In each of Problems 1 through 6 determine (without solving the problem) an interval in which
the solution of the given initial value problem is certain to exist.

5. \((4 - t^2)y' + 2ty = 3t^2, \quad y(1) = -3\)

This is linear so we can use the Linear Existence and Uniqueness
Theorem.

\[
p(t) = \frac{2t}{(2-t)(2+t)} \quad \text{and} \quad g(t) = \frac{3t^2}{(2-t)(2+t)}
\]

are not continuous at \(t = \pm 2\)

Possible intervals are \((-\infty, -2), (-2, 2), (2, \infty)\)

\(t_0 = 1\) is in \((-2, 2)\)

6. \((\ln t)y' + y = \cot t, \quad y(2) = 3\)

\[
y' + \frac{1}{\ln t} y = \cot t
\]

\[
p(t) = \frac{1}{\ln t} \quad \text{is not defined for} \quad t \leq 0
\]

\[
g(t) = \frac{\cot t}{\ln t} \quad \text{is not cont. at} \quad t = 1,
\]

\(t \in \{n\pi\} \quad \text{at} \quad t = \text{any multiple of} \pi
\]

\(t_0 = 2\) is in \((1, \pi)\)

In each of Problems 7 through 12 state where in the \(xy\)-plane the hypotheses of Theorem 2.4.2
are satisfied.

7. \(y' = \frac{t - y}{2t + 3y}\)

\[
\frac{2y'}{2y} = \frac{(2t+3y)^2 - (t-y) \cdot 3}{(2t+3y)^2} = -\frac{7t}{(2t+3y)^2}
\]

Both \(y'\) and \(\frac{2y'}{2y}\) are not continuous where \(2t + 3y = 0\)

or where \(y = -\frac{3}{2}t\). Thus, the Theorem is satisfied

for any initial condition where \(y_0 \neq -\frac{3}{2}t\).
9. \( v' = \frac{\ln |y|}{1 - t^2 + y^2} \)

\[
\frac{dy'}{dy} = \frac{1 - t^2 + y^2 - 2y^2 \ln |y|}{y(1 - t^2 + y^2)^2}
\]

Both \( y' \) and \( \frac{dy'}{dy} \) do not exist where \( ty = 0 \); where \( t=0 \) or \( y=0 \).
Additionally, both are not defined where \( 1 - t^2 + y^2 = 0 \)
\[
\Rightarrow 1 = t^2 - y^2 \quad \text{This is a hyperbola with vertices (±1, 0)}
\]

The regions are \( 1 - t^2 + y^2 > 0 \) and \( 1 - t^2 + y^2 < 0 \)

In each of Problems 13 through 16 solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value \( y_0 \).

14. \( y' = 2ty^2, \quad y(0) = y_0 \)

This is separable (not linear)

Note that \( \frac{y(t)}{t} = 0 \) is a constant soln.

\[
y - \frac{2y^2}{t} \Rightarrow 5y - \frac{2y^2}{t} = \frac{5yt}{t} + c
\]

\[
\Rightarrow -\frac{1}{y} = t^2 + c \Rightarrow \frac{1}{y} = c - t^2 \Rightarrow y(t) = \frac{1}{c - t^2}
\]

\( y(0) = y_0 \Rightarrow y_0 = \frac{1}{c} \) so \( c = \frac{1}{y_0} \) for \( y_0 \neq 0 \)

Thus, when \( y_0 \neq 0 \), the specific soln is \( y(t) = \frac{1}{y_0 - t^2} = \frac{y_0}{y_0 - t^2} \)

If \( y_0 = 0 \), then the specific soln is the constant soln \( y(t) = 0 \).
To see how the interval on which the solution exists depend on \( y_0 \), consider these three cases:

1) \( y_0 > 0 \).

Then \( 1 - y_0 t^2 \) could be 0 and \( y(t) = \frac{y_0}{1 - y_0 t^2} \) has a discontinuity at \( 1 - y_0 t^2 = 0 \) \( \Rightarrow y_0 t^2 = 1 \) \( \Rightarrow t^2 = \frac{1}{y_0} \) \( \Rightarrow |t| = \frac{1}{\sqrt{y_0}} \). This gives us three intervals to consider: \( (-\infty, -\frac{1}{\sqrt{y_0}}) \), \( (-\frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{y_0}}) \), \( (\frac{1}{\sqrt{y_0}}, \infty) \).

To determine which interval(s) are part of the domain in this case, consider that \( y_0 > 0 \) \( \Rightarrow y > 0 \) \( \Rightarrow y(t) = \frac{y_0}{1 - y_0 t^2} > 0 \) \( \Rightarrow 1 - y_0 t^2 > 0 \) \( \Rightarrow t^2 < \frac{1}{y_0} \) \( \Rightarrow |t| < \frac{1}{\sqrt{y_0}} \) \( \Rightarrow -\frac{1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}} \).

2) \( y_0 = 0 \).

In this case the soln is \( y(t) = 0 \) which is defined for \( -\infty < t < \infty \).

3) \( y_0 < 0 \).

In this case the denominator \( 1 - y_0 t^2 \) of \( y(t) \) will always be positive and never = 0. So \( y(t) \) will have no discontinuities and will be defined for \( -\infty < t < \infty \).