Section 3.2, Fundamental Solutions of Linear Homogeneous Equations

Existence and Uniqueness Theorem for Second Order Linear I.V.P.s
Consider the initial value problem \( y'' + p(t)y' + q(t)y = g(t) \), \( y(t_0) = y_0 \), \( y'(t_0) = y'_0 \), where \( p, q, \) and \( g \) are continuous on an open interval \( I \). Then there is a unique solution \( y(t) = \Phi(t) \) of this problem, and the solution exists throughout the interval \( I \).

This theorem says three things:
1. The initial value problem has a solution; in other words, a solution exists.
2. The initial value problem has only one solution; that is, the solution is unique.
3. The solution \( \Phi \) is defined throughout the interval \( I \) where the coefficients and \( g(t) \) are continuous and at least twice differentiable there.

Example 1.

a) Find all intervals in which an initial value problem \((t - 3)y'' + 5y' - y = t \), \( y(t_0) = y_0 \), \( y'(t_0) = y'_0 \) is certain to have a unique twice differentiable solution.

\[
y'' + \frac{\frac{5}{t-3}y'}{t-3} - \frac{1}{t-3}y = \frac{t}{t-3} \quad \text{if } p(t), q(t) \text{ and } g(t) \text{ are not continuous at } t = 3
\]

standard form \( \Rightarrow \) two intervals: \(( -\infty, 3), (3, \infty)\)

b) What is the longest interval in which an initial value problem \((t - 3)y'' + 5y' - y = t \), \( y(t_0) = y_0 \), \( y'(t_0) = y'_0 \) is certain to have a unique twice differentiable solution if \( t_0 = 4 \)?

\(( 3, \infty)\)

c) What is the longest interval in which an initial value problem \((t - 3)y'' + 5y' - y = t \), \( y(t_0) = y_0 \), \( y'(t_0) = y'_0 \) is certain to have a unique twice differentiable solution if \( t_0 = -2 \).

\((-\infty, 3)\)

d) What does the Existence and Uniqueness Theorem for Second Order Linear I.V.P.s tell us about a solution of the initial value problem \((t - 3)y'' + 5y' - y = t \), \( y(t_0) = y_0 \), \( y'(t_0) = y'_0 \) is certain to have a unique twice differentiable solution if \( t_0 = 3 \)?

No information

Example 2. Prove that for \( ay'' + by' + cy = 0 \) the interval of solution is always \((-\infty, \infty)\).

In standard form: \( y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \)

\( p(t) = \frac{b}{a} \), \( q(t) = \frac{c}{a} \), \( g(t) = 0 \) are constants so there are no discontinuities, so the interval of solution guaranteed by the Existence and Uniqueness Theorem is \((-\infty, \infty)\).

Sec. 3.2, Boyce & DiPrima, p.1
The operator \( L \)

The \( n \)th order linear o.d.e. \( y^{(n)} + p(t)y^{(n-1)} + q(t)y = g(t) \) can be represented using the **linear differential operator** \( L[y] = y^{(n)} + p(t)y^{(n-1)} + q(t)y \). Using the linear differential operator \( L[y] \) we can write \( y^{(n)} + p(t)y^{(n-1)} + q(t)y = g(t) \) as \( L[y] = g(t) \). The corresponding homogeneous equation can be written as \( L[y] = 0 \). Of course the exact composition of \( L[y] \) will vary from one linear o.d.e. to another.

**Example 3.** The o.d.e. \( \frac{d^2x}{dt^2} - e^t \frac{dx}{dt} + tx = \sin t \) could be written as \( L[x] = \sin t \). What is \( L \) in this case?

\[
L[x] = x'' - e^t x' + tx
\]

\( L[y] \) can be used to operate on functions.

**Example 4.** Given \( y^{(n)} - 5y' + 4y = 0 \).

a) Identify \( L[y] \) and use it to operate on \( y = e^{4t} \). Is \( y = e^{4t} \) a solution of \( L[y] = 0 \)?

\[
L[y] = y'' - 5y' + 4y \quad \text{Fm.} \quad y = e^{4t} \quad \begin{align*}
y' &= 4e^{4t} \\
y'' &= 16e^{4t}
\end{align*}
\]

So \( y = e^{4t} \) is a soln of \( L[y] = 0 \)

b) Find \( L[y] \).

\[
\begin{align*}
y &= e^{4t} \\
y' &= 4e^{4t} \\
y'' &= 16e^{4t}
\end{align*}
\]

Based on the result, we can conclude that \( y = t \) is a solution of \( L[y] = \frac{4t}{-5} \)

**Example 5.** Determine which, \( y_1(t) = t \) or \( y_2(t) = t^2 \), is a solution of \( y'' + \frac{1}{t}y' - 2y = 0 \), \( 0 < t < \infty \)

\[
\begin{align*}
y_1(t) &= t \\
y_1' &= 1 \\
y_1'' &= 0
\end{align*}
\]

\[
\begin{align*}
y_2(t) &= t^2 \\
y_2' &= 2t \\
y_2'' &= 2
\end{align*}
\]

\[
L[y_1] = t^2 + \frac{1}{t} + 2t \quad \text{so} \quad y_1 = t \quad \text{is a soln of} \quad L[y] = 0
\]

\[
L[y_2] = t^2 + 2t + 2t - 2t^2 = 4t^2 \neq 0 \quad \text{so} \quad y_2 = t^2 \quad \text{is not a soln of} \quad L[y] = 0
\]

* The operator \( L \) is introduced in Section 3.2, pages 143–4 of our text.

† Sometimes \( L \) is represented using the **differential operator** \( D \). For example, \( Dx = x' \) and \( D^2x = x'' \). If \( x = e^t \), then \( Dx = 3e^t \) and \( D^2x = 9e^t \). We could also write \( D(e^t) = 3e^t \) and \( D^2(e^t) = 9e^t \). Using the \( D \) operator, \( L \) can be written as \( L = D^2 + p(t)D + q(t) \). The corresponding homogeneous differential equation can be written as \( L[y] = [D^2 + p(t)D + q(t)]y = 0 \).

Sec. 3.2, Boyce & DiPrima, p.2
The linear differential operator $L[y]$ satisfies the following principles:

1. **Principle of superposition**: $L[y_1 + y_2] = L[y_1] + L[y_2]$.

**Example 6.** If $L[y_1] = 3$ and $L[y_2] = 5$, what is $L[2y_1 + 5y_2]$?

\[
L[2y_1 + 5y_2] = 2L[y_1] + 5L[y_2] = 2 \cdot 3 + 5 \cdot 5 = 31
\]

Every linear operator satisfies the **principles of superposition and proportionality**. It is this fact of linear operators that provides the theoretical basis for writing a general solution of $y'' + p(t)y' + q(t)y = 0$ as a linear combination of the fundamental set of solutions.

**The Wronskian** of a fundamental set of solutions $\{y_1(t), y_2(t)\}$ of $y'' + p(t)y' + q(t)y = 0$ is defined as the determinant $W(y_1(t), y_2(t)) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$.

**Example 7.** Find the Wronskian of $\{e^t, e^{-2t}\}$.

\[
W = \begin{vmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{vmatrix} = e^t(-2e^{-2t}) - e^{-2t}e^t = -2 - 2 = -4
\]

**Theorem.** If $y_1$ and $y_2$ are two solutions of the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$ and if there is a point $t_0$ where the Wronskian of $y_1$ and $y_2$ is nonzero, then the family of solutions $y(t) = c_1y_1(t) + c_2y_2(t)$ includes every solution of $L[y] = y'' + p(t)y' + q(t)y = 0$.

We may generalize this theorem as follows: If the Wronskian of $y_1$ and $y_2$ is nonzero on some interval $I$ then

1) The functions $\{y_1(t), y_2(t)\}$ form a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ on $I$, and
2) The general solution of $y'' + p(t)y' + q(t)y = 0$ is $y(t) = c_1y_1(t) + c_2y_2(t)$ with domain equal to $I$.

**Example 8.** On what interval $I$ does the set $\{e^t, e^{-2t}\}$ form a fundamental set of solutions for $ay'' + by' + cy = 0$? What is the domain of the general solution?

\[(-\infty, \infty) \text{ because } W(e^t, e^{-2t}) \neq 0 \text{ for all } t.\]

**Supplemental submit problem:**

If $L[y] = t^2y'' - ty' + y$, find $g(t)$ for which $y = t^3$ is a solution of $L[y] = g(t)$.

Sec. 3.2, Boyce & DiPrima, p.3