1. Statements a – e below refer to the second order non-homogeneous \( L[y] = g(t) \) with particular solution \( Y(t) \). The fundamental set of the corresponding \( L[y] = 0 \) is \( \{y_1(t), y_2(t)\} \) on some interval \( I \).

a. F or F \( y(t) = 0 \) is always a solution of \( L[y] = 0 \).

b. T or F The Wronskian of \( \{5y_1(t), 3y_2(t)\} \) will never equal zero on \( I \).

c. F or F \( L[y_1(t)] = g(t) \).

d. F or F \( L(c_1y_1(t) + c_2y_2(t)) = g(t) \).

e. T or F \( y_1(t) \) and \( y_2(t) \) are linearly-independent on \( I \).

2. Check (✓) the differential equations below for which you could use undetermined coefficients to find \( Y(t) \).

a. \( y'' - 2y' + 3y = t^{-1} \)  

b. \( t^2y'' - 2ty + 3y = 4 \)  

c. \( y'' + 4y = te^t \sin 3t \) ✓

3. On what intervals will the solutions of \( (t-1)y'' + y' = \ln t \) have their domains?

Use the EUL Theorem \( y'' + \frac{1}{t-1}y' = \frac{\ln t}{t-1} \), \( p(t) = \frac{1}{t-1} \), not cont at \( t=1 \) and not defined for \( t \leq 0 \)

\[ g(t) = \frac{\ln t}{t-1} \]  
not cont at \( t=1 \) and not defined for \( t \leq 0 \). Intervals are \( (0, 1), (1, \infty) \)

4. a) If \( \begin{cases} \begin{align*} \alpha x + \beta y &= 0 \\ \gamma x + \delta y &= 0 \end{align*} \end{cases} \) and \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 \), what do we know about the solution of the system?

There is a unique solution, namely \((0, 0)\).

b) If \( \begin{cases} \begin{align*} \alpha x + \beta y &= 0 \\ \gamma x + \delta y &= f \end{align*} \end{cases} \) and \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \), what do we know about the solution of the system?

There could be no solution or infinitely-many solutions.
5. Consider \( L = t^2 y'' - 2ty' - 4y \)
   a. Find \( L[y] \) if \( y(t) = t^3 \).
      \[
      y'(t) = 3t^2 \quad \text{and} \quad y(t) = t^3 \\
      y''(t) = 6t \quad \text{and} \quad L[y] = t^3 (t^3)'' - 2t (t^3)' - 4t^3
      = t^3 (6t) - 2t (3t^2) - 4t^3 = 6t^4 - 6t^3 - 4t^3
      = 2t^4 - 4t^3
      \]
   b. Based on your answer to part a, \( y(t) = t^3 \) is a solution of \( L[y] = -4t^3 \).

6. The fundamental set of the 2nd-order \( L[y] = 0 \) is \( \{t^2, t^3\} \).
   a. Evaluate the Wronskian of \( \{t^2, t^3\} \).
      \[
      W = \begin{vmatrix} t^2 & t^3 \\ t^3 & 3t^3 \end{vmatrix} = 3t^4 - 2t^4 = t^4 
      \]
   b. What is/are the interval(s) on which \( \{t^2, t^3\} \) form a linearly-independent set of solutions of \( L[y] = 0 \)?
      \( W = 0 \) at \( t = 0 \)
      \((-\infty, 0), (0, \infty)\)

7. Short answer. Fill in the blanks.
   a. By definition, the functions \( f_1(t), f_2(t) \) are linearly independent on some
      interval \( I \) if the equation \( k_1 f_1(t) + k_2 f_2(t) = 0 \) is true if and only if
      \[ k_1 = k_2 = 0 \]
   b. The functions \( f_1(t) = e^{2t} \) and \( f_2(t) = e^t \) are linearly independent on
      \((-\infty, \infty)\). \( k_1 e^{2t} + k_2 e^t = 0 \) is true \( \iff \) \( k_1 = k_2 = 0 \)
   c. The functions \( f_1(t) = e^{2t} \) and \( f_2(t) = 3e^{2t} \) are linearly independent on
      \((-\infty, \infty)\).
   d. The functions \( f_1(t) = t \) and \( f_2(t) = |t| \) are linearly dependent on \((-\infty, 0)\).
      \[ \text{On } (-\infty, 0), k_1 t + k_2 |t| = 0 \Rightarrow k_1 = k_2 = 0 \text{ which is true whenever } \begin{pmatrix} \text{nonzero} \end{pmatrix} k_1 = k_2 = 0 \]

8. For \( r = 1 - 2i \) use the Cauchy-Euler identity to express \( e^t \) in the form \( a + ib \)
   \[
   e^{(1-2i)t} = e^t e^{-2it} = e^t (\cos 2t - i \sin 2t) 
   \]
9. \(y'' + 4y = 4e^{2t} - 3t\) has a particular solution with the form 
\[Y(t) = A + Bt + Ce^{2t}.\]

a. Find the values of the coefficients \(A\), \(B\), and \(C\).

\[
\begin{align*}
Y' &= B + 2Ce^{2t} \\
Y'' &= 4Ce^{2t}
\end{align*}
\]
Plug into \((N)\):

\[
\frac{4Ce^{2t} + 4A + 4Bt + 4Ce^{2t}}{y'' + 4y} = \frac{4e^{2t} - 3t}{y'' + 4y}
\]

Compare coefficients:

\[
e^{2t} \text{ terms: } 4C + 4C = 4 \Rightarrow 8C = 4 \Rightarrow C = \frac{1}{2}
\]

\[
+ \text{ terms: } 4B = -3 \Rightarrow B = -\frac{3}{4}
\]

\[
\text{Constants: } 4A = 0 \Rightarrow A = 0
\]

\[
Y(t) = -\frac{3}{4}t + \frac{1}{2}e^{2t}
\]

b. What is the general solution of \(y'' + 4y = 4e^{2t} - 3t\)?

1) Solve \(y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow y_c(t) = C_1 \cos 2t + C_2 \sin 2t\)

2) The g. s. is 
\[
y(t) = y_c(t) + y_p(t) = C_1 \cos 2t + C_2 \sin 2t - \frac{3}{4}t + \frac{1}{2}e^{2t}
\]

10. Find the general solution of \(y'' - 4y' = t\). \((N)\)

1) Solve \(y' - 4y = 0\); \(y_c(t) = C_1 + C_2 e^{4t}\)

2) Using undetermined coefficients, a 1st guess for \(Y(t) = A + Bt\). However, \(t\) corresponds to a root of 0 which is also a root of \(P(r)\). So, the form of \(Y(t)\) is \(Y(t) = At + Bt^2\). 

Plug into \((N)\): 

\[
\begin{align*}
2B - 4A - 8Bt &= t \\
-4A &= \frac{1}{8}
\end{align*}
\]

Thus \(A = -\frac{1}{16}\)

\[
Y(t) = -\frac{1}{16}t - \frac{1}{8}t^2
\]

3) The g. s. is 
\[
y(t) = y_c(t) + y_p(t) = C_1 + C_2 e^{4t} - \frac{1}{16}t - \frac{1}{8}t^2
\]
11. Consider \( y'' - 3y' = 5e^{3t} + 3e^t \cos 2t \). Find the form of \( Y(t) \) using the method of Undetermined Coefficients. DO NOT SOLVE FOR THE COEFFICIENTS.

\[
\text{Let } Y(t) = \sum_{i=1}^{n} c_i e^{r_i t} + A_1 e^{2t} + A_2 e^{3t} + B_1 \cos 2t + B_2 \sin 2t.
\]

\[
Y(t) = \sum_{i=1}^{3} c_i e^{r_i t} + A_1 e^{2t} + A_2 e^{3t} + B_1 \cos 2t + B_2 \sin 2t.
\]

12. Consider \( t^2 y'' + ty' - y = 3t^2 \) on \((0, \infty)\) for which the fundamental set of \( L[y] = 0 \) is \( \{t, t'\} \). Use the method of Variation of Parameters to find a particular solution, \( Y(t) \), of \( t^2 y'' + ty' - y = 3t^2 \). For partial credit answer the following:

a. What is the form of \( Y(t) \)?

\[
Y(t) = u_1(t) t + u_2(t) t^{-1}
\]

b. What system of equations must be solved to completely find \( Y(t) \)?

\[
\begin{align*}
&u_1'(t) t + u_2'(t) t^{-1} = 0 \\
u_1(t) - u_2'(t) t^{-2} &= \frac{3t^2}{2}
\end{align*}
\]

\[
\begin{align*}
u_1'(t) &= 0 \\
u_2'(t) &= \frac{3t}{2}
\end{align*}
\]

\[
\begin{align*}
u_1(t) &= \frac{3t^2}{2} t^{-2} \\
u_2(t) &= \frac{3t^3}{2} t^{-2} = \frac{3}{2} t^2
\end{align*}
\]

\[
Y(t) = \frac{3}{2} t^2 t^{-2} - \frac{1}{2} t^3 t^{-2} = \frac{3}{2} t^2 - \frac{1}{2} t^2 = t^2
\]

The g.s. of \( y(t) = c_1 t + c_2 t^{-1} + t^2 \).
13. Suppose a 4-lb weight stretches a spring 6 inches beyond its natural length. An external force equal to \( \frac{1}{6} \cos 8t \) is acting on the spring. If the weight is started in motion from its equilibrium position with an upward velocity of 4 ft/sec and there is no damping, set up a initial value problem whose solution will give an equation of the motion of the mass + spring.

\[ m u'' + ku = \frac{1}{6} \cos 8t \]

To find \( m \): use \( w = mg \Rightarrow 4 = m \cdot \frac{3}{2} \Rightarrow m = \frac{8}{3} \text{ slug} \)

To find \( k \): use Hooke's Law: \(-4 = -\frac{1}{6} k \Rightarrow k = 8 \text{ lb/ft} \)

So \( \frac{1}{6} u'' + 8u = \frac{1}{6} \cos 8t \) or \( u'' + 64u = 4 \cos 8t \)

\( u(0) = 0 \) \(<\) began at equilibrium
\( u'(0) = -4 \) \(<\) initial velocity is directed upward (neg. direction)

14. A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyne-sec/cm. If the mass is pulled down an additional 2 cm and then released, set up an initial value problem to model the motion of the mass + spring.

\[ m = 20 \text{ g}, \quad \gamma = 400 \text{ dyne-sec/cm} \]

To find \( k \): we first have to find the force \( F \) exerted by the mass: \( w = mg = 20 \cdot 980 = 19600 \text{ dyne} \)

So \( F = -19600 = -k \cdot 5 \Rightarrow k = 3920 \text{ dyne/cm} \)

So we have \( 20u'' + 400u' + 3920u = 0 \)

or \( u'' + 20u' + 196u = 0 ; \quad u(0) = 2 \)

\[ u'(0) = 0 \]
15. Suppose the solution of a vibrating model (unforced) is \( u(t) = -2 \cos 4t + 2 \sin 4t \).
   a) Describe the type of motion of the system.
      \[ \text{Roots } s = \pm 4i \Rightarrow \text{free, undamped = harmonic no particular soln.} \]

   b) Put the solution in the amplitude-phase angle form.
      \[ \begin{align*}
      A &= c_1 = -2 \\
      B &= c_2 = 2
      \end{align*} \]
      \[ R = \sqrt{(-2)^2 + (2)^2} = 2 \sqrt{2} \]
      \[ \theta = \tan^{-1} \left( \frac{2}{-2} \right) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4} \]
      (because \( A = R \cos \theta < 0 \) and \( B = R \sin \theta > 0 \))

   c) Identify the period and frequency of the motion.
      \[ \text{Period } = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ rad.} \]
      \[ \text{Natural frequency } = \omega_0 = \frac{\pi}{\sqrt{2}} \text{ rad/s} \]

   d) If the mass attached to the spring is 4 kilograms, what is the spring constant of the spring?
      \[ m = \frac{k}{\omega_0^2} \Rightarrow k = \frac{m}{\omega_0^2} = 64 \text{ N/m} \]

16. Classify the type of motion of each equation below.
   a) \( u(t) = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t \)
      \( \text{no particular soln } \frac{\text{undamped}}{\text{free}} \)

   b) \( u(t) = c_1 e^{-3t} + c_2 e^{-3t} \)
      \( \text{no particular soln } \frac{\text{critically damped}}{\text{free}} \)

   c) \( u(t) = c_1 e^{-3t} + c_2 e^{-2t} \)
      \( \text{no particular soln } \frac{\text{overdamped}}{\text{free}} \)

   d) \( u(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{2} \sin 4t \)
      \( \text{particular soln } \frac{-1/2 \sin 4t}{\text{free}} \text{ undamped, free} \)

17. Suppose a mass-spring-damper system is modeled by \( u'' + 9u' + 14u = \frac{1}{2} \sin t , u(0) = 0, u'(0) = -1 \). If the specific solution is \( u(t) = -18e^{-2t} + .2e^{-3t} + .03 \sin t - .02 \cos t \), identify the steady-state portion of the solution and rewrite it in amplitude-phase angle form.

The steady-state portion is \( .03 \sin t - .02 \cos t \)
\[ R = \sqrt{(1.03)^2 + (-0.02)^2} \approx .036 \]
Be careful finding \( \theta \) because \( A = c_1 = -1.02, R \cos \theta \)
\[ R = .036 \]
\[ \text{sin quadrant II.} \]
\[ \theta = \tan^{-1} \left( \frac{0.03}{-0.02} \right) + \pi = -\frac{\pi}{2} + \pi = \frac{3\pi}{2} + \pi = 2\pi/16 \]
In amplitude phase angle form we have \( \frac{.036 \cos (t - 2\pi/16)}{\text{or } .036 \cos \left( t - \left[ \tan^{-1} \left( \frac{3}{2} \right) + \pi \right] \right)} \)