Section 5-3: Linear Programming in Two Dimensions – A Geometric Approach

Linear Programming: General Description

A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum value) of a linear **objective function** of the form $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ where the **decision variables** $x_1, x_2, \ldots, x_n$ are subject to **problem constraints** in the form of linear inequalities and equations. In addition, each decision variable must satisfy the **nonnegative constraint** $x_i \geq 0$, $i = 1, 2, \ldots, n$.

The set of points satisfying both the problem constraints and the nonnegative constraints is called the **feasible region**. Any point in the feasible region that produces the optimal value of the objective function over the feasible region is called an **optimal solution.**

Geometric Solution of a Linear Programming Problem with Two Decision Variables

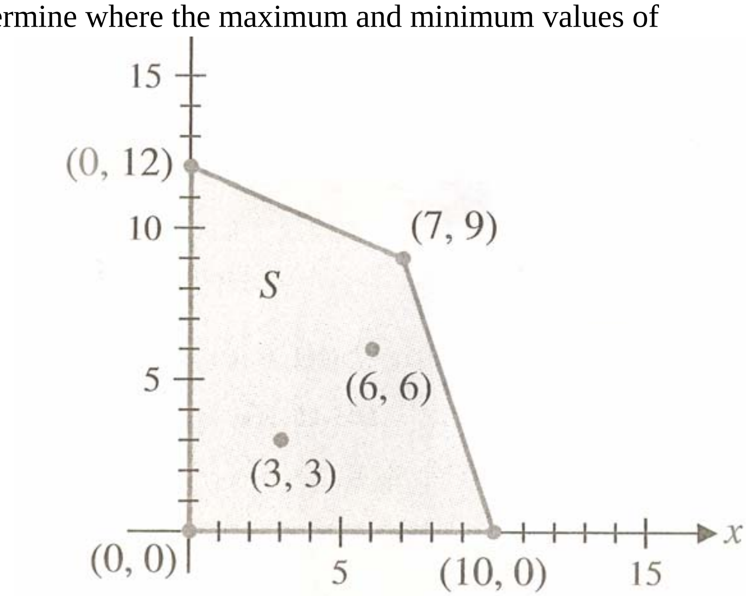
Step 1. Graph the feasible region. Then find the coordinates of each corner point.
Step 2. Make a table listing the value of the objective function at each corner point.
Step 3. Determine the optimal solution(s) from the table in Step 2.
Step 4. For an applied problem, interpret the optimal solution(s) in terms of the original problem.

Theorem 1. If the optimal value of the objective function in a linear programming problem exists then that value must occur at one (or more) of the corner points of the feasible region.

Q1 (based on #4, page 285)**.** Maximize and minimize $P = 9x + 3y$ subject to

$$\begin{aligned} y &\leq -\tfrac{3}{7}x + 12 \\ y &\leq -3x + 30 \\ x, y &\geq 0 \end{aligned}$$

A) Sketch constant profit lines through (3,3) and (6,6).
B) Without calculating, try to determine where the maximum profit occurs.
C) Construct a corner point table to determine where the maximum and minimum values of P occur



Note that in general, if two corner points are both optimal solutions to a linear programming problem, then any point on the line segment joining them is also an optimal solution.

Theorem 2. **Existence of Optimal Solutions [Note: this theorem does not cover all possibilities.]**

(A) If the feasible region for a linear programming problem is bounded, then both the maximum value and the minimum value of the objective function always exist.

(B) If the feasible region is unbounded and the coefficients of the objective function are positive, then the minimum value of the objective function exists, but the maximum value does not.

(C) If the feasible region is empty (that is, there are no points that satisfy all the constraints), then both the maximum value and the minimum value of the objective function do not exist.

Constructing the Model for an Applied Linear Programming Problem

Step 1. Introduce decision variables.

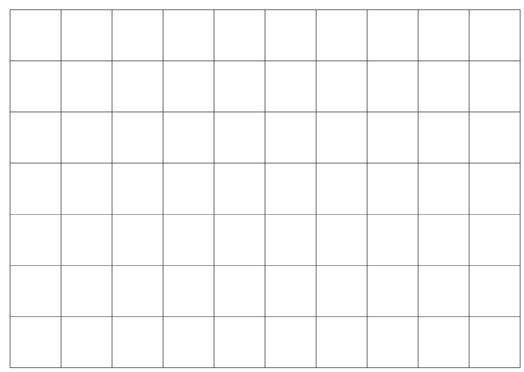
Step 2. Summarize relevant material in table form, relating the decision variables with the columns in the table, if possible.

Step 3. Determine the objective and write a linear objective function.

Step 4. Write problem constraints using linear equations and/or inequalities.

Step 5. Write nonnegative constraints.

Q2 (#38, pages 287-9) *Investment.* An investor has $24,000 to invest in bonds of AAA and B qualities. The AAA bonds yield on the average 6% and the B bonds yield 10%. The investor requires that at least three times as much money should be invested in AAA bonds as in B bonds. How much should be invested in each type of bond to maximize the return? What is the maximum return?

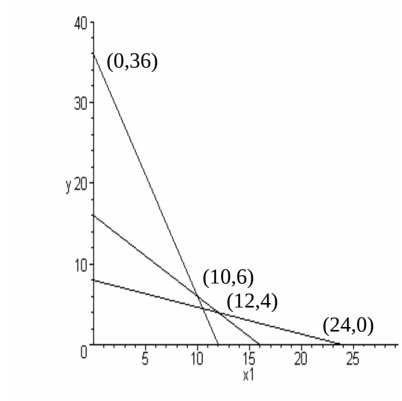


Q3 (#42, page 288). *Nutrition: people.* A dietitian in a hospital is to arrange a special diet composed of two foods, *M and N*. Each ounce of food *M* contains 30 units of calcium, 10 units of iron, 10 units of vitamin A, and 8 units of cholesterol. Each ounce of food *N* contains 10 units of calcium, 10 units of iron, 30 units of vitamin A, and 4 units of cholesterol. The minimum daily requirements are 360 units of calcium, 160 units of iron, and 240 units of vitamin A. How many ounces of each food should be used to meet the minimum requirements and at the same time minimize the cholesterol intake? What is the minimum cholesterol intake?

From Q3 of the Section 5.2 lecture we have the following chart.
The cholesterol row is new.

	Per ounce of Food M	Per ounce of Food N	Minimum Daily Requirement
Calcium	30 units	10 units	360 units
Iron	10 units	10 units	160 units
Vitamin A	10 units	30 units	240 units
Cholesterol	8 units	4 units	

The boundary lines corresponding to the constraints are shown.



The objective function is what we want to minimize or maximize.

For this problem we want to ____________________ ________________________

minimize or maximize? What objective function?

Subject to the constraints:

Identify the corner points and find the solution.

Corner point	Value of objective function

The solution is