1. (4 pts) On what interval(s) will the solutions of \((t^2 - 4)y'' + 4y' + \ln |t|\) have their domains?
\[
y'' + \frac{4}{t^2} y' = \frac{\ln |t|}{t^2 - 4} \quad \text{not defined } @ t = 0
\]
\[
\text{not cont. } @ t = \pm 2
\]
Interval(s): \((-\infty, -2), (-2, 0), (0, 2), (2, \infty)\)

2. (4 pts) Consider the system \[
\begin{align*}
ax + by &= 0 \\
\frac{ax}{c} + dy &= 0
\end{align*}
\]
Suppose \[
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} = 0.
\]
Circle one best answer.

a) We cannot answer definitively; there may be no solution or there may be infinitely many solutions including (0, 0).

b) There are infinitely many solutions including (0, 0)

c) There is a unique solution which is (0, 0).

d) There is a unique solution which is not necessarily (0, 0).

3. (8 pts) The statements below refer to \(L[y] = g(t)\) with particular solution \(Y(t) = -\frac{1}{2} \sin t\). The corresponding \(L[y] = 0\) has fundamental set \(\{1, t^{-2}\}\) on \((0, \infty)\).

a) \(\square\) or \(\bigcirc\) The functions \(\{1, t^{-2}\}\) are linearly-independent on \((0, \infty)\).

b) \(\square\) or \(\bigcirc\) The functions \(\{2, 5t^{-2} - 1\}\) also form a fundamental set of \(L[y] = 0\) on \((0, \infty)\).

c) \(\bigcirc\) or \(\square\) \[
L[5t^{-2}] = g(t)
\]
\[
\left| \begin{array}{c}
2 \\
5t^{-2}
\end{array} \right| = -20 t^{-3} \neq 0 \text{ on } (0, \infty)
\]

d) \(\square\) or \(\bigcirc\) \[
L\left[\frac{-1}{2} \sin t\right] = 0
\]

4. (4 pts) Suppose \(r = 4 - 3i\). Use the Cauchy-Euler Identity to express \(e^r\) as a sum \(a + ib\) (or as a difference \(a - ib\)).

\[
e^{4 - 3i} = e^4 (\cos 3 - i \sin 3) = e^4 (c + 3 - i \sin 3)
\]

5. (3 pts) Circle the differential equation(s) below that could be solved using the method of Undetermined Coefficients.

a) \(t^2 y'' - 3y' + y = 4t^2\)

b) \(y'' - 3y' + y = 3t^2\)

c) \(y'' - 3y' + y = 3 \sin t\)
6. (10 pts) a) (2) Find the fundamental set for \( y'' + 4y = 0 \).
\[
 r^2 + 4 = 0 \Rightarrow r = \pm 2i
\]
\[
 \{ \cos 2t, \sin 2t \}
\]

b) (6) Find the values of the constants \( A \) and \( B \) for which \( Y(t) = A + Be^t \) is a solution of \( y'' + 4y = 6 - 5e^t \). (N)
\[
 y' = Be^t \\
 y'' = Be^t
\]
\[
 \begin{align*}
 10e^t + 4A + 4Be^t &= 6 - 5e^t \\
 \text{Compare coefficients:} \\
 e^t \text{terms: } 5B &= -5 \Rightarrow B = -1 \\
 \text{constant: } 4A &= 6 \Rightarrow (A = \frac{3}{2})
\end{align*}
\]

\[
 A = \frac{3}{2} \\
 B = -1
\]

(2) The general solution of \( y'' + 4y = 6 - 5e^t \)
\[
 Y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{3}{2} - e^t
\]

7. (10 pts) Consider \( y'' - 3y' = 2te^{3t} - \cos t \).

a) (2) What is the fundamental set of \( y'' - 3y' = 0 \)?
\[
 r^2 - 3r = 0 \Rightarrow r(r - 3) = 0
\]
\[
 \{ 0, 3 \}
\]

b) (8) Find the simplest form of \( Y(t) \) using the method of Undetermined Coefficients. DO NOT SOLVE FOR THE COEFFICIENTS.
\[
 Y(t) = A e^{3t} + B e^{3t} + C \cos t + D \sin t
\]

<table>
<thead>
<tr>
<th>term</th>
<th>root</th>
<th>basis set</th>
<th>root in ( \mathbb{R} )?</th>
<th>adjusted set</th>
</tr>
</thead>
<tbody>
<tr>
<td>( te^{3t} )</td>
<td>3</td>
<td>( e^{3t}, te^{3t} )</td>
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<td>( te^{3t}, t e^{3t} )</td>
</tr>
<tr>
<td>( \cos t )</td>
<td>( \pm i )</td>
<td>( \cos t, \sin t )</td>
<td>no</td>
<td></td>
</tr>
</tbody>
</table>
8. (12 pts) The equation \( t^2y'' - 2ty' + 2y = 0 \) has the general solution \( y_e(t) = c_1t + c_2t^2 \) \( (t > 0) \).

Find the particular solution \( Y(t) \) for \( t^2y'' - 2ty' + 2y = 2 \) using Variation of Parameters.

For partial credit, answer the following:

a) (2) What is the form of \( Y(t) \)?

\[
Y(t) = u_1(t)t + u_2(t)t^2
\]

b) (2) Set up two equations that must be solved in order to completely find \( Y(t) \).

\[
\begin{align*}
u_1' &+ u_2' t^2 = 0 \\
u_1' &+ 2u_2' t = 2t^2
\end{align*}
\]

c) (1) Evaluate the Wronskian of the fundamental set of \( t^2y'' - 2ty' + 2y = 0 \).

\[
W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2
\]

d) (7) Find \( Y(t) \) [Minimal to no credit if answers to a) - c) are incorrect.]

\[
\begin{align*}
u_1' &= \frac{0 \cdot t^2 - 2t \cdot t^2}{t^2} = -\frac{2t^2}{t^2} = -2t^-2 \\
u_1 &= -2t^{-1} = \frac{-2}{t}
\end{align*}
\]

\[
\begin{align*}
u_2' &= \frac{t \cdot 0 - 2t \cdot 2t^2}{t^2} = \frac{2t^{-1}}{t^2} = 2t^{-3} \\
u_2 &= \frac{2t^{-2}}{-2} = -\frac{1}{t^2}
\end{align*}
\]

\[
Y = \frac{2}{t} \cdot t - \frac{1}{t^2} \cdot t^2 = 2 - 1 = 1
\]

(7)

\[
Y(t) = 1
\]
9. (3) The motion of a vibrating system is given by \( u(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t - 2t \).
   a) (2) Circle all the terms that describe the motion.

   [Underdamped]  [Overdamped]  [Critically-damped]  [Undamped]  [Free]  [Forced]

   b) (1) Circle the transient term(s) of \( u(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t - 2t \).

10. (10) The motion of a mass-spring vibrating system is given by \( u(t) = \sqrt{3} \sin 4t - \sqrt{6} \cos 4t \).
   a) (6) Write this equation in the amplitude-phase angle form.

   \[ K = \sqrt{(13)^2 + (16)^2} = \sqrt{49 + 256} = \sqrt{305} = 3 \]

   \[ \sin \theta = \frac{13}{\sqrt{305}}, \cos \theta = \frac{16}{\sqrt{305}} \]

   \[ a = \frac{13}{\sqrt{305}} + \eta_1 = 6.15 + \eta_1 = 3.757 \]

   b) (2) What is the period of this motion? \( \frac{2\pi}{\omega} \)

   \[ T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} \]

   c) (2) If the mass is 0.25 slug, what is the spring constant \( k \)?

   \[ \omega_m = \sqrt{\frac{k}{m}} \Rightarrow 4 = \sqrt{\frac{k}{0.25}} \Rightarrow k = 4 \times 4 \Rightarrow k = 16 \text{ lb/ft} \]

11. (7 pts) A 10-kg mass is attached to a spring, stretching it 0.7 m from its natural length. The mass is attached to a viscous damper with a damping constant of 9 Newton-sec/meter. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction. Set up an I.V.P (differential equation + initial conditions) to describe the motion of this system. DO NOT SOLVE.

   (5) The differential equation

   \[ 16u'' + 9u' + 140u = 0 \] or \[ u'' + 0.9u' + 14u = 0 \]

   To find \( k \):

   \[ \omega = 10(9.8) = 98 \]

   \[ -98 = -0.7k \Rightarrow k = 140 \]

   (2) The initial conditions

   \[ u(0) = 0 \]
   \[ u'(0) = -1 \]
MATH 083 NAME ____________________________ Class Time ________________ Class No. ____________
March 12, 2008

1. (4 pts) On what interval(s) will the solutions of \((t^2 - 1)y'' + (\ln |t|)y = 4\) have their domains?

\[
\begin{align*}
y'' + \frac{\ln |t|}{t^2 - 1}y &= \frac{4}{t^2 - 1} \\
\text{Interval(s):} &\quad (-\infty, -1), (-1, 0), (0, 1), (1, \infty)
\end{align*}
\]

2. (4 pts) Consider the system \[
\begin{align*}
ax + by &= 0 \\
\frac{\ln |t|}{t^2 - 1}y &= \frac{4}{t^2 - 1} \\
\text{Not defined at } t &= 0 \\
\text{Not constant.} &\quad t = \pm 1
\end{align*}
\]
Suppose \[
\begin{align*}
a &\neq 0 \\
b &\neq 0
\end{align*}
\]
Circle one best answer.

a) We cannot answer definitively; there may be no solution or there may be infinitely many solutions including (0, 0).

b) There are infinitely many solutions including (0, 0).

c) There is a unique solution which is (0, 0).

d) There is a unique solution which is not necessarily (0, 0).

3. (8 pts) The statements below refer to \(L[y] = g(t)\) with particular solution \(Y(t) = -3t^2\). The corresponding \(L[y] = 0\) has fundamental set \(\{t, t^{-1}\}\) on \((0, \infty)\).

5. \(\text{T or F}\) The functions \(\{2t, 5t^{-1} - t\}\) also form a fundamental set of \(L[y] = 0\) on \((0, \infty)\).

6. \(T\) or \(F\) The functions \(\{t, t^{-1}\}\) are linearly-dependent on \((0, \infty)\).

7. \(T\) or \(F\) \(L[5t^{-1}] = 0\)

d) \(T\) or \(F\) \(L[-3t^2] = g(t)\)

4. (4 pts) Suppose \(r = 2 - 5i\). Use the Cauchy-Euler Identity to express \(e^r\) as a sum \(a + ib\) (or as a difference \(a - ib\)).

\[
E^{2-5i} = E^2 E^{-5i} = E^2 (\cos 5 - i \sin 5)
\]

5. (3 pts) Circle the differential equation(s) below that could be solved using the method of Undetermined Coefficients.

\[
\begin{align*}
a) \quad y'' - 3y' + y &= 3\sin t \\
b) \quad t^2 y'' - 3y' + y &= 4t^2 \\
c) \quad y'' - 3y' + y &= 3t^2
\end{align*}
\]
6. (10 pts) a) (2) Find the fundamental set for \( y'' + 9y = 0 \).
\[
  r^2 + 9 = 0 \Rightarrow r = \pm 3i
\]
\[
\{ \cos 3t, \sin 3t \}
\]
b) (6) Find the values of the constants \( A \) and \( B \) for which \( Y(t) = A + Be^t \) is a solution of \( y'' + 9y = 6 - 10e^t \). (<N>)
\[
  y' = Be^t
\]
\[
  y'' = Be^t \]
\[
(N) \rightarrow Be^t + 9A + 9Be^t = 6 - 10e^t
\]
Choose coefficient \( e^t \) term
\[
B + 9B = -10 \Rightarrow B = -1
\]
Constants: \( 9A = 6 \Rightarrow A = \frac{2}{3} \)
\[
A = \frac{2}{3}; \quad B = -1
\]
c) (2) What is the general solution of \( y'' + 9y = 6 - 10e^t \)?
\[
y(t) = C_1 \cos 3t + C_2 \sin 3t + \frac{2}{3} - e^t
\]

7. (10 pts) Consider \( y'' - 2y' = 2te^{2t} - \sin t \).

a) (2) What is the fundamental set of \( y'' - 2y' = 0 \)?
\[
  r^2 - 2r = 0 \Rightarrow r(r - 2) = 0
\]
\[
\Rightarrow r = 0, \quad r = 2
\]
\[
\{ 1, e^{2t} \}
\]
b) (8) Find the simplest form of \( Y(t) \) using the method of Undetermined Coefficients. DO NOT SOLVE FOR THE COEFFICIENTS.
\[
\text{The form of } Y(t)
\]
\[
Y(t) = Ate^{2t} + Bt^2e^{2t} + C \sin t + D \cos t
\]

<table>
<thead>
<tr>
<th>Function</th>
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8. (12 pts) The equation \( t^2y'' - 2ty' + 2y = 0 \) has the general solution \( y_c(t) = c_1t + c_2t^2 \) \( (t > 0) \).

Find the particular solution \( Y(t) \) for \( t^2y'' - 2ty' + 2y = 2 \) using Variation of Parameters.

For partial credit answer the following:

a) (2) What is the form of \( Y(t) \)?

\[
Y(t) = u_1(t)t + u_2(t)t^2
\]

b) (2) Set up two equations that must be solved in order to completely find \( Y(t) \).

\[
\begin{align*}
\frac{u_1'}{u_1} + \frac{2u_2'}{u_2} &= 0 \\
\frac{u_1'}{u_1} + \frac{2u_2'}{u_2} &= \frac{2}{t^2}
\end{align*}
\]

c) (1) Evaluate the Wronskian of the fundamental set of \( t^2y'' - 2ty' + 2y = 0 \).

\[
W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2
\]

d) (7) Find \( Y(t) \) (Minimal to no credit if answers to a) – c) are incorrect.)

\[
\begin{align*}
u_1' &= \begin{vmatrix} 0 & t^2 \\ 2t^2 & 2t \end{vmatrix} = -\frac{2}{t^2} \Rightarrow u_1 = 2t^{-1} \\
u_2' &= \begin{vmatrix} t & 0 \\ 1 & 2t^2 \end{vmatrix} = \frac{2t^{-1}}{t^2} = 2t^{-3} \Rightarrow u_2 = \frac{2t^{-2}}{-2} = -t^{-2}
\end{align*}
\]

\[
Y(t) = 2t^{-1}t - t^{-2}t^2 = 2 - 1 = 1
\]

\[
(7)
Y(t) = 1
\]
9. (3) The motion of a vibrating system is given by \( u(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t - \sin 2t \).
   a) (2) Circle all the terms that describe the motion.

   *free*  *forced*  *overdamped*  *critically-damped*  *underdamped*  *undamped*

b) (1) Circle the steady-state term(s) of \( u(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t - \sin 2t \).

10. (10) The motion of a mass-spring vibrating system is given by \( u(t) = -2 \sin 2t - 3 \cos 2t \)
   a) (6) Write this equation in the amplitude-phase angle form.

   \[
   R = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \approx 3.606
   \]

   \[\delta \text{ is in Q III} \] so \( \delta = \tan^{-1} \left( \frac{-2}{-3} \right) + \pi \)

   \[\delta = \tan^{-1} \left( \frac{2}{3} \right) + \pi \approx 1.23 + \pi \]

   \[\approx 3.73\]

b) (2) What is the period of this motion? \( \pi \)

c) (2) If the mass is 0.25 slug, what is the spring constant \( k \)? \( 1 \) lb/s

\[
\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow \omega = \sqrt{\frac{1}{0.25}} \Rightarrow k = 1
\]

11. (7 pts) A 10-kg mass is attached to a spring, stretching it 0.7 m from its natural length. The mass is attached to a viscous damper with a damping constant of 8 Newton-sec/meter. The mass + spring is compressed and released into motion from a position 2 meters above the equilibrium position. Set up an I.V.P (differential equation + initial conditions) to describe the motion of this system.

   DO NOT SOLVE.

   (5) The differential equation

   \[
   10u'' + 8u' + 140u = 0
   \]

   (2) The initial conditions

   \[
   u(0) = -2 \\
   u'(0) = 0
   \]

To find \( k \)

\[
\omega = \sqrt{10 \cdot 9.8} = 9.8
\]

\[-98 = -k(0.7) \Rightarrow k = 140\]