Section 3.3, Linear Independence and the Wronskian

**Definition:** Two functions $f$ and $g$ are said to be **linearly dependent** on an interval $I$ if there exist two constants $k_1$ and $k_2$, not both zero, such that $k_1 f(t) + k_2 g(t) = 0$ (defining equation) for all $t$ in $I$. The functions $f$ and $g$ are said to be **linearly independent** on an interval $I$ if they are not linearly dependent, i.e., if $k_1 f(t) + k_2 g(t) = 0$ only if $k_1 = k_2 = 0$.

Note that $f$ and $g$ are linearly dependent on some interval $I$ if $g$ is a multiple of $f$.

[To test a set of functions for linear independence/dependence on $I$ where it is not obvious that one is a multiple of the other follow these steps: 1) Insert the functions in question into the defining equation. 2) Obtain two homogeneous equations using two different values for $t$ ($t \in I$) or letting the second equation be the derivative of the first. 3) Take the coefficient determinant of the two equations. If the determinant $\neq 0$ there is a unique solution, namely, $k_1 = k_2 = 0$ which tells you that the two functions are linearly independent on $I$. If the determinant $= 0$ this is inconclusive. In this case algebraically solve for the values of the $k_i$.]

**Example 1.** Determine whether the following sets of functions are linearly independent or dependent on $(-\infty, \infty)$.

a) $f(t) = t$, $g(t) = 2t$

b) $f(t) = t + 3$, $g(t) = 2t$

c) $f(t) = t$, $g(t) = |t|$
Example 2. Suppose the functions \( y_1(t) = t^2 + \frac{5}{t} \), \( y_2(t) = \frac{2}{t} \) are known to be solutions of the same 2nd-order \( L[y] = 0 \).

a) Use the Wronskian to determine the intervals on which these solutions are linearly independent.

b) What is the most simplified version of the fundamental set?

Example 3. Use the Wronskian to determine which set \( \{ e^{2t}, e^t \} \) or \( \{ e^{2t}, 3e^{2t} \} \) could not be a fundamental set of a 2nd order \( L[y] = 0 \). *

The superposition and proportionality principles that characterize \( L[y] = P(t)y'' + Q(t)y' + R(t)y = 0 \) tell us that any multiple of a solution or linear combination of solutions of \( L[y] = 0 \) is also a solution of \( L[y] = 0 \).

Example 4. Show that if \( y_1(t) \) and \( y_2(t) \) are solutions of \( L[y] = 0 \), then so is \( y(t) = c_1y_1(t) + c_2y_2(t) \).

* In Section 3.2 our text shows that if \( y_1 \) and \( y_2 \) are solutions of the 2nd-order \( L[y] = 0 \) and if \( W(y_1, y_2) \neq 0 \) for all \( t \in I \), then \( \{ y_1, y_2 \} \) is a fundamental set of solutions of \( L[y] = 0 \) on \( I \) from which we can write the general solution. When we put this together with the fact that, if \( W(y_1, y_2) \neq 0 \) on \( I \) then \( y_1 \) and \( y_2 \) are linearly-independent solutions of the 2nd-order \( L[y] = 0 \) on \( I \), we can see that a fundamental set must be a linearly-independent set of solutions of \( L[y] = 0 \) on \( I \).
Example 5. Given that $y_1(t) = e^{2t}$ and $y_2(t) = e^t$ are solutions of the same $2^{nd}$ order $L[y] = 0$, which functions below are also solutions of this same $L[y] = 0$?

a) $y_3(t) = 3e^{2t}$  
 b) $y_4(t) = 0$  
 c) $y_5(t) = e^{3t} = e^{2t}e^t$  
 d) $y_6(t) = 3e^{2t} - 4e^t$

PROPERTIES OF THE FUNDAMENTAL SET OF $L[y] = 0$

Fundamental sets in “closed form”

As it turns out we are able to find fundamental sets in “closed form” of $L[y] = 0$ only if $L[y]$ has constant coefficients or in limited cases where $L[y]$ has variable coefficients. Otherwise, we have to resort to series or other methods of representing general solutions of $L[y] = 0$.

How do we find fundamental sets of homogeneous linear equations $L[y] = 0$ in “closed form”?  

- If $L[y]$ has constant coefficients, or if $L[y]$ has a certain pattern of variable coefficients (Euler equations)†, the fundamental set is found by assuming the form of the solutions: In the case of constant coefficient $L[y]$ the form is $y = e^{rt}$. In the case of Euler equations the form is $y = t^r$.
- Inserting the form of the solution into $L[y] = 0$ yields a characteristic polynomial equation in the case of constant coefficient $L[y]$ and a similar kind of equation in the Euler case.
- The roots of these equations determine the functions of the fundamental sets of $L[y] = 0$.
- Note that if $L[y]$ is second order there will be two functions in the fundamental set. If $L[y]$ is third order there will be three functions in the fundamental set. Etc.

Equivalent Statements for Solutions of $2^{nd}$-Order $L[y] = 0$

Given the $2^{nd}$-Order $L[y] = 0$, $P(t)y''+Q(t)y'+R(t)y = G(t)$, the following statements are equivalent:

1. The fundamental set $\{ y_1(t), y_2(t) \}$ is a set of two solutions of $L[y] = 0$ defined on some $t$-interval $I$.
2. The two solutions of the fundamental set of $L[y] = 0$ are linearly independent on $I$.
3. The Wronskian of the solutions $\{ y_1(t), y_2(t) \}$ exists and will never equal zero on $I$.
4. The general solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$, is a linear combination of $\{ y_1(t), y_2(t) \}$.
5. The Existence and Uniqueness Theorem guarantees a unique solution for all initial conditions with $t_0 \in I$.
6. The specific solution of the I.V.P. of $L[y] = 0$ with $t_0 \in I$ will be defined throughout the interval $I$.

† We will consider Euler o.d.e.s in Section 5.5.