In each of Problems 7 through 12 determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

(1) \((x - 3)y'' + xy' + (\ln |x|)y = 0\), \(y(1) = 0\), \(y'(1) = 1\)

11. Write the equation as \(y'' + \frac{x}{x-3}y' + \frac{\ln |x|}{x-3}y = 0\). The coefficients are discontinuous at \(x = 0\) and \(x = 3\). Since \(x_0 \in (0, 3)\), the largest interval is \(0 < x < 3\).

13. Verify that \(y_1(t) = t^2\) and \(y_2(t) = t^{-1}\) are two solutions of the differential equation \(t^2 y'' - 2y = 0\) for \(t > 0\). Then show that \(c_1 t^2 + c_2 t^{-1}\) is also a solution of this equation for any \(c_1\) and \(c_2\).

13. \(y'_2 = 2\). We see that \(t^2(2) - 2(t^2) = 0\). \(y''_2 = 2t^{-3}\), with \(t^2(y''_2) - 2(y_2) = 0\). Let \(y_3 = c_1 t^2 + c_2 t^{-1}\), then \(y''_3 = 2c_1 + 2c_2 t^{-3}\). It is evident that \(y_3\) is also a solution.

Substituting \(y_3\) into \(t^2 y'' - 2y = 0\) we get
\[
= 2c_1 t^2 + 2c_2 t^{-1} - 2c_1 t^2 - 2c_2 t^{-1} = 0 \quad (0 = 0)
\]

In each of Problems 23 through 26 verify that the functions \(y_1\) and \(y_2\) are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

23. \(y'' + 4y = 0;\) \(y_1(t) = \cos 2t\), \(y_2(t) = \sin 2t\)

\[
y_1' = -2 \sin 2t
y_2'' = -4 \sin 2t
\]

\[
y_1'' = -4 \cos 2t
y_2'' = -4 \cos 2t
\]

Use the Wronskian to see if they form a f.s.

\[
W(y_1, y_2) = \begin{vmatrix}
\cos 2t & \sin 2t \\
-2 \sin 2t & 2 \cos 2t
\end{vmatrix}
= 2 \cos^2 2t + 2 \sin^2 2t = 2(\cos^2 2t + \sin^2 2t) = 2
\]

The Wronskian \(\neq 0\) on \((-\infty, \infty)\), so \(\{\cos 2t, \sin 2t\}\) form a f.s. on \((-\infty, \infty)\).
Consider the equation \( y'' - y' - 2y = 0 \).

(a) Show that \( y_1(t) = e^t \) and \( y_2(t) = e^{-t} \) form a fundamental set of solutions.

(b) Let \( y_3(t) = -2e^{2t} \), \( y_4(t) = y_1(t) + 2y_2(t) \), and \( y_5(t) = 2y_1(t) - 2y_2(t) \). Are \( y_3(t) \), \( y_4(t) \), and \( y_5(t) \) also solutions of the given differential equation?

(c) Determine whether each of the following pairs form a fundamental set of solutions: 
\( \{y_1(t), y_3(t)\} \), \( \{y_2(t), y_3(t)\} \), \( \{y_1(t), y_4(t)\} \), \( \{y_4(t), y_5(t)\} \).

### Solution

(a) \( y_1(t) = e^t \) is a solution because \( y_1'' - y_1' - 2y_1 = 0 \)
\[
\begin{align*}
y_1 &= e^t \\
y_1' &= e^t \\
y_1'' &= e^t
\end{align*}
\]
\[
e^{-t}(-e^{-t}) - 2e^{-t} = 0
\]

\( y_2(t) = e^{-t} \) is a solution because \( y_2'' - y_2' - 2y_2 = 0 \)
\[
\begin{align*}
y_2 &= e^{-t} \\
y_2' &= -e^{-t} \\
y_2'' &= e^{-t}
\end{align*}
\]
\[
4e^{2t} - 2e^{2t} - 2e^t = 0
\]

\[
W(e^t, e^{-t}) = \begin{vmatrix} e^t & e^{-t} \\ -e^t & -e^{-t} \end{vmatrix} = 2e^t + e^{-t} = 3e^t \neq 0 \text{ on } (-\infty, \infty)
\]

So \( \{e^t, e^{-t}\} \) form a f.s. of solutions on \( (-\infty, \infty) \).

(b) \( y_3 = -2e^{2t} \) is a solution by the principle of proportionality.

\( y_4 = y_1(t) + 2y_2(t) \) is a solution by the principle of proportionality and superposition.

\( y_5 = 2y_1(t) - 2y_2(t) \) is a solution by the principle of proportionality and superposition.

(c) \[
W(y_1, y_3) = \begin{vmatrix} e^t & -2e^{2t} \\ -e^{-t} & 4e^{2t} \end{vmatrix} = -4e^t - e^{-t} = -5e^t \neq 0
\]
\( \{y_1, y_3\} \) form a f.s.

\[
W(y_2, y_3) = \begin{vmatrix} e^{2t} & -2e^{2t} \\ 2e^{2t} & 4e^{2t} \end{vmatrix} = -4\neq 0 \Rightarrow \{y_2, y_3\} \text{ do not form a f.s.}
\]

\[
W(y_1, y_4) = \begin{vmatrix} e^t & e^t + 2e^{2t} \\ -e^{-t} & -e^{-t} + 4e^{2t} \end{vmatrix} = e^t(-e^{-t} + 4e^{2t}) + e^t(e^{-t} + 2e^{2t})
\]
\( \{y_1, y_4\} \) form a f.s.
Section 3.2 submit problems (3)

#27 (c) continued

\[
W(y_4, y_5) = \begin{vmatrix} e^{-t} + 2e^{2t} & 2e^{-t} + 4e^{2t} \\ -e^{-t} + 4e^{2t} & -2e^{-t} + 8e^{2t} \end{vmatrix} = \\
= (e^{-t} + 2e^{2t})(-2e^{-t} + 8e^{2t}) - (-e^{-t} + 4e^{2t})(2e^{-t} + 4e^{2t}) \\
= -2e^{-2t} + 4e^{4t} - 16e^{-4t} - (-2e^{-2t} + 4e^{4t} - 16e^{4t}) \\
= 0
\]

\[\Rightarrow \{y_4, y_5\} \text{ is a fundamental set.} \]

Supplemental submit problem:

If \( L[y] = t^2y'' - ty' + y \), find \( g(t) \) for which \( y = t^3 \) is a solution of \( L[y] = g(t) \).

\[
y = t^3 \\
y' = 3t^2 \\
y'' = 6t \\
L[y] = t^2(6t) - t(3t^2) + t^3 = 4t^3
\]

So \( y = t^3 \) is a solution of \( L[y] = 4t^3 \).