The following questions are merely meant to give you some idea of the kinds of questions I could ask on your exam. The best way you can prepare for this exam is to 1) review all lecture notes, especially all examples worked in class; 2) review your homework problems, especially those worked in class. THERE WILL BE SEVERAL VERSIONS OF THE EXAM.

Please note that to receive full credit for a correct answer on your exam you must show work, where appropriate, to support your answer. In addition, the notation you use must be correct. For example, constant solutions must be expressed in the form $y(t) = k$ or $y(x) = k$. Please come to class with a sharp pencil and eraser.

1. Complete the chart.

<table>
<thead>
<tr>
<th>Differential equation</th>
<th>separable? (Y if yes)</th>
<th>Autonomous? (Y if yes)</th>
<th>linear? (Y if yes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $y'' = 2xy' + \sin x$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>2. $\frac{dT}{dt} = k(T - 20)$</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>3. $\frac{dx}{dt} = x^3$</td>
<td>Y</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. a) What is the interval in which the solution of $(t^2 - 9)y' + ty = 1$, $y(5) = 0$ is certain to exist?

$$\frac{y'}{t^2 - 9} + \frac{y}{t} = \frac{1}{t^2 - 9}$$

$p(t) = \frac{1}{t^2 - 9}, \quad \mu(t) = e^{\frac{t^2}{2}}$,

$$\frac{d}{dt} \left[ \frac{y(t)}{t^2 - 9} \right] = e^{\frac{t^2}{2}} \ln t^2 - 9$$

$$y(t) = e^{\frac{t^2}{2}} \ln t^2 - 9 + c$$

b) Find and completely simplify the integrating factor of $(t^2 - 9)y' + ty = 1$.

$$p(t) = \frac{1}{t^2 - 9}, \quad \mu(t) = e^{\frac{t^2}{2}}$$

$$\frac{d}{dt} \left[ \frac{y(t)}{t^2 - 9} \right] = e^{\frac{t^2}{2}}$$

$$y(t) = e^{\frac{t^2}{2}} \ln t^2 - 9 + c$$

3. a) Find an explicit solution of $y' + p(t)y = 2t^3 e^{-t}$, if the integrating factor is $\mu(t) = t^4$.

$$(y + p(t)y) t^{-4} = 2t^3 e^{-t} \Rightarrow \frac{d}{dt} [y + p(t)y] t^{-4} = 2t^3$$

$$\int \frac{d}{dt} [y + p(t)y] t^{-4} dt = \int 2t^3 e^{-t} dt + C \Rightarrow y(t) t^{-4} = e^{-t} + C$$

$$\Rightarrow y(t) = (e^{-t} + C)t^4$$

b) Find an explicit solution of $y' + p(t)y = 2t^3 e^{-t}$, $y(1) = 1$ if the integrating factor is $\mu(t) = t^{-3}$.

$$(y' + p(t)y) t^{-3} = 2t^3 e^{-t} \Rightarrow \frac{d}{dt} [y(t) t^{-3}] = 2t^3$$

$$\Rightarrow y(t) t^{-3} = 2t^3 \Rightarrow y(t) t^{-3} = 2t^3$$

$$\Rightarrow y(t) t^{-3} = 2t^3 \Rightarrow y(t) t^{-3} = 2t^3$$

$$\Rightarrow y(t) t^{-3} = 2t^3$$

$$\Rightarrow y(t) = \left[ \int 2t^3 ds + 2 \right] t^3$$
4. a) Solve \( \frac{dy}{dt} = 2ty^2 \) for an explicit (one parameter) solution.

\[
y^{-2} \, dy = 2t \, dt \quad \Rightarrow \quad \int y^{-2} \, dy = 2 \int t \, dt + C \quad \Rightarrow \quad -\frac{1}{y} = t^2 + C
\]
\[
\Rightarrow \quad \frac{1}{y} = C - t^2 \quad \Rightarrow \quad y(t) = \frac{1}{C - t^2} \quad \text{(also constant)} \quad y(t) = \frac{-1}{C + t^2}
\]

b) Note that \( y(3) = 0 \) yields a contradiction when plugged into \( y(t) = \frac{1}{C - t^2} \). Also note that \( y' = 0 \Rightarrow y(t) = 0 \) is a constant (in this case singular) solution of \( y' = 2ty^2 \).

b) What is the specific solution of \( \frac{dy}{dt} = 2ty^2, y(3) = 0? \)

5. a) What two conditions must be met by \( y' = \frac{y^3}{t^2 - 9}, \, y(t_0) = y_0 \), in order to satisfy the hypotheses of the General Existence and Uniqueness Theorem?

\[
y' = \frac{y^3}{t^2 - 9} \quad \text{and} \quad \frac{dy}{dt} = \frac{3y^2}{t^2 - 9} \quad \text{must be continuous over an open rectangle containing} \, (t_0, y_0)
\]

b) Where does \( y' = \frac{y^3}{t^2 - 9} \) fail to satisfy the hypotheses of the General Existence and Uniqueness Theorem?

\[
y \text{ is not continuous @ } t = \pm 3 \quad \text{and} \quad \frac{dy}{dt} \text{ is also not continuous @ } y = 0
\]

b) What does the General Existence and Uniqueness Theorem say about the following initial conditions for \( y' = \frac{y^3}{t^2 - 9} \)?

\[
(0,2) \quad \underline{A} \quad (2,0) \quad \underline{C} \quad (-3, 1) \quad \underline{C}
\]

A. There is a unique solution.
B. There is no solution.
C. No information.

Study guide for Test 1, p. 2
6. a) Sketch a phase line for \( y' = y(y-1)^2(y-2) \).

\[
\begin{array}{c|cccc}
 y' & 0 & 1 & 2 & 3 \\
 \hline
 y & - & + & - & + \\
 y^2 & - & + & + & - \\
\end{array}
\]

are equilibrium solutions.  

b) Identify the equilibrium solutions of \( y' = y(y-1)^2(y-2) \) as stable, unstable, or semistable.

\( y = 0 \) is stable
\( y = 1 \) is semistable
\( y = 2 \) is unstable

c) The limiting value of \( y(t) \) for \( 1 < y_0 < 2 \) is: \( \lim_{t \to \infty} y(t) = 1 \)

7. Solve \( yy' = \cos x \), \( y(0) = -2 \) for an explicit solution.

This is separable and non-linear.
\[
y \, dy = \cos x \, dx \\
\Rightarrow \int y \, dy = \int \cos x \, dx \\
\Rightarrow \frac{y^2}{2} = \sin x + C \\
\Rightarrow y^2 = 2 \sin x + 2C \\
\Rightarrow y(t) = -\sqrt{2 \int_0^x \cos s \, ds + 2C} \\
bc \ y_0 < 0
\]

8. A large tank is partially filled with 100 gallons of water in which 10 lb. of salt is dissolved. Pure water is pumped into the tank at a rate of 5 gal./min. The well-mixed solution is pumped out at the faster rate of 7 gal./min. Set up an initial value problem for this system.
a) What is the differential equation?

\[
\frac{dQ}{dt} = -\frac{Q}{100 - t}
\]

b) What is the initial condition? \( Q(0) = 10 \)

c) For how long is the model valid? 50 min \( \left( \frac{100}{2} \right) \)

Study guide for Test 1, p. 3
9. Empirical studies show that the rate at which new symbols can be memorized is proportional to the difference between the number of symbols already memorized, \( y(t) \), and a certain maximum amount of symbols, \( M \). Set up a differential equation for this model.

\[
\frac{dy}{dt} = k(y - M)
\]

10. When a cake is removed from a baking oven its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take to cool to the room temperature of 70°F? Set up an I.V.P. and solve.

Newton's law: \( \frac{dT}{dt} = k(T - 70) \) \( \Rightarrow \) \( \frac{dT}{T - 70} = kdt \) \( \Rightarrow \) \( \int \frac{dT}{T - 70} = \int kdt + C \)

\( \Rightarrow \ln|T - 70| = kt + C \Rightarrow T = 70 + Ce^{kt} \)

In this context \( T = 300 \) \( \Rightarrow \) \( T(0) = 300 \Rightarrow 300 = 70 + C \Rightarrow C = 230 \Rightarrow T = 70 + 230e^{kt} \)

To find \( c \) use \( T(3) = 200 \) \( \Rightarrow \) \( 200 = 70 + 230e^{3k} \) \( \Rightarrow e^{3k} = \frac{130}{230} \)

\( \Rightarrow 3k = \ln\left(\frac{13}{23}\right) \Rightarrow k = \frac{1}{3} \ln\left(\frac{13}{23}\right) \approx -1.9 \)

So \( T(t) = 70 + 230e^{-1.9t} \)