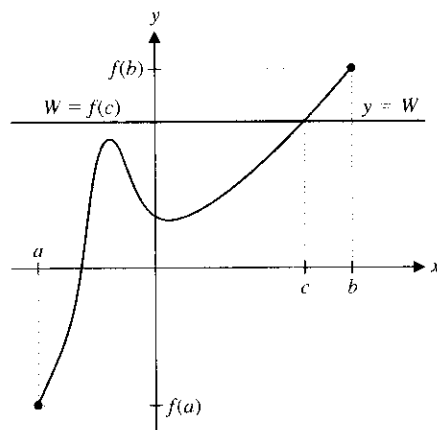


FIGURE 1.32a

An illustration of the Intermediate Value Theorem.

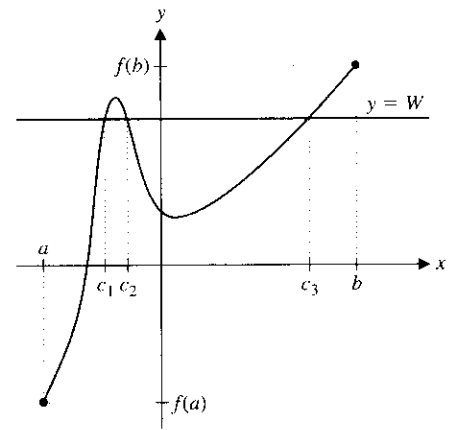


FIGURE 1.32b

More than one value of c .

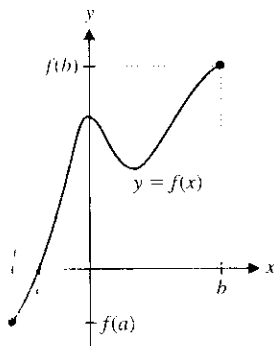


FIGURE 1.33

Intermediate Value Theorem where c is a zero of f .

COROLLARY 4.2

Suppose that f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs [i.e., $f(a) \cdot f(b) < 0$]. Then, there is at least one number $c \in (a, b)$ for which $f(c) = 0$. (Recall that c is then a zero of f .)

Notice that the corollary is simply the special case of the Intermediate Value Theorem where $W = 0$ (see Figure 1.33). The Intermediate Value Theorem and Corollary 4.2 are examples of **existence theorems**; they tell you that there *exists* a number c satisfying some condition, but they do *not* tell you what c is.

In example 4.9, we see how Corollary 4.2 can help us locate the zeros of a function.

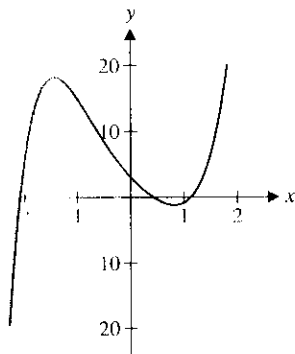


FIGURE 1.34

$f(x) = x^5 + 4x^2 - 9x + 3$.

EXAMPLE 4.9 Finding Zeros by the Method of Bisections

Find the zeros of $f(x) = x^5 + 4x^2 - 9x + 3$.

Solution If f were a quadratic polynomial, you could certainly find its zeros. But, what can you do here? You certainly don't have any formulas for finding zeros of polynomials of degree 5. The only alternative is to try to somehow approximate the zeros. A good starting place would be to use your graphing utility to draw a graph of $y = f(x)$ like the one in Figure 1.34. There are three zeros visible on the graph. (Can you convince yourself that f has *only* three zeros?) Note that since f is a polynomial, it is continuous everywhere. Hence, Corollary 4.2 says that there must be a zero on any interval on which the function changes sign. From the graph, you can see that there must be zeros between -3 and -2 , between 0 and 1 and between 1 and 2 . Without the graph, we could conclude this by computing say, $f(0) = 3$ and $f(1) = -1$. Although we've now found intervals that contain zeros, the question remains as to how we can *find* the zeros themselves.

You could certainly use a rootfinding program to obtain an accurate approximation, but that's not the point. The issue here is not so much to get an answer as it is to understand how you might find one. We suggest the following simple yet effective rootfinding method, called the **method of bisections**.

To find the zero between 0 and 1, we make a *guess* as to its location. A reasonable guess might be the midpoint, 0.5. Since $f(0.5) = -0.469 < 0$ and $f(0) = 3 > 0$, there's a sign change between 0 and 0.5 and so, the zero is there. Next, look at the midpoint of $[0, 0.5]$. Since $f(0.25) = 1.00098 > 0$, the zero is on the interval $(0.25, 0.5)$ and so on. We continue to narrow the interval on which there's a zero until the interval is sufficiently small so that any point in the interval can serve as an adequate approximation to the actual zero. We do this in the following table.

a	b	$f(a)$	$f(b)$	Midpoint	$f(\text{midpoint})$
0	1	3	-1	0.5	-0.469
0	0.5	3	-0.469	0.25	1.00098
0.25	0.5	1.00098	-0.469	0.375	0.195
0.375	0.5	0.195	-0.469	0.4375	-0.156
0.375	0.4375	0.195	-0.156	0.40625	0.015
0.40625	0.4375	0.015	-0.156	0.421875	-0.072
0.40625	0.421875	0.015	-0.072	0.4140625	-0.029
0.40625	0.4140625	0.015	-0.029	0.41015625	-0.007
0.40625	0.41015625	0.015	-0.007	0.408203125	0.004

If you were to continue this process through 20 more steps, you would ultimately arrive at the approximate zero $x = 0.40892288$, which is accurate to at least eight decimal places. ■

REMARK 4.4

This method of bisections certainly is a tedious process, at least if you're working it with pencil and paper. Even though this is not a particularly efficient method of approximating zeros, it is interesting because it's reliable and it's likely the first general method that you have seen. We all have access to a computer or calculator rootfinding utility that is much more accurate and efficient. Such tools are very useful, but their inner workings remain unseen by the user. Our purpose in this calculus text is to provide you with some tools for understanding, as well as with the ability to calculate some answers. Among other things, this understanding should leave you more capable of appreciating programs such as rootfinding utilities. We discuss some more powerful methods for rootfinding in Chapter 3.



EXERCISES 1.4



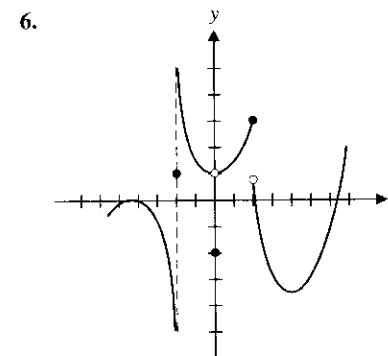
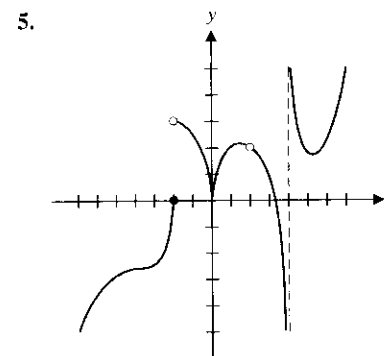
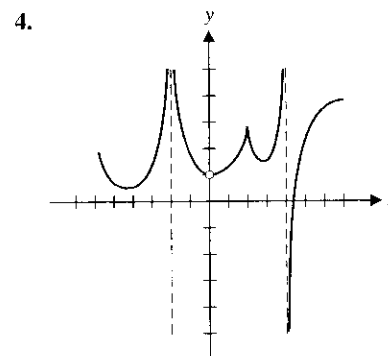
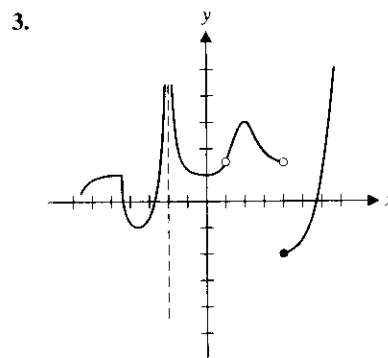
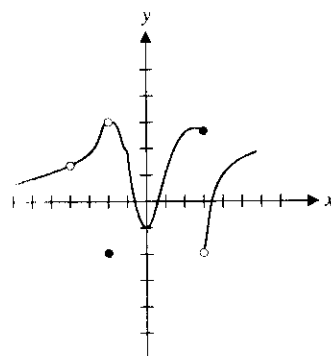
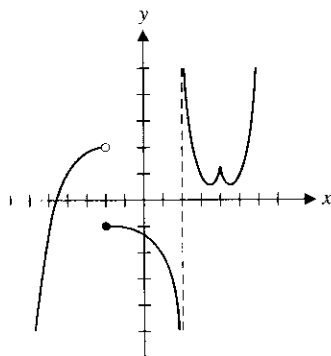
WRITING EXERCISES

1. Think about the following "real-life" functions, each of which is a function of the independent variable time: the height of a falling object, the velocity of an object, the amount of money in a bank account, the cholesterol level of a person, the heart rate of a person, the amount of a certain chemical present in a test tube and a machine's most recent measurement of the cholesterol level of a person. Which of these are continuous functions? For each function you identify as discontinuous, what is the real-life meaning of the discontinuities?
2. Whether a process is continuous or not is not always clear-cut. When you watch television or a movie, the action seems to be continuous. This is an optical illusion, since both movies and television consist of individual "snapshots" that are played back at many frames per second. Where does the illusion of

continuous motion come from? Given that the average person blinks several times per minute, is our perception of the world actually continuous? [A famous experiment in cognitive psychology showed that the human brain first decides whether a stimulus is important enough to merit conscious consideration. If so, the brain “predates” the stimulus so that the person correctly identifies when the stimulus actually occurred (hence, the different reaction times we have to stimuli)].

- 1 When you sketch the graph of the parabola $y = x^2$ with pencil or pen, is your sketch (at the molecular level) actually the graph of a continuous function? Is your calculator or computer's graph actually the graph of a continuous function? On many calculators, you have the option of a connected or disconnected graph. At the pixel level, does a connected graph show the graph of a function? Does a disconnected graph show the graph of a continuous function? Do we ever have problems correctly interpreting a graph due to these limitations? In the exercises in section 1.6 we examine one case where our perception of a computer graph depends on which choice is made.
- 4 For each of the graphs in Figures 1.22–1.25, describe (with an example) what the formula for $f(x)$ might look like to produce the given discontinuity.

In exercises 1–6, use the given graph to identify all x -values at which the function is discontinuous.



In exercises 7–12, explain why each function is discontinuous at the given point by indicating which of the three conditions in Definition 4.1 are met and which are not met.

7. $f(x) = \frac{x}{x-1}$ at $x = 1$

8. $f(x) = \frac{x^2 - 1}{x - 1}$ at $x = 1$

9. $f(x) = \sin \frac{1}{x}$ at $x = 0$

10. $f(x) = e^{1/x}$ at $x = 0$

11. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ at $x = 2$

12. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ at $x = 2$

In exercises 13–24, find all discontinuities of $f(x)$. For each discontinuity that is removable, define a new function that removes the discontinuity.

13. $f(x) = \frac{x-1}{x^2-1}$

14. $f(x) = \frac{4x}{x^2+x-2}$

15. $f(x) = \frac{4x}{x^2+4}$

16. $f(x) = \frac{3x}{x^2-2x-4}$

17. $f(x) = x^2 \tan x$

18. $f(x) = \csc x$

19. $f(x) = x \cot x$

20. $f(x) = \frac{\sec x}{x^2}$

21. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

22. $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

23. $f(x) = \begin{cases} 3x-1 & \text{if } x \leq -1 \\ x^2+5x & \text{if } -1 < x < 1 \\ 3x^3 & \text{if } x \geq 1 \end{cases}$

24. $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ \sin x & \text{if } 0 < x \leq \pi \\ x - \pi & \text{if } x > \pi \end{cases}$

In exercises 25–32, determine the intervals on which $f(x)$ is continuous.

25. $f(x) = \sqrt{x+3}$

26. $f(x) = \sqrt{x^2-4}$

27. $f(x) = \sqrt[3]{x+2}$

28. $f(x) = (x-1)^{3/2}$

29. $f(x) = \sin(x^2+2)$

30. $f(x) = \cos\left(\frac{1}{x}\right)$

31. $f(x) = \ln(x+1)$

32. $f(x) = \ln(4-x^2)$

In exercises 33–35, determine values of a and b that make the given function continuous.

33. $f(x) = \begin{cases} 2\frac{\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b \cos x & \text{if } x > 0 \end{cases}$

34. $f(x) = \begin{cases} ae^x + 1 & \text{if } x < 0 \\ \sin^{-1} \frac{x}{2} & \text{if } 0 \leq x \leq 2 \\ x^2 - x + b & \text{if } x > 2 \end{cases}$

35. $f(x) = \begin{cases} a(\tan^{-1} x + 2) & \text{if } x < 0 \\ 2e^{bx} + 1 & \text{if } 0 \leq x \leq 3 \\ \ln(x-2) + x^2 & \text{if } x > 3 \end{cases}$

36. Prove Corollary 4.1.

37. Suppose that a state's income tax code states that the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 0.14x & \text{if } 0 < x < 10,000 \\ c + 0.21x & \text{if } 10,000 \leq x \end{cases}$$

Determine the constant c that makes this function continuous at all x . Give a rationale why such a function should be continuous.

38. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ a + 0.12x & \text{if } 0 < x \leq 20,000 \\ b + 0.16(x - 20,000) & \text{if } x > 20,000 \end{cases}$$

such that $T(x)$ is continuous at all x .

39. In example 4.8, find b and c to complete the table.

40. In example 4.8, show that $T(x)$ is continuous at $x = 6000$.

T In exercises 41–46, use the Intermediate Value Theorem to verify that $f(x)$ has a zero in the given interval. Then use the method of bisections to find an interval of length $1/32$ that contains the zero.

41. $f(x) = x^2 - 7$, $[2, 3]$

42. $f(x) = x^3 - 4x - 2$, $[2, 3]$

43. $f(x) = x^3 - 4x - 2$, $[-1, 0]$

44. $f(x) = x^3 - 4x - 2$, $[-2, -1]$

45. $f(x) = \cos x - x$, $[0, 1]$

46. $f(x) = e^x + x$, $[-1, 0]$

A function is continuous from the right at $x = a$ if $\lim_{x \rightarrow a^+} f(x) = f(a)$. In exercises 47–50, determine if $f(x)$ is continuous from the right at $x = 2$.

47. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 1 & \text{if } x \geq 2 \end{cases}$

48. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

$$49. f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$$

$$50. f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x \geq 2 \end{cases}$$

51. Define what it means for a function to be **continuous from the left** at $x = a$ and determine which of the functions in exercises 47–50 are continuous from the left at $x = 2$.

52. Suppose that $f(x) = \frac{g(x)}{h(x)}$ and $h(a) = 0$. Determine whether each of the following statements is definitely true, definitely false, or maybe true/maybe false. Explain. (a) $\lim_{x \rightarrow a} f(x)$ does not exist. (b) $f(x)$ is discontinuous at $x = a$.

53. The sex of newborn Mississippi alligators is determined by the temperature of the eggs in the nest. The eggs fail to develop unless the temperature is between 26°C and 36°C . All eggs between 26°C and 30°C develop into females and eggs between 34°C and 36°C develop into males. The percentage of females decreases from 100% at 30°C to 0% at 34°C . If $f(T)$ is the percentage of females developing from an egg at $T^\circ\text{C}$, then $f(T) = \begin{cases} 100 & \text{if } 26 \leq T \leq 30 \\ g(T) & \text{if } 30 < T < 34 \text{ for some function } g(T) \\ 0 & \text{if } 34 \leq T \leq 36 \end{cases}$. Explain why it is reasonable that $f(T)$ be continuous. Determine a function $g(T)$ such that $0 \leq g(T) \leq 100$ for $30 < T < 34$ and the resulting function $f(T)$ is continuous. [Hint: It may help to draw a graph first and make $g(T)$ linear.]

54. If $f(x) = \begin{cases} x^2, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases}$ and $g(x) = 2x$, show that

$$\lim_{x \rightarrow 0} f(g(x)) \neq f\left(\lim_{x \rightarrow 0} g(x)\right).$$

55. If you push on a large box resting on the ground, at first nothing will happen because of the static friction force that opposes motion. If you push hard enough, the box will start sliding, although there is again a friction force that opposes the motion. Suppose you are given the following description of the friction force. Up to 100 pounds, friction matches the force you apply to the box. Over 100 pounds, the box will move and the friction force will equal 80 pounds. Sketch a graph of friction as a function of your applied force based on this description. Where is this graph discontinuous? What is significant physically about this point? Do you think the friction force actually ought to be continuous? Modify the graph to make it continuous while still retaining most of the characteristics described.

56. For $f(x) = 2x - \frac{400}{x}$, we have $f(-1) > 0$ and $f(2) < 0$. Does the Intermediate Value Theorem guarantee a zero of $f(x)$ between $x = -1$ and $x = 2$? What happens if you try the method of bisections?

57. On Monday morning, a saleswoman leaves on a business trip at 7:13 A.M. and arrives at her destination at 2:03 P.M. The following morning, she leaves for home at 7:17 A.M. and arrives

at 1:59 P.M. The woman notices that at a particular stoplight along the way, a nearby bank clock changes from 10:32 A.M. to 10:33 A.M. on both days. Therefore, she must have been at the same location at the same time on both days. Her boss doesn't believe that such an unlikely coincidence could occur. Use the Intermediate Value Theorem to argue that it *must* be true that at some point on the trip, the saleswoman was at exactly the same place at the same time on both Monday and Tuesday.

58. Suppose you ease your car up to a stop sign at the top of a hill. Your car rolls back a couple of feet and then you drive through the intersection. A police officer pulls you over for not coming to a complete stop. Use the Intermediate Value Theorem to argue that there was an instant in time when your car was stopped (in fact, there were at least two). What is the difference between this stopping and the stopping that the police officer wanted to see?
59. Suppose a worker's salary starts at \$40,000 with \$2000 raises every 3 months. Graph the salary function $s(t)$; why is it discontinuous? How does the function $f(t) = 40,000 + \frac{2000}{3}t$ (t in months) compare? Why might it be easier to do calculations with $f(t)$ than $s(t)$?

60. Prove the final two parts of Theorem 4.2.

61. Suppose that $f(x)$ is a continuous function with consecutive zeros at $x = a$ and $x = b$; that is, $f(a) = f(b) = 0$ and $f(x) \neq 0$ for $a < x < b$. Further, suppose that $f(c) > 0$ for some number c between a and b . Use the Intermediate Value Theorem to argue that $f(x) > 0$ for all $a < x < b$.

- T** 62. Use the method of bisections to estimate the other two zeros in example 4.9.

63. Suppose that $f(x)$ is continuous at $x = 0$. Prove that $\lim_{x \rightarrow 0} xf(x) = 0$.

64. The **converse** of exercise 63 is not true. That is, the fact $\lim_{x \rightarrow 0} xf(x) = 0$ does not guarantee that $f(x)$ is continuous at $x = 0$. Find a counterexample; that is, find a function such that $\lim_{x \rightarrow 0} xf(x) = 0$ and $f(x)$ is not continuous at $x = 0$.

65. If $f(x)$ is continuous at $x = a$, prove that $g(x) = |f(x)|$ is continuous at $x = a$.

66. Determine whether the converse of exercise 65 is true. That is, if $|f(x)|$ is continuous at $x = a$, is it necessarily true that $f(x)$ must be continuous at $x = a$?

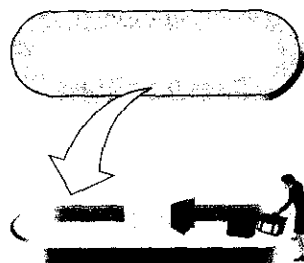
EXPLORATORY EXERCISES

- T** 1. In the text, we discussed the use of the method of bisections to find an approximate solution of equations such as $f(x) = x^3 + 5x - 1 = 0$. We can start by noticing that $f(0) = -1$ and $f(1) = 5$. Since $f(x)$ is continuous, the Intermediate Value Theorem tells us that there is a solution between $x = 0$ and

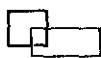
$x = 1$. For the method of bisections, we guess the midpoint, $x = 0.5$. Is there any reason to suspect that the solution is actually closer to $x = 0$ than to $x = 1$? Using the function values $f(0) = -1$ and $f(1) = 5$, devise your own method of guessing the location of the solution. Generalize your method to using $f(a)$ and $f(b)$, where one function value is positive and one is negative. Compare your method to the method of bisections on the problem $x^3 + 5x - 1 = 0$; for both methods, stop when you are within 0.001 of the solution, $x \approx 0.198437$. Which method performed better? Before you get overconfident in your method, compare the two methods again on $x^3 + 5x^2 - 1 = 0$. Does your method get close on the first try? See if you can determine graphically why your method works better on the first problem.

2. You have probably seen the turntables on which luggage rotates at the airport. Suppose that such a turntable has two long straight parts with semicircles on the end (see the figure). We will model the left/right movement of the luggage. Suppose the straight part is 40 ft long, extending from $x = -20$ to $x = 20$. Assume that our luggage starts at time $t = 0$ at location $x = -20$ and that it takes 60 s for the luggage to reach $x = 20$. Suppose the radius of the circular sections is 5 ft and

it takes the luggage 30 s to complete the half-circle. We model the straight-line motion with a linear function $x(t) = at + b$. Find constants a and b so that $x(0) = -20$ and $x(60) = 20$. For the circular motion, we use a cosine (Why is this a good choice?) $x(t) = 20 + d \cdot \cos(et + f)$ for constants d , e and f . The requirements are $x(60) = 20$ (since the motion is continuous), $x(75) = 25$ and $x(90) = 20$. Find values of d , e and f to make this work. Find equations for the position of the luggage along the backstretch and the other semicircle. What would the motion be from then on?



Luggage carousel.



1.5 LIMITS INVOLVING INFINITY

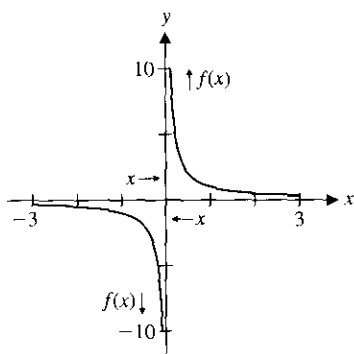


FIGURE 1.35

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

In this section, we revisit some old limit problems in order to give more informative answers and examine some related questions.

EXAMPLE 5.1 A Simple Limit Revisited

Examine $\lim_{x \rightarrow 0} \frac{1}{x}$.

Solution Of course, we can draw a graph (see Figure 1.35) and compute a table of function values easily, by hand (see the tables in the margin on the next page).

While we would say that the limits $\lim_{x \rightarrow 0^+} \frac{1}{x}$ and $\lim_{x \rightarrow 0^-} \frac{1}{x}$ do not exist, the behavior of the function is clearly quite different for $x > 0$ than for $x < 0$. Specifically, as $x \rightarrow 0^+$, $\frac{1}{x}$ increases without bound, while as $x \rightarrow 0^-$, $\frac{1}{x}$ decreases without bound. Although neither one-sided limit exists, the limiting behavior on each side of $x = 0$ is different. To communicate as much as we can about the behavior of the function near $x = 0$, we write

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad (5.1)$$

and

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty. \quad (5.2)$$