Section 3.6 Nonhomogeneous Linear Equations and the Method of Undetermined Coefficients

I. How to Solve Nonhomogeneous Equations, \( L[y] = g(t) \).

1. Find the general solution \( y_c(t) = c_1y_1(t) + c_2y_2(t) \) of the corresponding homogeneous equation \( L[y] = 0 \). This solution \( y_c(t) \) is often referred to as the complementary solution of \( L[y] = g(t) \).
2. Use the method of Undetermined Coefficients (or Variation of Parameters, see Section 3.7) to obtain a particular solution \( Y(t) \) of the nonhomogeneous equation \( L[y] = g(t) \).
3. The general solution of \( L[y] = g(t) \) is \( y(t) = y_c(t) + Y(t) \).

Example 1. Suppose the particular solution of \( y'' + 4y = \sin 2t \) is

\[ Y(t) = -\frac{1}{2}t \cos 2t - \frac{1}{16}t \sin 2t \]

Find the general solution of \( y'' + 4y = \sin 2t \).

Find \( y_c(t) \):

\[ y'' + 4y = 0 \]

has \( p(r) = r^2 + 4 = 0 \) \( \Rightarrow \) roots are \( \pm 2i \)

\( y_c = \{ \cos 2t, \sin 2t \} \) and \( y_c(t) = C_1 \cos 2t + C_2 \sin 2t \)

The g.s. of \( y'' + 4y = \sin 2t \) is \( y(t) = y_c(t) + Y(t) \)

\[ y(t) = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{2} t \cos 2t - \frac{1}{16} t \sin 2t \]

II. The method of Undetermined Coefficients can only be used to find a particular solution \( Y(t) \) of \( L[y] = g(t) \) if \( L \) is a constant-coefficient operator and if the terms of \( g(t) \) are constant multiples of functions found in fundamental sets of constant coefficient \( L[y] = 0 \). If \( L \) does not have constant coefficients or if \( g(t) \) is not a linear combination of nonnegative integer powers of \( t \) times exponentials times sines or cosines, then the method of Undetermined Coefficients will not work to find a particular solution \( Y(t) \) and the method of Variation of Parameters may be tried.

Example 2. For which o.d.e.s below is the method of Undetermined Coefficients not suitable for finding \( Y(t) \)?

a. \( y'' + y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \)\( \sqrt{\frac{1}{2b < 0 \text{ or } t > 0}} \)

b. \( 2y'' - 6y' + 4y = 6e^{2t} \)

c. \( [(t^2 + t)D^2 + (2 - t^2)D - (2 + t)]y = (t + 1)^2, \quad t > 0 \)\( \sqrt{\text{Variation of Parameters}} \)}
The method of Undetermined Coefficients

- The method of Undetermined Coefficients is sometimes referred to as the method of “judicious guessing” because you have to make an educated “guess” about the form of the particular solution, \( Y(t) \).
- The name of the method comes from the fact that the form of \( Y(t) \) will involve unknown (undetermined) coefficients that subsequently have to be determined.
- Given the non-homogeneous equation \( L[y] = g(t) \), the steps in the method of Undetermined Coefficients to find \( Y(t) \) are as follows:
  1) Find the roots of the characteristic equation \( P(r) = 0 \) of \( L[y] = 0 \).
  2) “Guess” the form of the particular solution. In general, \( Y(t) \) will be a linear combination of functions of the same types as appear in the \( g(t) \) terms.

To determine the functions in \( Y(t) \) the following four general rules apply:

- A function in \( g(t) \) of the form \( t^n f(t) \) (\( n \) a non-negative integer) will give rise to \( n+1 \) terms in \( Y(t) \). The terms will correspond to values of \( t^i f(t) \), where \( i \) takes on all integer values from 0 to \( n \).
- If functions that appear in \( g(t) \) are in the fundamental set of \( L[y] = 0 \) and correspond to a root of the characteristic equation of multiplicity 1, multiply the functions by \( t \) when forming corresponding \( Y(t) \) terms.
- If functions that appear in \( g(t) \) are in the fundamental set of \( L[y] = 0 \) and correspond to a root of the characteristic equation of multiplicity 2, multiply the functions by \( t^2 \) when forming corresponding \( Y(t) \) terms.

### Table of Functions

<table>
<thead>
<tr>
<th>Functions in ( g(t) ) terms</th>
<th>Basic function(s)</th>
<th>Function root(s)</th>
<th>Corresponding functions in ( Y(t) ) terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( kt^n )</td>
<td>1, ( t )</td>
<td>0</td>
<td>1, ( t ), ..., ( t^n )</td>
</tr>
<tr>
<td>( kt^n e^{rt} )</td>
<td>( e^{rt} )</td>
<td>( r )</td>
<td>( e^{rt}, te^{rt}, ..., t^n e^{rt} )</td>
</tr>
<tr>
<td>( kt^n \cos \mu t ) or ( \sin \mu t )</td>
<td>( \cos \mu t ), ( \sin \mu t )</td>
<td>( \pm i \mu t )</td>
<td>( \cos \mu t, \sin \mu t, t \cos \mu t, ... )</td>
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<tr>
<td>( \pm i \mu t )</td>
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<tr>
<td>( kt^n e^{\sigma t} \cos \mu t ) or ( \sin \mu t )</td>
<td>( e^{\sigma t} \cos \mu t ), ( e^{\sigma t} \sin \mu t )</td>
<td>( \lambda \pm i \mu t )</td>
<td>( t e^{\sigma t} \cos \mu t, t e^{\sigma t} \sin \mu t, ... )</td>
</tr>
<tr>
<td>( t e^{\sigma t} \cos \mu t, t e^{\sigma t} \sin \mu t )</td>
<td></td>
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</tr>
</tbody>
</table>

- Cross out any repeated functions in \( Y(t) \) and any functions that already appear in the fundamental set of \( L[y] = 0 \).

3) Substitute the form of \( Y(t) \) into \( L[y] = g(t) \).
4) Compare coefficients to find determine the coefficients of \( Y(t) \).

\[ m = \text{multiplicity} \]

Sec. 3.6, Boyce & DiPrima, p. 2
In the examples below we will refer to the non-homogeneous equation as \( N \).

**Example 3.** Consider \( y'' - 4y' = g(t) \).

a) If \( g(t) = 6e^t \), find \( Y(t) \) and the specific solution corresponding to \( y(0) = 0, y'(0) = 1 \).

1) Find \( y_c \): \( r^2 - 4r = 0 \) \( \Rightarrow \) \( r(r - 4) = 0 \) \( \Rightarrow \) \( r_1 = 0, r_2 = 4 \) \( \Rightarrow \) \( y_c(t) = c_1 + c_2 e^{4t} \)

so \( \frac{y_c}{c_1 + c_2 e^{4t}} \)

2) \( g(t) \) is \( \text{root of} \) \( \text{multiplicity of} \) \( \text{root in} \) \( \text{or in} \) \( \text{or in} Y(t) \)

| \( g(t) \) | \( \text{root} \) | \( \text{multiplicity of} \) \( \text{root in} \) \( \text{or in} \) \( \text{or in} \) \( Y(t) \) |
|---|---|---|---|---|---|
| \( e^t \) | \( 1 \) | \( 0 \) | \( e^t \) | \( \Rightarrow \) \( -2A = 6 \) \( \Rightarrow \) \( A = -3 \) \( \Rightarrow \) \( Y(t) = -2e^t \)

The g.s. of \( N \) is \( \frac{y(t) = c_1 + c_2 e^{4t} - 2e^t}{y(t) = y_c(t)} \)

\[ \begin{align*}
y(0) = 0 & \Rightarrow c_1 + c_2 - 2 = 0 \Rightarrow c_1 + c_2 = 2 \\
y'(0) = 1 & \Rightarrow 4c_2 - 2 = 1 \Rightarrow 4c_2 = 3 \Rightarrow c_2 = \frac{3}{4} \\
& \Rightarrow c_1 = \frac{5}{4}
\end{align*} \]

The specific solution is \( Y(t) = 5e^t + \frac{3}{4} e^{4t} - 2e^t \)

b) If \( g(t) = 4te^t \), find the general solution of \( y'' - 4y' = g(t) \).

2) \( g(t) \) is \( \text{root of} \) \( \text{multiplicity of} \) \( \text{root in} \) \( \text{or in} \) \( \text{or in} \) \( Y(t) \)

| \( g(t) \) | \( \text{root} \) | \( \text{multiplicity of} \) \( \text{root in} \) \( \text{or in} \) \( \text{or in} \) \( Y(t) \) |
|---|---|---|---|---|---|
| \( te^t \) | \( 1 \) | \( 0 \) | \( e^t, te^t \) | \( \Rightarrow \) \( Y(t) = Ae^t + Bte^t \)

3) \( y' = Ae^t + Bte^t + Ce^t = (A + B)e^t + Bte^t \)

\[ y'' = Ae^t + Bte^t + Ce^t + Be^t = (A + 2B)e^t + Bte^t \]

Substitute into \( N \):

\[ (A + 2B)e^t + Bte^t - 4(A + B)e^t - 4Bte^t = 4te^t \]

4) Compare coefficients:

\[ te^t \text{ terms: } B - 4B = 4 \Rightarrow -3B = 4 \Rightarrow B = -\frac{4}{3} \]

\[ e^t \text{ terms: } A + 2B - 4A - 4B = 0 \Rightarrow -3A - 2B = 0 \]

\[ \Rightarrow -3A - 2\left(-\frac{4}{3}\right) = 0 \]

\[ \Rightarrow -3A - \frac{8}{3} = 0 \Rightarrow A = \frac{8}{3} \]

so \( Y(t) = \frac{8t}{3} e^t - \frac{4}{3} te^t \)

The g.s. is \( \frac{y(t) = c_1 + c_2 e^{4t} + \frac{8t}{3} e^t - \frac{4}{3} te^t}{y(t) = \text{or in} Y(t)} \)

Sec. 3.6, Boyce & DiPrima, p. 3
c) If \( g(t) = 5 - 4t \), find the particular solution of \( y'' - 4y' = g(t) \).

\[
\begin{array}{c|c|c|c}
\text{Root} & \text{Root} & \text{Root} \\
\text{Value} & \text{Value} & \text{Value} \\
1 & t & 1 \\
\hline
t & 0 & t^2 \\
\hline
\end{array}
\]

\[ y(t) = A + 2Bt^2 \]

\[
\begin{aligned}
y' &= A + 2Bt \\
y'' &= 2B \\
y'' - 4y' &= 5 - 4t \\
(2B - 4A) - 8Bt &= 5 - 4t \\
2B - 4A &= 5 \\
2B - 4A &= 5 \\
-4B &= -4t \\
A &= 1
\end{aligned}
\]

Thus, \( y(t) = -t + \frac{1}{2}t^2 \)

d) What happens if you try to find the particular solution of part c) above with the (incorrect) form for the particular solution, \( Y(t) = A + Bt \)?

\[
\begin{aligned}
y' &= B \\
y'' &= 0 \\
-4B &= 5 - 4t \\
0 - 4B &= 5 - 4t \\
B &= -\frac{1}{2}
\end{aligned}
\]

When you try to compare coefficients of the \( t \)-terms, you obtain a contradiction: \( 0 = -4 \)

Moral: If in comparing coefficients you run into a contradiction, something is wrong!
e) If \( g(t) = 4\sin 2t \), find the particular solution of \( y'' - 4y' = g(t) \).

\[
\begin{array}{c|c|c|c|c}
\text{root} & \text{multiplicity} & \text{frees in } y(t) & \text{frees in } g(t) \\
\hline
\sin 2t & 1 & 0 & 0 \\
\hline
\end{array}
\]

\[
y(t) = \text{Asin} 2t + \text{Bcos} 2t \\
y' = 2\text{Acos} 2t - 2\text{Asin} 2t \\
y'' = -4\text{Acos} 2t - 4\text{Bsin} 2t
\]

\[
\begin{cases}
Y'' - 4Y' - 16Y = 0 \\
-9A + 8B = 1 \\
-8A - 4B = 0
\end{cases}
\]

\[
\begin{align*}
\text{Sine terms: } & -9A + 8B = 1 \\
\text{Cosine terms: } & -8A - 4B = 0
\end{align*}
\]

\[
A = \frac{-1}{5} \\
B = \frac{2}{5}
\]

\[
y(t) = \frac{-1}{5} \sin 2t + \frac{2}{5} \cos 2t
\]

Example 4. Find the form of \( Y(t) \) for \( y'' - 4y' + 5y = 3te^{2t} \cos t - t^3 + t^2 e^t \)

1) Find \( r(t) = t^2 - 4t + 5 = 0 \Rightarrow r = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i \)

\[
\begin{array}{c|c|c|c|c}
\text{root} & \text{multiplicity} & \text{frees in } y(t) & \text{frees in } g(t) \\
\hline
te^{2t} \cos t & 2 \pm i & 1 & t(e^{2t} \cos t, e^{2t} \sin t, t e^{2t} \cos t, t e^{2t} \sin t) \\
t^3 & 0 & 0 & t, t^3, t^3 \\
t^2 e^t & 1 & 0 & e^t, t e^t, t^2 e^t
\end{array}
\]

\[
Y(t) = \text{Ate}^{2t} \cos t + \text{Bte}^{2t} \sin t + \text{Cte}^{2t} \cos t + \text{Dte}^{2t} \sin t + E + P6 + Gt^3 + Ht^3 + Ie^t + Jte^t + Kt^2 e^t
\]

Sec. 3.6, Boyce & DiPrima, p. 5