Section 2.1, Linear Equations with Variable Coefficients

The method of integrating factors, due to Leibniz, is a technique for finding the general solution of a nonhomogeneous linear o.d.e.

A first order linear o.d.e. can be put into the form \( \frac{dy}{dt} + p(t)y = g(t) \). If \( g(t) \neq 0 \), this o.d.e. is said to be nonhomogeneous. If \( g(t) = 0 \), the o.d.e. is homogeneous.

To solve a first-order nonhomogeneous linear o.d.e. follow these steps:

1. Put into the form \( \frac{dy}{dt} + p(t)y = g(t) \). \((S)\)

   The integrating factor is defined to be \( \mu(t) = e^{\int p(t) dt} \).

   [Note: When evaluating the integral \( \int p(t) dt \) we do not need to add \( C \) because, when we multiply both sides of \((S)\) by the integrating factor in Step 2 below, these constants will cancel out.]

2. Multiply both sides of \((S)\) by \( \mu(t) \) to get

   \[ \mu(t) \left( \frac{dy}{dt} + p(t)y \right) = \mu(t)g(t). \]

   \((M)\)

   Note that generally the left-hand side of \((M)\) cannot be integrated. The key to getting beyond this step is to recognize that the left-hand side of \((M)\) is actually the product rule derivative of \( \mu(t)y(t) \), i.e.,

   \( \frac{d}{dt} [\mu(t)y(t)] = \mu(t)[y' + p(t)y] \).

   \( \frac{d}{dt} [\mu(t)y(t)] \) can be integrated because it is a derivative.

   Thus we substitute \( \frac{d}{dt} [\mu(t)y(t)] \) for the left-hand side of \((M)\) to obtain

   \[ \frac{d}{dt} [\mu(t)y(t)] = \mu(t)g(t). \]

   \((D)\)

3. Now integrate both sides of \((D)\), \( \int \frac{d}{dt} [\mu(t)y(t)] dt = \int \mu(t)g(t) dt + C \), to obtain the implicit general solution.

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1 We can state this as a Lemma: \( \frac{d}{dt} [\mu(t)y(t)] = \mu(t)[y' + p(t)y] \).

The proof is as follows: \( \frac{d}{dt} [\mu(t)y(t)] = \mu(t)y'(t) + \mu'(t)y(t) \), where \( \mu'(t) = e^{\int p(t) dt} \). Note that

   \[ \mu'(t) = \frac{d}{dt} e^{\int p(t) dt} = e^{\int p(t) dt} p(t) = \mu(t)p(t) \].

   Thus, \( \mu(t)y'(t) + \mu'(t)y(t) = \mu(t)y'(t) + \mu(t)p(t)y(t) = \mu(t)[y' + p(t)y] \).
Example 1. Find the general solution of $y' = y + e^t$.

1) Put in standard form: $y' - y = e^t$. $p(t) = -1$ so $\mu(t) = e^{\int -1 \, dt} = e^{-t}$

2) $e^{-t}(y' - y) = e^{-t}e^{-t} = 1$

3) $\frac{d}{dt} \left( e^{-t} y(t) \right) = \int 1 \, dt + C$

$y(t)e^{-t} = t + C$

$y(t) = (t + C)e^t$

Example 2. a) Find the general solution of $y' = \frac{2y}{t+1} + 3(t+1)^2$.

1) $y' - \frac{2y}{t+1} = 3(t+1)^2$

$\mu(t) = e^{\int \frac{2}{t+1} \, dt} = e^{2 \ln |t+1|} = e^{\ln |t+1|^2} = (t+1)^2$

2) $\frac{1}{(t+1)^2} \left( y - \frac{2y}{t+1} \right) = 3(t+1) \frac{1}{(t+1)^2} = 3$

$\frac{d}{dt} \left[ \frac{1}{(t+1)^2} y(t) \right] = 3$

3) $\int \frac{d}{dt} \left[ \frac{1}{(t+1)^2} y(t) \right] \, dt = 3 \int \, dt + C \Rightarrow \frac{y(t)}{(t+1)^2} = 3t + C$

$\Rightarrow y(t) = (3t + C)(t+1)^2$

b) Find the specific solution of $y' = \frac{2y}{t+1} + 3(t+1)^2$; $y(1) = 8$.

$y(1) = (3 + c) \cdot 1^2 = 8 \Rightarrow 4c = -4 \Rightarrow c = -1$

$\Rightarrow y(t) = (3t - 1)(t+1)^2$
Solving I.V.P.s for which the solution must be left in integral form (because the integral involved cannot be integrated)

Given \( y' + p(t)y = g(t), \quad y(t_0) = y_0. \)
1. Follow step 1 above to obtain the integrating factor.

2. Follow step 2 above, making the substitution to obtain the equation (D).

3. Set up the integrals of step 3 with \( t_0 \) as the lower limit and \( t \) as the upper limit. Do not add C. Change the name of the variable \( t \) in the integrand to another name (such as \( s \)).

\[
\int_s \frac{d}{ds} [\mu(s)x(s)]ds = \int_s \mu(s)g(s)ds.
\]
Evaluate the integrals (if possible). Substitute \( y_0 \) for \( y(t_0) \) in the antiderivative of the left-hand integral.

Note: If it is not possible to obtain a closed-form antiderivative of either integral, a numerical approach will be required - or use your calculator to obtain the value of the integral for specified values of \( t \).

**Example 3.** Find the specific solution of \( y' = 1 + 2ty, \quad y(0) = 3. \)

1) \( y' - 2ty = 1 \quad \Rightarrow \quad p(t) = -2t \quad \Rightarrow \quad \mu(t) = e^{\int -2tdt} = e^{-t^2} \)

2) \( e^{-t^2} (y' + 2ty) = e^{-t^2} \quad \Rightarrow \quad \text{cannot integrate} \)

\[
\frac{d}{dt} \left[ e^{-t^2} y(t) \right] = e^{-t^2}
\]

3) \( \int_s^t \frac{d}{ds} [e^{-s^2} y(s)]ds = \int_s^t e^{-s^2} ds \quad \Rightarrow \quad \int_s^t e^{-s^2} ds \quad \Rightarrow \quad e^{-s^2} y(s) - y(s) = \int_s^t e^{-s^2} ds \)

\( \Rightarrow \quad e^{-t^2} y(t) = \int_s^t e^{-s^2} ds + 3 \quad \Rightarrow \quad y(t) = e^{t^2} \left[ \int_s^t e^{-s^2} ds + 3 \right] \)

Graph of the specific solution of \( y' = 1 + 2ty, \quad y(0) = 3. \)

Boyce & DiPrima, Section 2.1 lecture, p. 3