Section 2.4, Differences Between Linear and Nonlinear Equations

Review:
A first-order initial value problem (I.V.P.) is a first order o.d.e. \( y' = f(t, y) \), together with an initial condition, \( y(t_0) = y_0 \).
A solution of the o.d.e. \( y' = f(t, y) \) is a function \( \phi(t) \) defined for \( t \) in some interval \((a, b)\) which satisfies \( y' = f(t, y) \), that is, \( \phi'(t) = f(t, \phi(t)) \).
A solution of the I.V.P. \( y' = f(t, y) \), \( y(t_0) = y_0 \), is a solution \( \phi(t) \) of \( y' = f(t, y) \) defined on \((a, b)\) with \( t_0 \in (a, b) \) and with \( \phi(t_0) = y_0 \). In other words, the solution satisfies both the o.d.e. on \((a, b)\) and the initial condition.

Existence and Uniqueness Theorem for Linear First Order Equations

If the functions \( p \) and \( q \) are continuous on an open interval \( I: \alpha < t < \beta \) containing the point \( t = t_0 \), then there exists a unique function \( y = \phi(t) \) that satisfies the differential equation \( y'' + p(t)y' = q(t) \) for each \( t \) in \( I \), and that also satisfies the initial condition \( y(t_0) = y_0 \), where \( y_0 \) is an arbitrary prescribed initial value. The interval \((a, b)\) is referred to as the **interval of definition** of the specific solution.

Example 1. Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist (i.e., find the interval of definition of the I.V.P.).

a) \( ty'' + 2y' = 4t^2 \), \( y(1) = y_0 \).
   \[ p(t) = \frac{2}{t} \quad \text{is not continuous at } t = 0. \]
   This gives two intervals \((-\infty, 0) \cup (0, \infty)\). \( t_0 = 1 \in (-\infty, 0) \]

b) \( ty'' + 2y' = 4t^2 \), \( y(-1) = y_0 \).
   \[ t_0 = -1 \in (-\infty, 0) \]

c) Find the one-parameter (general) solution of \( ty'' + 2y' = 4t^2 \). Note that all solutions have a discontinuity at \( t = 0 \) with one exception. What is the one exception?
   \[ p(t) = \frac{2}{t} \Rightarrow \mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln |t|} = e^{2 \ln t} = t^2 \]
   \[ q(t) = 4t^2 \Rightarrow \frac{d}{dt} \left[ t^2 y(t) \right] = 4t^3 \]
   \[ s = \int t^2 y(t) dt = 4t^3 + C \Rightarrow t^2 y(t) = 4t^3 + C \]
   \[ y(t) = t^2 + \frac{C}{t^2} \] or \[ y(t) = -t^2 + \frac{C}{t^2} \]
   The solutions are discontinuous at \( t = 0 \) except when \( C = 0 \) (for example, when \( y(1) = 1 \)).

The Existence and Uniqueness Theorem specifies the smallest intervals on which solutions may be defined. Some specific solutions may be defined on larger intervals.

Boyce & DiPrima, Section 2.4 lecture, p.1
General Existence and Uniqueness Theorem for First Order Equations

Suppose that a first-order o.d.e. can be written in the form \( \frac{dy}{dt} = f(t, y) \). Suppose both \( f(t, y) \) and \( \frac{\partial f}{\partial y}(t, y) \) are continuous in a rectangular region \( R = \{(t, y) \mid \alpha < t < \beta \text{ and } \gamma < y < \delta \} \) (where any of the bounds may be infinite) containing the point \((t_0, y_0)\). Then, in some interval, \( t_0 - h < t < t_0 + h \) contained in \( \alpha < t < \beta \), there exists a unique solution \( y = \phi(t) \) of the initial value problem \( y' = f(t, y), \ y(t_0) = y_0 \).

Note that if \( \frac{dy}{dt} = f(t, y) \) is linear the General Existence and Uniqueness Theorem reduces to the Existence and Uniqueness Theorem for Linear First Order Equations.

Example 2. Answer the following questions:

a) Suppose a first order I.V.P. fails to satisfy one or both hypotheses, i.e., \( f(t, y) \) and/or \( \frac{\partial f}{\partial y}(t, y) \) are discontinuous at \((t_0, y_0)\). What does the theorem tell us in this case? Nothing. There may be a unique solution, no solution, or infinitely many solutions.

b) Is it possible to have a unique solution to a first order I.V.P. even though the hypotheses of the General Existence and Uniqueness Theorem are not satisfied?
   Yes.

c) Is it possible for the graphs of two solutions of a differential equation to intersect in a region where both hypotheses of the General Existence and Uniqueness Theorem are satisfied?
   No. The point of intersection would be a point where two solutions exist, contrary to the conclusion of the theorem.

Example 3. The one-parameter family of solutions of \( \frac{dy}{dt} = -\frac{t}{y} \) is \( y^2 + t^2 = C \).

a) Sketch the region(s) in the \( ty \)-plane where the hypotheses of the General Existence and Uniqueness Theorem are satisfied.
   Note that \( \frac{dy}{dt} = \frac{t}{y} \). Both \( y' \) and \( \frac{\partial f}{\partial y} = \frac{-t}{y} \) are not cont. at \( y = 0 \).

b) Describe all those points \((t_0, y_0)\) for which the theorem does not guarantee a unique solution.
   All \((t_0, y_0)\) where \( y_0 = 0 \); i.e., all points \((t_0, 0)\).

c) What is the explicit solution of the I.V.P. \( \frac{dy}{dt} = -\frac{t}{y}, \ f(3) = -4 \)?
   \( \frac{dy}{dt} = -\frac{t}{y} \). \( f(3) = -4 \).
   So \( y^2 + t^2 = 25 \) \( \Rightarrow \ y(t) = \pm \sqrt{25 - t^2} \).
   Choose the negative radicand because \( y(0) = -4 \).

d) What is the domain of the solution of the I.V.P. of part c? How does this domain depend on the initial condition? The initial condition determines the radius of the solution. (The explicit solution is a semicircle)
   Domain is \([-5, 5] \).

Boyce & DiPrima, Section 2.4 lecture, p.2
Example 4. Consider \( t^2 y' = y^2 \) with the one-parameter family of solutions \( y(t) = \frac{t}{1 + Ct} \) (see Example 1 of the lecture). 

a) Sketch the region(s) in the \( ty \)-plane where the hypotheses of the General Existence and Uniqueness Theorem are satisfied. 
\[ y' = \frac{y^2}{t^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{2y}{t^2} \text{ are not const. at } t = 0. \]

Two regions: 
- To the right and
- To the left of

The \( y \)-axis.

b) What does the General Existence and Uniqueness Theorem tell us about a solution to the I.V.P., \( t^2 y' = y^2, y(0) = 1 \)?

A. There is a unique solution.
B. There is no solution.
C. The Theorem gives no information about this I.V.P.

\[ t_0 = 0 \]

\[ l = \frac{0}{1 + 0} \rightarrow \text{(a contradiction). There is no solution.} \]

c) Analytically determine how many solutions there are to the I.V.P. of part b. If there is a unique solution, what is it? When we substitute \( y(t) = \frac{t}{1 + Ct} \) into \( y(t) = \frac{t}{1 + Ct} \), we get

\[ l = \frac{0}{1 + 0} \rightarrow \text{(a contradiction). There is no solution.} \]

d) What does the General Existence and Uniqueness Theorem tell us about a solution to the I.V.P., \( t^2 y' = y^2, y(1) = 0 \)?

A. There is a unique solution.
B. There is no solution.
C. The Theorem gives no information about this I.V.P.

c) Analytically determine how many solutions there are to the I.V.P. of part d. If there is a unique solution, what is it? Note that \( t^2 y' = y^2 \) has a constant solution \( y(t) = 0, \ y(0) = 0 \) yields a contradiction when substituted into \( y(t) = \frac{t}{1 + Ct} \). However, the solution \( y(t) = 0 \) passes through the pt. \((1,0)\) and is a singular solution.

Example 5. The hypothesis of the Linear Existence and Uniqueness Theorem is not satisfied for the \( y \)-I.V.P.
\[ y' + 1 = 0. \]

a) Is it possible that the hypotheses of the General Existence and Uniqueness Theorem will be satisfied for the I.V.P., \( (t - 1)y' = 0, y(1) = 0 \)?  
\( \text{No, the Linear Theorem is a special case of the General Theorem.} \)

b) Determine whether there are no solutions, more than one solution, or a unique solution satisfying this I.V.P.

Separating:
\[ \frac{dy}{y} = \frac{dt}{t-1} \Rightarrow \ln|y| = \ln|t-1| + C \Rightarrow y(t) = C(t-1) \]

\( y(1) = 0 \) for infinitely many values of \( C \), so there are infinitely many solutions.
Example 6. Consider the I.V.P. \((t^2 - 4)y' = y^{1/3}, \quad y(t_0) = \alpha.\)

a) Is \(y(t) = 0\) a solution of \((t^2 - 4)y' = y^{1/3}\)? **Yes**

b) Use the General Existence-Uniqueness Theorem to choose answers A-C below for the following initial conditions.

\[
\text{Note: } y' = \frac{y^{1/3}}{t^2 - 4} \text{ is not cont at } t = \pm 2; \quad \frac{dy}{dt} = \frac{1}{3y^{2/3}(t^2 - 4)} \text{ is not cont at } y = 0 \quad \text{and at } t = \pm 2
\]

i) \((-2, 2) \quad B\)  
ii) \((0, 2) \quad A\)  
iii) \((0, 0) \quad B\)

A. There is exactly one solution.
B. Unable to determine.
C. There is no solution.

Properties of Linear Equations
1. Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. A particular (specific) solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.
2. While there may be constant solutions, there will be no singular solutions.
3. An explicit solution \(y = \phi(t)\) can be found.
4. Discontinuities in the solution can only occur at points of discontinuity of the coefficients \(p\) or \(g\). Thus, if the coefficients are continuous for all \(t\), then the solution exists and is continuous for all \(t\). It is possible, however, for solutions to be continuous even at points of discontinuity of the coefficients (see Example 1c above). Thus, discontinuities of \(p\) and \(g\) give us the smallest interval(s) on which the solutions may be defined.

Properties of Nonlinear Equations
1. The one-parameter solution may not be a general solution.
2. There may be singular, constant solutions corresponding to \(y\)-values where \(y'\) is not defined.
3. It may be impossible to find an analytic solution, let alone an explicit analytic solution.
4. The interval of domain can only be determined if an explicit analytic solution can be found. In this case the discontinuities of the solution may depend in an essential way on the initial conditions as well as on the differential equation (see Example 3d above). In other words, the interval of domain may have no simple relationship to the function \(f\) in the differential equation \(\frac{dy}{dt} = f(t, y)\).