Section 3.1, Second Order Linear Equations

A general 2\textsuperscript{nd} order linear equation can be written: \( P(t)y'' + Q(t)y' + R(t)y = G(t) \).

To put this equation into standard form we divide by \( P(t) \) to obtain
\[
y'' + p(t)y' + q(t)y = g(t)
\]

This equation is nonhomogeneous if \( g(t) \neq 0 \) identically for all \( t \), and homogeneous if \( g(t) = 0 \). The nonhomogeneous term, \( g(t) \), is sometimes referred to as a “forcing function.”

**Example 1.** Which of the following o.d.e.s are linear? Of the linear equations, which are homogeneous?

<table>
<thead>
<tr>
<th>o.d.e.</th>
<th>linear?</th>
<th>homogeneous?</th>
</tr>
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<tbody>
<tr>
<td>( \frac{d^2y}{dt^2} + 2t^2y^2 = 0 )</td>
<td></td>
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<tr>
<td>( \frac{d^3y}{dx^3} = -y \sin x )</td>
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<tr>
<td>( t^2y'' - ty' - sint = 0 )</td>
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<tr>
<td>( x\frac{d^2x}{dt^2} + t\frac{dx}{dt} = t )</td>
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**Constant Coefficient homogeneous 2\textsuperscript{nd} order linear o.d.e.s**

In the case where the coefficients of \( P(t)y'' + Q(t)y' + R(t)y = 0 \) are constants we have a constant-coefficient 2\textsuperscript{nd} order homogeneous linear equation \( ay'' + by' + cy = 0 \).

Recall that the 1\textsuperscript{st} order constant-coefficient homogeneous linear equation \( y' = ry \) has the general solution \( y(t) = ce^{rt} \). Analogously, it makes sense that solutions of second-order constant-coefficient linear homogeneous o.d.e.s will involve exponentials and will have two arbitrary constants.

**Example 2.** Assume that \( y = e^r \) is a solution of \( ay'' + by' + cy = 0 \). For what \( r \) values is this assumption valid?
As we saw in Example 3, \( ay'' + by' + cy = 0 \) has two solutions of the form \( y = e^{rt} \) corresponding to the two roots of the equation \( ar^2 + br + c = 0 \). The equation \( ar^2 + br + c = 0 \) is called the \textbf{characteristic equation} of \( ay'' + by' + cy = 0 \). It is easy to obtain the characteristic equation from \( ay'' + by' + cy = 0 \) by setting \( y'' = r^2, y' = r \) and \( y = 1 \).

Suppose the characteristic equation has two different real roots, \( r_1 \) and \( r_2 \). Then the two solutions corresponding to \( r_1 \) and \( r_2 \) are \( y_1 = e^{r_1 t} \) and \( y_2 = e^{r_2 t} \). We say that \( y_1 = e^{r_1 t} \) and \( y_2 = e^{r_2 t} \) form a \textbf{fundamental set of solutions} \{ \( e^{r_1 t}, e^{r_2 t} \) \}. Note the correlation between the fact that we began with a 2nd-order linear constant-coefficient homogeneous o.d.e and the fact that the fundamental set has two solutions. The \textbf{general solution} of \( ar^2 + br + c = 0 \) is a \textbf{linear combination} of the two solutions of the fundamental set, i.e., the general solution is \( y(t) = c_1 y_1 + c_2 y_2 \) or \( y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \).

\textit{Example 3.} Find the fundamental set and general solution of \( y'' - 5y' + 4y = 0 \).

\textit{Example 4.} Find the fundamental set and general solution of \( y'' = y \).

\footnote{Our text introduces the notion of fundamental set on page 148 but does not use set notation.}
Example 5. Find the specific solution of the initial value problem, \( y'' - 5y' = 0 \), \( y(0) = 1 \), \( y'(0) = 3 \). Determine the behavior of this solution as \( t \to \infty \).

Working backwards

Example 6. The general solution of the constant-coefficient \( ar^2 + br + c = 0 \) is \( y(t) = c_1 e^{-2t} + c_2 e^{2t} \). If \( a = 1 \), find \( b \) and \( c \).

Example 7. The fundamental set of \( ar^2 + br + c = 0 \) is \{1, e^{3t}\}. If \( a = 1 \), find \( b \) and \( c \).