Section 3.2, Fundamental Solutions of Linear Homogeneous Equations

Existence and Uniqueness Theorem for Second Order Linear I.V.P.s
Consider the initial value problem $y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$, where $p$, $q$, and $g$ are continuous on an open interval $I$. Then there is a unique solution $y(t) = \Phi(t)$ of this problem, and the solution exists throughout the interval $I$.

This theorem says three things:
1. The initial value problem has a solution; in other words, a solution exists.
2. The initial value problem has only one solution; that is, the solution is unique.
3. The solution $\Phi$ is defined throughout the interval $I$ where the coefficients and $g(t)$ are continuous and is at least twice differentiable there.

Example 1.

a) Find all intervals in which an initial value problem $(t-3)y'' + 5y' - y = t, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$ is certain to have a unique twice differentiable solution.

b) What is the longest interval in which an initial value problem $(t-3)y'' + 5y' - y = t, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$ is certain to have a unique twice differentiable solution if $t_0 = 4$?

c) What is the longest interval in which an initial value problem $(t-3)y'' + 5y' - y = t, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$ is certain to have a unique twice differentiable solution if $t_0 = -2$?

d) What does the Existence and Uniqueness Theorem for Second Order Linear I.V.P.s tell us about a solution of the initial value problem $(t-3)y'' + 5y' - y = t, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$ is certain to have a unique twice differentiable solution if $t_0 = 3$?

Example 2. Prove that for $ay'' + by' + cy = 0$ the interval of solution is always $(-\infty, \infty)$.
The operator $L^*$

The $n^{th}$ order linear o.d.e. $y^{(n)} + p(t)y^{(n-1)} + q(t)y = g(t)$ can be represented using the linear differential operator $L[y] = y^{(n)} + p(t)y^{(n-1)} + q(t)y$. Using the linear differential operator $L[y]$ we can write $y^{(n)} + p(t)y^{(n-1)} + q(t)y = g(t)$ as $L[y] = g(t)$. The corresponding homogeneous equation can be written as $L[y] = 0$. Of course the exact composition of $L[y]$ will vary from one linear o.d.e. to another.

Example 3. The o.d.e. $\frac{d^2x}{dt^2} - e^t \frac{dx}{dt} + tx = \sin t$ could be written as $L[x] = \sin t$. What is $L$ in this case?

$L[y]$ can be used to operate on functions.

Example 4. Given $y^{(n)} - 5y^{(n-1)} + 4y = 0$.

a) Identify $L[y]$ and use it to operate on $y = e^{4t}$. Is $y = e^{4t}$ a solution of $L[y] = 0$?

b) Find $L[t]$.

Based on the result, we can conclude that $y = t$ is a solution of $L[y] = \underline{\text{________}}$

Example 5. Determine which, $y_1(t) = t$ or $y_2(t) = t^2$, is a solution of $t^2y'' + 2ty' - 2y = 0, \quad 0 < t < \infty$

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* The operator $L$ is introduced in Section 3.2, pages 143-4 of our text.

† Sometimes $L$ is represented using the differential operator, $D$. For example, $Dx = x'$ and $D^2x = x''$. If $x = e^{3t}$, then $Dx = 3e^{3t}$ and $D^2x = 9e^{3t}$. We could also write $D(e^{3t}) = 3e^{3t}$ and $D^2(e^{3t}) = 9e^{3t}$. Using the $D$ operator, $L$ can be written as $L = D^2 + p(t)D + q(t)$. The corresponding homogeneous differential equation can be written as $L[y] = [D^2 + p(t)D + q(t)]y = 0$. 

Sec. 3.2, Boyce & DiPrima, p.2
The linear differential operator \( L[y] \) satisfies the following principles:

1. **Principle of superposition**: \( L[y_1 + y_2] = L[y_1] + L[y_2] \).
2. **Principle of proportionality**: \( L[c y] = c L[y] \).

**Example 6.** If \( L[y_1] = 3 \) and \( L[x_2] = 5 \), what is \( L[2y_1 + 5y_2] \)?

Every linear operator satisfies the principles of superposition and proportionality. It is this fact of linear operators that provides the theoretical basis for writing a general solution of \( y'' + p(t)y' + q(t)y = 0 \) as a linear combination of the fundamental set of solutions.

The **Wronskian** of a fundamental set of solutions \( \{ y_1(t), y_2(t) \} \) of \( y'' + p(t)y' + q(t)y = 0 \) is defined as the determinant 

\[
W(y_1(t), y_2(t)) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.
\]

**Example 7.** Find the Wronskian of \( \{ e^{2t}, e^{-2t} \} \).

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**Theorem.** If \( y_1 \) and \( y_2 \) are two solutions of the differential equation 

\( L[y] = y'' + p(t)y' + q(t)y = 0 \) and if there is a point \( t_o \) where the Wronskian of \( y_1 \) and \( y_2 \) is nonzero, then the family of solutions \( y(t) = c_1 y_1(t) + c_2 y_2(t) \) includes every solution of \( L[y] = y'' + p(t)y' + q(t)y = 0 \).

We may generalize this theorem as follows: If the Wronskian of \( y_1 \) and \( y_2 \) is nonzero on some interval \( I \) then

1. The functions \( \{ y_1(t), y_2(t) \} \) form a fundamental set of solutions of \( y'' + p(t)y' + q(t)y = 0 \) on \( I \), and
2. The general solution of \( y'' + p(t)y' + q(t)y = 0 \) is \( y(t) = c_1 y_1(t) + c_2 y_2(t) \) with domain equal to \( I \).

**Example 8.** On what interval \( I \) does the set \( \{ e^{2t}, e^{-2t} \} \) form a fundamental set of solutions for \( ay'' + by' + cy = 0 \)? What is the domain of the general solution?

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**Supplemental submit problem:**
If \( L[y] = t^3\ y'' - ty' + y \), find \( g(t) \) for which \( y = t^3 \) is a solution of \( L[y] = g(t) \).