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COMPOSITION OF CUMULATIVE DISTRIBUTION FUNCTIONS

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Abstract

A distribution F is generated by composing a cumulative distribution function H with another cumulative distribution function G or a function of such cumulative distribution function G. Composition is applied to eight distributions H for three choices of G. Some properties of F are explored and an illustrative example is given.

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1 Introduction

Composition of cumulative distribution functions (CDFs) with other CDFs, or functions of CDFs, produces new families of distributions. For example, if $H(.)$ and $G(.)$ are (absolutely continuous) CDFs, then

$$F(x) = H[G(x)], \quad (1)$$

is a new CDF and

$$\bar{F}(x) = H[-\ln G(x)], \quad (2)$$

is a new survival function (SF), where $\bar{F}(x) = 1 - F(x)$.

1.1 Composition Using the Function $G(x)$

If the random variable (r.v.) X is positive, then (1) can be written as

$$F(x) = H[G(x)] = \int_0^{G(x)} h(y)\, dy, \; x > 0, \quad (3)$$

where $h(.)$ is the probability density function (PDF) corresponding to the CDF $H(.)$. In this case, $H(.)$ is positive on the interval $(0,1)$, since $0 \le y \le G(x) \le 1$, for all values of x. So, while $G(.)$ is positive on $(0,\infty)$, $H(.)$ is positive on $(0,1)$.

Jones [12] chose $h(.)$ to be beta PDF with parameters (a,b). So that

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} y^{a-1}(1-y)^{b-1}\, dy, \; x > 0. \quad (4)$$

The CDF $F(x)$, in (4), is known as the beta G family of distributions. Several beta G families of distributions can be generated by specifying G. For example, Eugene et al [10] , Nadarajah and Gupta [15], Nadarajah and Kotz ([16], [17]) , Cardeiro et al [9] , Barreto-Sousa et al [7], Paranoíba et al [18] and Cardeiro and Brito [8] , defined the beta normal, beta Fréchet, beta Gumbel , beta exponential , beta Weibull, beta exponentiated exponential, beta Burr type XII and beta power distributions , respectively.

1.2 Composition Using the Function $-\ln G(.)$

If the r.v. X is positive, it follows, from (2), that

$$\bar{F}(x) = H[-\ln G(x)] = \int_0^{-\ln G(x)} h(y)\, dy, \; x > 0. \quad (5)$$

The PDF, $f(x)$, corresponding to (5), is given by

$$f(x) = \lambda_G^*(x)\, h(-\ln G(x)), \tag{6}$$

where

$$\lambda_G^*(x) = \frac{g(x)}{G(x)}, \tag{7}$$

is known as the reversed proportional hazard rate function (RPHRF) and $g(x)$ is the PDF corresponding to the CDF $G(x)$.

AL-Hussaini and Hussein [5] showed that any CDF, F(x) can be written in terms of the hazard rate function (HRF) $\lambda_F(x)$ and RPHRF $\lambda_F^*(x)$ as follows

$$F(x) = \frac{\lambda_F(x)}{\lambda_F(x) + \lambda_F^*(x)},$$

where the HRF $\lambda_F(x)$ is given by $\lambda_F(x) = \dfrac{f(x)}{\bar{F}(x)}$.

Equivalently, the SF of F is given by $\bar{F}(x) = \dfrac{\lambda_F^*(x)}{\lambda_F(x) + \lambda_F^*(x)}$ and the corresponding PDF is given by $f(x) = \dfrac{\lambda_F(x)\lambda_F^*(x)}{\lambda_F(x) + \lambda_F^*(x)}$.

In this paper, we shall concentrate on the study of the SF $\bar{F}(x)$, given by $\bar{F}(x) = H[-\ln G(x)]$, or equivalently, the CDF $F(x)$, as $F(x) = 1 - \bar{F}(x)$.

Three cases for $G(x)$ shall be considered:

(i) $G(x)$ is the exponentiated exponential (EE) CDF, given by

$$G(x) = (1 - e^{-\beta x})^{\alpha}, \; x > 0, \; (\alpha, \beta > 0) \tag{8}$$

The EE distribution was studied by several authors such as AL-Hussaini [2], [3]) and AL-Hussaini and Hussein ([5], [6]) and the references therein.

Substitution of (8) in (5) yields

$$\bar{F}(x) = H[-\alpha \ln(1 - e^{-\beta x})] = \int_0^{-\alpha \ln(1 - e^{-\beta x})} h(y)\, dy, \; x > 0. \tag{9}$$

(ii) $G(x)$ is Fréchet CDF, given by

$$G(x) = \exp[-\alpha\, x^{-\beta}]\,,\ x > 0,\ (\alpha, \beta > 0)\,. \tag{10}$$

The distribution was named after Maurice Fréchet [11].
From (10) and (5),

$$\bar{F}(x) = H[-\ln G(x)] = H(\alpha\, x^{-\beta}). \tag{11}$$

(iii) $G(x)$ is type 1 extreme value CDF, given by

$$G(x) = \exp[\alpha\, e^{-\beta x}],\ -\infty < x < \infty\,,\ (\alpha,\ \beta > 0\ ). \tag{12}$$

See Johnson et al [13]. It follows, from (12) and (5), that

$$\bar{F}(x) = H[-\ln G(x)] = H(\alpha\, e^{-\beta x}). \tag{13}$$

Remark 1

The function $H(.)$ is arbitrarily chosen. Tables 1-(i), 1-(ii) and 1-(iii) display eight chosen CDFs H(.) and the corresponding generated F(.) functions, obtained by applying (9), (11) and (13) and writing $F(x) = 1 - \bar{F}(x)$. The chosen H functions are Weibull, exponential, Rayleigh, Burr type XII, Lomax, compound Rayleigh, Gompertz and compound Gompertz CDFs.

Remark 2

All of the generated CDFs $F(x \mid \delta) \equiv F(x)$, in Tables 1-(i) to 1-(iii) can be written in the general form

$$F(x \mid \delta) = \exp[-\delta\, B(x)]\quad,\quad (\delta > 0) \tag{14}$$

where $B(0) = \infty$ and $B(\infty) = 0$ when , $x > 0$, as in Tables 1-(i) and 1-(ii), while

$B(-\infty) = \infty$ and $B(\infty) = 0$ when $-\infty < x < \infty$, as in Table 1-(iii).

Remark 3

It may be remarked that if H(.) is not in closed form, such as the gamma CDF, the resulting $F(.)$ will not be in closed form. This may be observed if

$$H(x) = \frac{\gamma^{\delta}}{\Gamma(\delta)} \int_0^x y^{\delta-1} e^{-\gamma y}\, dy\,,$$

it then follows, from (9), that

$$F(x) = 1 - \frac{\gamma^{\delta}}{\Gamma(\delta)} \int_0^{-\alpha \ln(1-e^{-\beta x})} y^{\delta-1} e^{-\gamma y}\, dy\,.$$

This is the CDF obtained by Ristić and Balakrishnan [19], if $\gamma = 1$.

Table 1-(i): $F(x)=1-H[-\alpha\ln(1-e^{-\beta x})]$

H(x)	$F(x\mid\delta)\equiv F(x)$	B(x)
1. Weibull (γ,η) :		
$1-e^{-\gamma x^{\eta}}$	$\exp[-\delta\{-\ln(1-e^{-\beta x})\}^{\eta}],\ \delta=\gamma\alpha^{\eta}$	$\{-\ln(1-e^{-\beta x})\}^{\eta}$
2. Exponential (γ)		
$1-e^{-\gamma x}$	$(1-e^{-\beta x})^{\delta},\ \delta=\gamma\alpha$	$-\ln(1-e^{-\beta x})$
3. Rayleigh (γ)		
$1-e^{-\gamma x^{2}}$	$\exp[-\delta\{\ln(1-e^{-\beta x})\}^{2}],\ \delta=\gamma\alpha^{2}$	$\{-\ln(1-e^{-\beta x})\}^{2}$
4. Burr type XII (γ,δ,η):		
$1-(1+\eta x^{\gamma})^{-\delta}$	$[1+\zeta\{-\ln(1-e^{-\beta x})\}^{\gamma}]^{-\delta},\ \zeta=\eta\alpha^{\gamma}$	$\ln[1+\zeta\{-\ln(1-e^{-\beta x})\}^{\gamma}]$
5. Lomax (δ,η):		
$1-(1+\eta x)^{-\delta}$	$[1-\zeta\ln(1-e^{-\beta x})]^{-\delta},\ \zeta=\eta\alpha$	$\ln[1-\zeta\ln(1-e^{-\beta x})]$
6. Compound Rayleigh (δ,η):		
$1-(1+\eta x^{2})^{-\delta}$	$[1+\zeta\{\ln(1-e^{-\beta x})\}^{2}]^{-\delta},\ \zeta=\eta\alpha^{2}$	$\ln[1+\zeta\{-\ln(1-e^{-\beta x})\}^{2}]$
7. Gompertz (δ,η) :		
$1-\exp[-\delta(e^{\eta x}-1)]$	$\exp(-\delta[(1-e^{-\beta x})^{-\zeta}-1]),\ \zeta=\eta\alpha$	$[(1-e^{-\beta x})^{-\zeta}-1]$
8. Compound Gompertz (γ,δ,η) :		
$1-[1+\frac{e^{\gamma x}-1}{\eta}]^{-\delta}$	$\left[1+\frac{(1-e^{-\beta x})^{-\zeta}-1}{\eta}\right]^{-\delta},\ \zeta=\alpha\gamma$	$\ln\left[1+\frac{(1-e^{-\beta x})^{-\zeta}-1}{\eta}\right]$

Table 1-(ii): $F(x) = 1 - H[\alpha x^{-\beta}]$

H(x)	$F(x \mid \delta) \equiv F(x)$	B(x)
1. Weibull (γ, η) :		
$1 - e^{-\gamma x^{\eta}}$	$\exp[-\delta x^{-\zeta}]$, $\delta = \gamma \alpha^{\eta}$, $\zeta = \beta\eta$	$x^{-\zeta}$
2. Exponential (γ)		
$1 - e^{-\gamma x}$	$\exp[-\delta x^{-\beta}]$, $\delta = \gamma\alpha$	$x^{-\beta}$
3. Rayleigh (γ)		
$1 - e^{-\gamma x^2}$	$\exp[-\delta x^{-2\beta}]$, $\delta = \gamma\alpha^2$	$x^{-2\beta}$
4. Burr type XII (γ, δ, η):		
$1 - (1 + \eta x^{\gamma})^{-\delta}$	$[1 + \zeta x^{-\kappa}]^{-\delta}$, $\zeta = \eta \alpha^{\gamma}$, $\kappa = \beta\gamma$	$\ln[1 + \zeta x^{-\kappa}]$
5. Lomax (δ, η):		
$1 - (1 + \eta x)^{-\delta}$	$[1 + \zeta x^{-\beta}]^{-\delta}$, $\zeta = \eta\alpha$	$\ln[1 + \zeta x^{-\beta}]$
6. Compound Rayleigh (δ, η):		
$1 - (1 + \eta x^2)^{-\delta}$	$[1 + \zeta x^{-2\beta}]^{-\delta}$, $\zeta = \eta \alpha^2$	$\ln[1 + \zeta x^{-2\beta}]$
7. Gompertz (δ, η) :		
$1 - \exp[-\delta(e^{\eta x} - 1)]$	$\exp(-\delta[e^{\zeta x^{-\beta}} - 1])$, $\zeta = \eta\alpha$	$e^{\zeta x^{-\beta}} - 1$
8. Compound Gompertz (γ, δ, η) :		
$1 - [1 + \frac{e^{\gamma x} - 1}{\eta}]^{-\delta}$	$\left[1 + \frac{e^{\zeta x^{-\beta}} - 1}{\eta}\right]^{-\delta}$, $\zeta = \alpha\gamma$	$\ln\left[1 + \frac{e^{\zeta x^{-\beta}} - 1}{\eta}\right]$

Table 1-(iii): $F(x) = 1 - H[\alpha e^{-\beta x}]$

H(x)	$F(x \mid \delta) \equiv F(x)$	B(x)
1. Weibull (γ, η) :		
$1 - e^{-\gamma x^{\eta}}$	$\exp[-\delta e^{-\zeta x}]$, $\delta = \gamma \alpha^{\eta}$, $\zeta = \beta \eta$	$e^{-\zeta x}$
2. Exponential (γ)		
$1 - e^{-\gamma x}$	$\exp[-\delta e^{-\beta x}]$, $\delta = \gamma \alpha$	$e^{-\beta x}$
3. Rayleigh (γ)		
$1 - e^{-\gamma x^2}$	$\exp[-\delta e^{-2\beta x}]$, $\delta = \gamma \alpha^2$	$e^{-\zeta x}$
4. Burr type XII (γ, δ, η):		
$1 - (1 + \eta x^{\gamma})^{-\delta}$	$[1 + \theta e^{-\xi x}]^{-\delta}$, $\theta = \eta \alpha^{\gamma}$, $\xi = \beta \gamma$	$\ln[1 + \theta e^{-\xi x}]$
5. Lomax (δ, η):		
$1 - (1 + \eta x)^{-\delta}$	$[1 + \theta e^{-\beta x}]^{-\delta}$, $\theta = \eta \alpha$	$\ln[1 + \theta e^{-\beta x}]$
6. Compound Rayleigh (δ, η):		
$1 - (1 + \eta x^2)^{-\delta}$	$[1 + \theta e^{-2\beta x}]^{-\delta}$, $\theta = \eta \alpha^2$	$\ln[1 + \theta e^{-2\beta x}]$
7. Gompertz (δ, η) :		
$1 - \exp[-\delta(e^{\eta x} - 1)]$	$\exp(-\delta[e^{\theta e^{-\beta x}} - 1])$, $\theta = \eta \alpha$	$e^{\theta e^{-\beta x}} - 1$
8. Compound Gompertz (γ, δ, η) :		
$1 - [1 + \frac{e^{\gamma x} - 1}{\eta}]^{-\delta}$	$\left[1 + \frac{e^{\varphi e^{-\beta x}} - 1}{\eta}\right]^{-\delta}$, $\varphi = \alpha \gamma$	$\ln\left[1 + \frac{e^{\varphi e^{-\beta x}} - 1}{\eta}\right]$

2. Some Properties of the Composed Distribution Function F

The PDF corresponding to the CDF (14) is given by

$$f(x|\delta) = \delta\, A(x) \exp[-\delta B(x)], \tag{15}$$

where

$$A(x) = -\, B'(x) > 0. \tag{16}$$

2.1 Quantile

The qth quantile x_q of F is given by the solution of $q = F(x_q) = 1 - H[-\ln G(x_q)]$. So that

$$x_q = F^{-1}(q) \;\text{ or }\; x_q = G^{-1}[\exp\{-H^{-1}(1-q)\}]. \tag{17}$$

where $F^{-1}(.)$, $G^{-1}(.)$ and $H^{-1}(.)$ are the inverse functions of $F(.)$, $G(.)$ and $H(.)$.

It follows that the

$$\text{median} = F^{-1}(\tfrac{1}{2}) \quad \text{or} \quad = G^{-1}[\exp\{-H^{-1}(\tfrac{1}{2})\}].$$

2.2 Moments

Except, probably, in few cases, the moments of the composed distributions F should be computed numerically. In fact, using (2.2), the rth moment, r=1, 2, . . ., is given, by

$$E(X^r) = \int_0^\infty x^r f(x|\delta)\, d\delta = \int_0^\infty x^r\, \delta A(x)\, e^{-\delta B(x)}\, d\delta = \int_0^\infty [B^{-1}(\frac{z}{\delta})]^r e^{-z}\, dz, \tag{18}$$

where $B^{-1}(.)$ is the inverse function of $B(.)$.

2.3 Mode

The mode, x , of the PDF (2.2) is such that

$$0 = f'(x|\delta) \Rightarrow \delta[\, B'(x)\,]^2 - B''(x) = 0$$

and that

$$f''(x|\delta) < 0 \Rightarrow \delta^2 [B'(x)]^3 - 3\delta B'(x)\, B''(x) + B'''(x) > 0.$$

2.4 Hazard rate function (HRF)

The HRF of $F(.)$, denoted by $\lambda_F(x)$, is given, from (14) and (15), by

$$\lambda_F(x) = \frac{f(x)}{\overline{F}(x)} = \frac{\delta A(x) \exp[-\delta B(x)]}{1-\exp[-\delta B(x)]} = \frac{\delta A(x)}{\exp[\delta B(x)]-1}. \tag{19}$$

Example 1

As an illustrative example, consider the case in which H is Burr type XII and G is EE with CDFs

$$H(x) = 1-(1+\eta x^{\gamma})^{-\delta} \quad \text{and} \quad G(x) = (1- e^{-\beta x})^{\alpha}. \tag{20}$$

The composition of H with $-\ln G(x) = -\alpha \ln(1-e^{-\beta x})$, then yields, from (2),

$\overline{F}(x) = H(-\ln G(x)) = H[-\alpha \ln(1-e^{-\beta x})] = 1-[1+\zeta\{-\ln(1-e^{-\beta x})\}^{\gamma}]^{-\delta}$, $\eta = \delta\alpha^{\delta}$, or, equivalently,

$$F(x) \equiv F(x|\delta) = [1+\zeta\{-\ln(1-e^{-\beta x})^{\gamma}]^{-\delta},$$

as given in case 4 of Table 1-(i). The corresponding PDF can be written as (15), where

$$B(x) = \ln[1+\zeta\{-\ln(1-e^{-\beta x})\}^{\gamma}] = \ln(1+\zeta y^{\gamma}), \tag{21}$$

and

$$A(x) = \frac{\zeta\gamma\{-\ln(1-e^{-\beta x})\}^{\gamma-1}}{1+\zeta\{-\ln(1-e^{-\beta x})\}^{\gamma}}\left(\frac{\beta e^{-\beta x}}{1-e^{-\beta x}}\right) = \frac{\zeta\gamma\beta y^{\gamma-1}(e^{y}-1)}{1+\zeta y^{\gamma}}, \tag{22}$$

$$y = -\ln(1-e^{-\beta x}). \tag{23}$$

- The q^{th} quantile, is given, from (14), by

$$x_q = F^{-1}[q|\delta] = -\frac{1}{\beta}\ln\left[1-\exp\left(-\left\{\frac{1}{\zeta}[q^{-1/\delta}-1]^{1/\gamma}\right\}\right)\right].$$

$\Rightarrow$

$$\text{Median} = -\frac{1}{\beta}\ln\left[1-\exp\left(-\left\{\frac{1}{\zeta}[2^{1/\delta}-1]^{1/\gamma}\right\}\right)\right].$$

- Mode

The mode x satisfies the equation $0 = f'(x|\delta) \Rightarrow \delta[B'(x)]^2 - B''(x) = 0$, provided that $f''(x|\delta) < 0$, where $B(x)$ is given by (21). After some algebraic mani-pulations, it can be shown that the mode satisfies the following equation

$$e^y = \frac{\gamma(\delta\zeta y^\gamma - 1) + (1+\zeta y^\gamma)}{\gamma(\delta\zeta y^\gamma - 1) + (1-y)(1+\zeta y^\gamma)} \quad , \tag{24}$$

where y is given by (23).

- Moments

The r^{th} moment is given by

$$E(X^r) = \int_0^\infty x^r \, \delta A(x) \exp[-\delta B(x)] \, dx = \int_0^\infty \left[B^{-1}\left(\frac{z}{\delta}\right) \right]^r e^{-z} \, dz,$$

where $B(.)$ is given by (21). So that, for r=1, 2, . . .,

$$E(X^r) = \int_0^\infty \left[-\frac{1}{\beta} \ln\left\{ 1 - \exp\left(-\left[\frac{1}{\zeta}(e^z - 1) \right]^{1/\gamma} \right) \right\} \right]^r e^{-z} \, dz. \tag{25}$$

For given values of β, ζ, γ, this integral can be numerically computed.

- Hazard rate function

It follows, from (14) and (15), that

$$\lambda_F(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\delta A(x) \, e^{-\delta B(x)}}{1 - e^{-\delta B(x)}} = \frac{\delta A(x)}{e^{\delta B(x)} - 1}.$$

Substitution of A(x) and B(x), given by (22) and (21), then yields

$$\lambda_F(x) = \frac{\delta\zeta\gamma\beta y^{\gamma-1}(e^y - 1)}{(1+\zeta y^\gamma)[(1+\zeta y^\gamma)^\delta - 1]} \quad , \tag{26}$$

where y is given by (23).

Several shapes of HRFs are obtained for different values of the parameters. For example, we can observe the following cases:

$(\delta = 0.5, \zeta = 0.2, \gamma = 2, \beta = 2)$ a decreasing HRF ,

$(\delta = 4, \zeta = 2, \gamma = 1.5, \beta = 0.5)$ an increasing HRF ,

$(\delta = 0.1, \zeta = 1.2, \gamma = 3, \beta = 0.6)$ a decreasing-increasing-decreasing HRF,

$(\delta = 0.5, \zeta = 13, \gamma = 0.1, \beta = 2)$ a decreasing-increasing HRF ,

$(\delta = 5, \zeta = 1.2, \gamma = 0.6, \beta = 0.6)$ an upside down bathtub HRF,

$(\delta = 5, \zeta = 1.2, \gamma = 0.6, \beta = 0.6)$ a decreasing-increasing HRF.

Example 2

Consider the case in which H is Burr type XII and G is type 1 extreme value with CDFs

$H(x) = 1 - (1 + \eta\, x^{\gamma})^{-\delta}$ and $G(x) = \exp(-\alpha\, e^{-\beta x})$.

The composition of H with $-\ln G(x) = \alpha e^{-\beta x}$, then yields, from (2),

$\bar{F}(x) = H(-\ln G(x)) = H(\alpha e^{-\beta x}) = 1 - [1 + \zeta\, e^{-\theta x}]^{-\delta}$, $\zeta = \eta \alpha^{\gamma}$, $\theta = \beta\gamma.$, or, equivalently,

$$F(x) \equiv F(x \mid \delta) = [1 + \zeta e^{-\theta x})]^{-\delta}, \; -\infty < x < \infty. \tag{27}$$

This is the Burr type II (ζ, θ, δ) CDF as obtained in (4) of Table 1-(iii). So, the Burr type II SF may also be obtained by composing Burr type XII CDF with – ln type 1 extreme value CDF.

The PDF, corresponding to (27), is given by

$$f(x) = \zeta\theta\delta e^{-\theta x} [1 + \zeta e^{-\theta x})]^{-\delta - 1}, \; -\infty < x < \infty, \tag{28}$$

which can be written in the form (15), where

$$B(x) = \ln[1 + \zeta e^{-\theta x})] \text{ and } A(x) = -B'(x) = \frac{\zeta\theta e^{-\theta x}}{1 + \zeta e^{-\theta x}}. \tag{29}$$

- The q^{th} quantile, is given, from (14), by

$$x_q = F^{-1}[q \mid \delta] = -\frac{1}{\theta} \ln\left[\left(q^{-1/\delta} - 1\right)/\zeta\right].$$

$\Rightarrow$

$$\text{Median}=\ x_{1/2} = -\frac{1}{\theta}\ln[(2^{1/\delta}-1)/\zeta]\ .$$

- Mode

The mode x is given by

$$x=\frac{1}{\theta}\ln(1+\delta-\zeta)\ ,\ \zeta<1+\delta\,.$$

- Moments

The r^{th} moment is given by

$$E(X^r)=\int_{-\infty}^{\infty} x^r\ \delta A(x)\exp[-\delta B(x)]\,dx=\int_0^{\infty}\left[B^{-1}\left(\frac{z}{\delta}\right)\right]^r e^{-z}\,dz\,,$$

where $B(.)$ is given by (29). So that, for r=1, 2, . . .,

$$E(X^r)=\int_0^{\infty}\left[-\frac{1}{\theta}\ln[\left(e^{z/\delta}-1\right)/\zeta]\right]^r e^{-z}\,dz\,. \qquad (30)$$

For given values of $\theta,\ \zeta,\delta$, this integral can be numerically computed.

- Hazard rate function

It follows, from (14) and (15), that

$$\lambda_F(x)=\frac{f(x)}{\bar{F}(x)}=\frac{\delta A(x)\,e^{-\delta B(x)}}{1-e^{-\delta B(x)}}=\frac{\delta A(x)}{e^{\delta B(x)}-1}\,.$$

Substitution of A(x) and B(x), given by (29), then yields

$$\lambda_F(x)=\frac{\delta\,\zeta\,\theta\,e^{-\theta x}}{(1+\zeta e^{-\theta x})[(1+\zeta e^{-\theta x})^{\delta}-1]} \qquad (31)$$

It may be noticed that the domain on which the distribution F is defined is the whole real line. So truncated version of the distribution (to be defined on the positive half of the real line) may be used in life testing.

3. Concluding Remarks

Generation of cumulative distribution functions by composition with other cumulative distributions or functions of such distributions could add an extra parameter to a distribution.

Adding a parameter or more to a distribution makes it more flexible to fitting data. Other methods were considered by Marshall and Olkin [14] and AL-Hussaini and Ghitany [4], among others. Exponentiation of a distribution function is another way of adding a parameter. AL-Hussaini [1] reviewed the research done on the EE distributions.

Indeed, many other distributions could be generated by composition. This may be attained by considering other functions rather than the distributions considered here for H and / or using other distributions than the EE, Fréchet or type 1 extreme value distrib-utions for G.

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