Section 2.2, Separation of Variables

Note: The method described below follows Example 2 (pp. 44-5) and Example 3 (pp. 46-7) of the text.

A first-order ordinary differential equation \( \frac{dy}{dx} = f(x, y) \) is separable if it can be put into the form \( M(x) + N(y) \frac{dy}{dx} = 0 \) or \( \frac{dy}{dx} = \frac{M(x)}{N(y)} \) or \( \frac{dy}{dx} = h(y)g(x) \). In other words, a first-order o.d.e., \( \frac{dy}{dx} = f(x, y) \), is separable if the rate expression \( f(x, y) \) is a product of \( x \)-factors and/or \( y \)-factors.

Separable differential equations can be either linear or nonlinear, autonomous or non-autonomous. They can be solved through integration after a simple algebraic manipulation to separate the variables.

Steps in solving:

1. **Look for any constant solutions.** These will occur at \( y \)-values for which \( y' = 0 \).
   
   Note that if \( y_0 \) is a value such that \( y'(x, y_0) = 0 \), then \( y(x) = y_0 \) is a solution of the I.V.P.
   
   \[
   \frac{dy}{dx} = \frac{M(x)}{N(y)}, \quad y(x_0) = y_0.
   \]

2. **Separate the variables.**
   
   Write \( y' \) in differential form and multiply by \( dx \):
   
   \[
   \frac{dy}{dx} = \frac{M(x)}{N(y)} \quad \Rightarrow \quad N(y)dy = M(x)dx
   \]

3. **Integrate both sides** as follows: \( \int N(y)dy = \int M(x)dx + C \)
   
   Note that the solution obtained will be a one-parameter family of solutions \( \Phi(x, y) = C \) which gives \( y \) as an implicit function of \( x \).

The complete set of solutions of \( \frac{dy}{dx} = \frac{M(x)}{N(y)} \) will consist of all the constant solutions together with all the solutions of the one-parameter family of solutions.

A constant solution \( y(x) = y_0 \) is said to be a **singular solution** if it does not satisfy \( \Phi(x, y) = C \) (i.e., \( y(x) = y_0 \) cannot be substituted into \( \Phi(x, y) = C \) to yield an identity).
Example 1. Consider $x^2 y' = y^2$

a) Find and graph the constant solution of $x^2 y' = y^2$.

b) Find a one-parameter solution of $x^2 y' = y^2$.

c) Find the solution of the I.V.P., $x^2 y' = y^2$, $y(1) = 0$.

d) Show that $y(x) = 0$ is a singular solution of $x^2 y' = y^2$.

e) Find the solution of the I.V.P., $x^2 y' = y^2$, $y(2) = 1$. Determine the interval in which the solution is valid.

f) What is the long term behavior of the specific solution of $x^2 y' = y^2$, $y(2) = 1$?
Example 2.

a) Solve the I.V.P. $3x + xy^2 - (y + x^2) \frac{dy}{dx} = 0$, $y(1) = -3$ for an explicit solution.

b) Graph the explicit specific solution of part a)

Direction field for $3x + xy^2 - (y + x^2) \frac{dy}{dx} = 0$
Method for solving the I.V.P. \( \frac{dy}{dx} = \frac{M(x)}{N(y)} \), \( y(x_0) = y_0 \) when one of the integrals of Step 3 above cannot be integrated.

After separating, solve the integral equation \( \int_{y_0}^{y(x)} N(v)dv = \int_{x_0}^{x} M(s)ds \).

*Example 3.* Find a specific solution for the I.V.P., \( xyy' = \sin x \), \( y(2) = 1 \).
[Note: The implicit specific solution of this I.V.P. was featured in Example 4 of the Section 1.3 lecture.]

Solution of \( xyy' = \sin x \), \( y(2) = 1 \).