1. (9 pts) Determine whether each differential equation is separable and/or linear and/or autonomous.

<table>
<thead>
<tr>
<th>Differential equation</th>
<th>(If 1st order) separable?</th>
<th>Autonomous? ((\forall ) if yes)</th>
<th>Linear? ((\forall ) if yes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (y'' = 3x^2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. (y' = \frac{x - y}{x + y})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. (\frac{dl}{dt} = f(2 - I))</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td></td>
</tr>
</tbody>
</table>

2. (8 pts) Consider the o.d.e. \((t - 5)y' - 2y = \ln |t - 2|\).
   a) (4) Without solving, determine an interval in which a solution to the I.V.P., \((t - 5)y' - 2y = \ln |t - 2|, y(5) = 0\), is certain to exist.
      \(p(t) = -\frac{2}{t - 5}\) is not cont @ \(t = 5\).
      \(g(t) = \frac{\ln |t - 2|}{t - 5}\) is not cont @ \(t = 5, 2\).
      Three possible intervals:
      \((-\infty, 2), (2, 5), (5, \infty)\) \(t_0 = 2\) is in \((2, 5)\).

   b) (4) Find and completely simplify the integrating factor of \((t - 5)y' - 2y = \ln |t - 2|\).
      \[\mu(t) = e^{-2\int \frac{1}{t-5} dt} = e^{-2 \ln |t-5|} = \frac{1}{(t-5)^2}\]

3. (6 pts) The o.d.e. \(y' + p(t)y = 4e^{4t}\) has an integrating factor \(\mu(t) = e^{-2t}\) Find an explicit one-parameter family of solutions.
   \[e^{-2t}(y + p(t)y) = 4e^{-2t}e^{4t}\]
   \[\frac{d}{dt}[e^{-2t}y(t)] = 1e^{2t}\]
   \[\int \frac{d}{dt}[e^{-2t}y(t)] dt = \int 1e^{2t} dt + C\]
   \[e^{-2t}y(t) = \frac{1}{2}e^{2t} + C\] \(\Rightarrow y(t) = e^{2t}(2e^{2t} + C)\) or \(y(t) = 2e^{4t} + Ce^{2t}\)

4. (4 pts) Determine the values of \(r\) for which \(y = e^{rt}\) is a solution of \(y'' - 5y' + 6y = 0\).
   \[
   \begin{align*}
   y &= e^{rt} \\
   y' &= re^{rt} \\
   y'' &= r^2e^{rt}
   \end{align*}
   \[
   \begin{align*}
   r^2e^{rt} - 5re^{rt} + 6e^{rt} &= 0 \\
   (r^2 - 5r + 6)e^{rt} &= 0 \\
   \Rightarrow r^2 - 5r + 6 &= 0 \Rightarrow (r - 3)(r - 2) = 0
   \end{align*}
   \]
   Values of \(r\)
   \[2, 3\]
5. (7 pts) Find all possible solutions of $y' = 2ty^2$. 

Separable: 

$$\int y^{-2} dy = \int 2t dt + C$$

$$\frac{1}{y} = \frac{t^2}{2} + C$$

$$y = \frac{1}{\frac{t^2}{2} + C}$$

All possible solutions

$$y(t) = \frac{1}{\frac{t^2}{2} + C} \text{ or } \frac{-1}{\frac{c+t^2}{2}}$$

$$y(t) = 0$$

6. (7 pts) Solve the I.V.P., $y' = \frac{e^{t^2}}{y}$, $y(0) = -1$ for an explicit solution.

Separable: 

$$\int y dy = \int e^{t^2} dt$$

$$\frac{y^2}{2} - \frac{1}{2} = \int e^{t^2} dt$$

$$\frac{y^2}{2} = \int e^{t^2} dt + 1$$

Explicit solution

$$y(t) = -\sqrt{2 \int e^{t^2} dt + 1}$$

(You may need to use a computer algebra system to evaluate the integral.)

7. (8 pts) A tank initially contains 100g of dye dissolved in 400 liters of water. At $t = 0$ pure water begins to run into the tank at a rate of 10 liters per minute. Simultaneously, the dye solution leaves the tank at the faster rate of 15 liters per minute. Set up an initial value problem (o.d.e. plus initial condition) to model the amount, $Q(t)$, of dye in the tank at any time $t$ for which the model is valid.

<table>
<thead>
<tr>
<th>Rate In = 0 (pure water)</th>
<th>Rate Out = $\frac{Q(t)}{400-5t}$</th>
<th>$\frac{dQ}{dt} = -15\frac{Q(t)}{400-5t}$</th>
<th>The o.d.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The initial condition</td>
<td>$Q(0) = 100$</td>
<td>$Q(0) = 100$</td>
<td></td>
</tr>
</tbody>
</table>

8. (8 pts) The model for the behavior (growth and decline) of the population of a large urban area is given by $\frac{dy}{dt} = 2y - y^2$, where the $y^2$ term represents the decline of the population as a result of emigration and death. In this model $y(t)$ represents the size of the present population (in millions), while $t$ is measured in years. The one-parameter solution of this model is given by $y(t) = \frac{2}{1 + C e^{-2t}}$.

a) (5) Sketch a phase line on which you identify the equilibrium solution(s) and show the behavior of solutions corresponding to this model.

<table>
<thead>
<tr>
<th>$y'$</th>
<th>$y = 0$</th>
<th>$y = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$=$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Phase line

b) (3) Express as a limit and find the long term behavior of the population when the initial population size is 3 (million).

$$\lim_{t \to \infty} y(t) = 2$$
9. (10 pts) The o.d.e. $ty' + 2y = y^2$ has a one-parameter family of solutions, $y(t) = \frac{2}{1 + Ce^t}$.

a) (2) List all constant solution(s) of $ty' + 2y = y^2$.

\[ y' = y \frac{2-2y}{t} = 0 \text{ when } y = 0, 2 \]

b) (2) What does the General Existence and Uniqueness Theorem say about a solution to the I.V.P., $ty' + 2y = y^2$, $y(1) = 0$? Circle the best choice.

- A. There is no solution
- B. There is a unique solution.
- C. No information

\[ y(t) = 0, y(t) = 0 \]

Your answer/conclusion

A. There is no solution
B. There is a unique solution.
C. No information

(2) If there is a unique solution to the I.V.P. of part b), what is it? Otherwise show analytically that there is no solution or infinitely many solutions.

- A. There is no solution
- B. There is a unique solution.
- C. No information

\[ y(t) = 0 \text{ is the unique soln.} \]

(2) What does the General Existence and Uniqueness Theorem say about a solution to the I.V.P., $ty' + 2y = y^2$, $y(0) = 1$?

Circle the best choice.

- A. There is no solution
- B. There is a unique solution.
- C. No information

\[ y'(t) \text{ are not cont. at } t = 0 \]

Your answer/conclusion

A. There is no solution
B. There is a unique solution.
C. No information

(2) If there is a unique solution to the I.V.P. of part d), what is it? Otherwise show analytically that there is no solution or infinitely many solutions.

\[ y(t) = 0 \text{ is the unique soln.} \]

10. (8 pts) Empirical studies suggest that each person has a certain maximum number (M) of symbols that he or she can remember at any one time: the rate at which new symbols can be memorized is proportional to the difference between the maximum and the number of symbols $y(i)$ already memorized. Suppose that for a given person $M = 800$. Initially the person has memorized no symbols. At the end of an hour the person has memorized 400 symbols.

a) (4) Set up an initial value problem (o.d.e. plus initial condition) to model this situation.

\[ \frac{dy}{dt} = r(800 - y) \text{ or } \frac{dy}{dt} = r(y - 800) \]

\[ y(0) = 0 \]

\[ \frac{dy}{dt} = r(800 - y) \text{ or } \frac{dy}{dt} = r(y - 800) \]

\[ y(0) = 0 \]

b) (4) The solution of the model of part a) is $y(t) = 800 + Ce^{rt}$ (t in hours). Find the values of $C$ and $r$.

\[ \frac{dy}{dt} = 800 + C e^{rt} \]

\[ y(0) = 0 \Rightarrow 800 + C = 0 \Rightarrow C = -800 \]

\[ y(t) = 800 - 800 e^{rt} \]

\[ y(1) = 400 \Rightarrow 400 = 800 - 800 e^{r} \Rightarrow -400 = -800 e^{r} \Rightarrow e^{r} = \frac{1}{2} \Rightarrow r = \ln \frac{1}{2} = -\ln 2 \]

\[ C = -800 \]

\[ r = \frac{\ln 1}{2} \text{ or } -\ln 2 \]

\[ r = -0.6931 \]
EXAM 1D

MATH 083  NAME       Key        Class Time        Class No.
Feb. 8, 2008,         Sections 1.1 – 1.3, 2.1 – 2.4, Boyce & DiPrima

1. (9 pts) Determine whether each differential equation is separable and/or linear and/or autonomous.

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<tr>
<th>Differential equation</th>
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<tr>
<td>(\frac{dQ}{dt} = Q(2 - Q))</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td></td>
</tr>
<tr>
<td>(y' = y(t - y))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(y'' = \sin x)</td>
<td>(\checkmark)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. (8 pts) Consider the o.d.e. \((t - 4)y' - 2y = \ln |t - 1|\).
   a) (4 pts) Without solving, determine an interval in which a solution to the I.V.P., \((t - 4)y' - 2y = \ln |t - 1|, y(2) = 0\) is certain to exist.
      \[ p(t) = \frac{-2}{t - 4} \text{ not cont. @ } t = 4 \]
      \[ q(t) = \frac{\ln |t - 1|}{t - 4} \text{ not cont. @ } t = 1, 4 \]
      \[ \text{The four possible intervals are: } (-\infty, 1), (1, 4), (4, \infty) \]
      \[ b_0 = 2 \text{ is in (1, 4)} \]

   b) (4 pts) Find and completely simplify the integrating factor of \((t - 4)y' - 2y = \ln |t - 1|\).
      \[ \mu(t) = e^{2 \int \frac{dt}{t - 4}} = e^{\ln (t - 4)^{-2}} = (t - 4)^{-2} \]

3. (6 pts) The o.d.e. \(y' + p(t)y = 6e^{4t}\) has an integrating factor \(\mu(t) = e^{-t}\) Find an explicit one-parameter family of solutions.
   \[ e^{-t} (y' + p(t)y) = 6e^{-t} e^{4t} \]
   \[ \frac{d}{dt} [y(t)e^{-t}] = 6 e^{3t} \]
   \[ \int \frac{d}{dt} [y(t)e^{-t}] dt = 6 \int e^{3t} dt + C \]
   \[ y(t)e^{-t} = \frac{6}{3} e^{3t} + C \]
   \[ y(t) = e^{t} (2e^{3t} + C) \text{ or } y(t) = 2e^{4t} + Ce^{t} \]

4. (4 pts) Determine the values of \(r\) for which \(y = e^{rt}\) is a solution of \(y'' - 7y' + 6y = 0\).
   \[ \begin{align*}
   y &= e^{rt} \\
   y' &= re^{rt} \\
   y'' &= r^2 e^{rt}
   \end{align*} \]
   \[ r^2 e^{rt} - 7re^{rt} + 6e^{rt} = 0 \]
   \[ (r^2 - 7r + 6)e^{rt} = 0 \]
   \[ r^2 - 7r + 6 = 0 \Rightarrow (r - 1)(r - 6) = 0 \]

Values of \(r\): \(1, 6\)
5. (7 pts) Find all possible solutions of $y' = e^t y^2$.

Separable

$y^{-2} y'-e^t=0$

$\int \frac{1}{y^2} \, dy = \int e^t \, dt$

$\frac{-1}{y} = e^t + C$

$y' = 0 \Rightarrow y(t) = 0$ is a constant solution

All possible solutions

$y(t) = \frac{1}{c-e^t}$ or $\frac{-1}{e^t - c}$

$y(t) = 0$

6. (7 pts) Solve the I.V.P., $y' = \frac{e^t}{y^2}, y(0) = -2$ for an explicit solution.

Separable can't be integrated analytically

$y \, dy = e^t \, dt$

$\int y \, dy = \int e^t \, dt$

$\frac{-2}{y} = e^t + C$

$y = 2 \frac{-e^t}{e^t - 2} \quad (\because -2$ because $y = -2)$

Explicit solution

$y(t) = -\sqrt{2 \int e^t \, ds + 4}$

7. (8 pts) A tank initially contains 50g of dye dissolved in 200 liters of water. At $t = 0$ pure water begins to run into the tank at a rate of 8 liters per minute. Simultaneously, the dye solution leaves the tank at the faster rate of 10 liters per minute. Set up an initial value problem (o.d.e. plus initial condition) to model the amount, $Q(t)$, of dye in the tank at any time $t$ for which the model is valid.

Rate In = 0 (pure water)

Rate Out = $\frac{Q(0)}{200-2t}, 10$

$= \frac{50}{100-2t}$

8. (8 pts) The model for the behavior (growth and decline) of the population of a large urban area is given by $\frac{dy}{dt} = 3y - y^2$, where the $y^2$ term represents the decline of the population as a result of emigration and death. In this model $y(t)$ represents the size of the present population (in millions), while $t$ is measured in years. The one-parameter solution of this model is given by $y(t) = \frac{3}{1 + ce^{-3t}}$.

a) (5) Sketch a phase line on which you identify the equilibrium solution(s) and show the behavior of solutions corresponding to this model.

$y' = y(3-y) = 0$ when $y = 0, 3$

$y = 0$

b) (3) Express as a limit and find the long-term behavior of the population when the initial population size is 2 (million).

The long-term behavior (limit and value)

$\lim_{t \to \infty} y(t) = 3$
9. (10 pts) The o.d.e. $ty' + 4y = 4y^2$ has a one-parameter family of solutions, $y(t) = \frac{1}{1+ct^2}$.

a) (2) List all constant solution(s) of $ty' + 4y = 4y^2$.

\[ y' = \frac{4y^2 - 4y}{t} = \frac{4y(y - 1)}{t} \]

When $y = 0$, $1$

Constant solution(s)

\begin{align*}
y(t) &= 0, \quad y(t) = 1
\end{align*}

Circle the best choice.

\begin{align*}
y' &\neq \text{c} y, \text{ are not} \\
\text{const} &\neq t = 0
\end{align*}

A. There is no solution
B. There is a unique solution.
C. No information

b) (2) What does the General Existence and Uniqueness Theorem say about a solution to the I.V.P. $ty' + 4y = 4y^2$, $y(0) = 2$?

C. No information

\begin{align*}
y'(t) &= \frac{1}{1+ct^2} \\
y(t) &= \frac{1}{1+ct^2} \Rightarrow y'(t) = \frac{1}{1+ct^2} \Rightarrow \text{No solution}
\end{align*}

Your answer/conclusion

A. There is no solution
B. There is a unique solution.
C. No information

d) (2) What does the General Existence and Uniqueness Theorem say about a solution to the I.V.P., $ty' + 4y = 4y^2$, $y(2) = 0$?

Circle the best choice.

\begin{align*}
y' &\neq \text{c} y, \text{ are cont. (2,0)} \\
\text{const} &\neq t = 0
\end{align*}

A. There is no solution
B. There is a unique solution.
C. No information

e) (2) If there is a unique solution to the I.V.P. of part d), what is it? Otherwise show analytically that there is no solution or infinitely many solutions.

\begin{align*}
y(t) &= \frac{1}{1+ct^2} \\
y(t) &= \frac{1}{1+ct^2} \Rightarrow \text{No solution}
\end{align*}

Your answer/conclusion

A. There is no solution
B. There is a unique solution.
C. No information

10. (8 pts) Empirical studies suggest that each person has a certain maximum number (M) of symbols that he or she can remember at any one time: the rate at which new symbols can be memorized is proportional to the difference between the maximum and the number of symbols $y(t)$ already memorized. Suppose that for a given person $M = 800$. Initially the person has memorized no symbols. At the end of an hour the person has memorized 200 symbols.

a) (4) Set up an initial value problem (o.d.e. plus initial condition) to model this situation.

\[ \frac{dy}{dt} = r(800 - y) \quad \text{I.V.P.} \]

\[ y(0) = 0 \]

\[ \frac{dy}{dt} = r(800 - y) \quad \text{I.V.P.} \]

\[ y(0) = 0 \]

b) (4) The solution of the model of part a) is $T(t) = 800 + Ce^{rt}$ ($t$ in hours). Find the values of $C$ and $r$.

\[ T(0) = 800 + C \Rightarrow C = -800 \text{ so } y(t) = 800 - 800e^{rt} \]

\[ 200 = 800 - 800e^{rt} \Rightarrow -600 = e^{rt} \]

\[ \Rightarrow \ln \frac{2}{3} = r \]

\[ C = -800 \]

\[ r = \ln \frac{2}{3} \approx -0.2877 \]