Semi Bayes Estimation of the Parameters of Binomial Distribution in the Presence of Outliers

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Abstract

Semi Bayes estimation of the Binomial distribution of $n$ based on some prior distribution is studied in the presence of outliers when probability of success is known and unknown. By simulation study it is conjectured that some prior distributions are more useful with respect to the generalized variance of the semi bayes estimator of $n$ and $p$.

Key Words: Binomial distribution, Bayes estimators, outliers.

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1 Introduction

Hamedani and Walter (1988) obtained Bayes estimate of the binomial parameter $n$ based on a general prior distribution of $n$. As special cases such as improper prior and Poisson prior were also considered and formulae for the marginal and posterior distributions were obtained.

The standard problem associated with the binomial distribution is that of estimating its probability of success, $p$. A much less well studied and considerably harder problem is that of estimating the number, $n$. The estimation of the parameter $n$ is not so straightforward nor well-known and does not appear to have a very large literature.

Consider a problem of shooter in Target archery. A shooter needs to aim at the target and hit precisely. To shoot the arrow, the shooter has to adopt the correct stance. He should place the body at a perpendicular angle to the shooting line with the feet apart at shoulder-width distance. The type of stance that the shooter adopts is vital to his performance. There are four types of stances, even, open, close and oblique stance. A shooter practices various stances by marking the exact placement of his feet on the shooting line. Deviation of even by a couple of inches can wreck a shooters aiming and sighting; and it is needless to say that it can begin to plague him with accuracy problems.

Let $X_1, X_2, ..., X_m$ be the number of successful attempts made by the shooter while hitting the target. The shooter has changed his stance on some attempts, say $k$ (known) which results into different probability of success. It implies that out of $m$ attempts, in $(m - k)$ attempts the probability of success is $p$ and in the remaining $k$ attempts the probability of success is changed to $\alpha p$, where $\alpha > 0$, $0 < \alpha p < 1$ and $\alpha$ is unknown.

Thus, we assume that the random variables $X_1, X_2, ..., X_m$ are such that $k$ of them are distributed with probability mass function $g(x, n, p, \alpha)$, where

$$g(x, n, p, \alpha) = C(n, x) (\alpha p)^x q_1^{n-x}, x = 0, 1, ..., n, \quad 0 < p < 1,$$

$$0 < \alpha p < 1, \quad q_1 = 1 - \alpha p, \quad (1)$$

and the remaining $(m - k)$ random variables are distributed with probability mass function $f(x, n, p)$, where

$$f(x, n, p) = C(n, x) p^x q^{n-x}; \quad x = 0, 1, ..., n, \quad 0 < p < 1, \quad q = 1 - p. \quad (2)$$
In this paper, we have obtained the Bayes estimate of \( n \) when the prior distribution of \( n \) is (i) improper prior (ii) Poisson (iii) Negative binomial, when \( p \) is known and unknown.

## 2 Improper prior Of \( n \)

We will consider the joint distribution of \((X_1, X_2, ..., X_m)\) in the presence of \( k \) outliers.

\[
f(x_1, x_2, ... x_m, | n, p) = [C(m, k)]^{-1} p^T q^{mn-T} (\frac{q_1}{q})^{(nk)} G(x, \alpha, p) \prod_{i=1}^{m} C(n, x_i)
\]

where \( C(m, k) = \frac{m!}{(m-k)k!} \),

\[
G(x, \alpha, p) = \sum_{A_1=1}^{m-k+1} \sum_{A_2=A_1+1}^{m-k+2} ... \sum_{A_k=A_{k-1}+1}^{m} (\frac{\alpha q}{q_1})^{\sum_{i=1}^{k} x_{A_i}},
\]

and \( T = \sum_{i=1}^{m} x_i \), see Dixit(1989).

Now we will consider the prior distribution of \( n \) as \( g(n) \), where

\[
g(n) = 1 \quad \forall \ n.
\]

The joint distribution of \((X_1, X_2, ..., X_m)\) is

\[
f(x_1, x_2, ... x_m, p) = \sum_{n=x_m}^{\infty} [C(m, k)]^{-1} \prod_{i=1}^{m} C(n, x_i) p^T q^{mn-T} (\frac{q_1}{q})^{(nk)} \sum_{A_1=1}^{m-k+1} \sum_{A_2=A_1+1}^{m-k+2} ... \sum_{A_k=A_{k-1}+1}^{m} (\frac{\alpha q}{q_1})^{\sum_{i=1}^{k} x_{A_i}} X g(n),
\]

where \( X = \sum_{i=1}^{k} x_{A_i} \), and

\[
\sum_{A_1=1}^{m-k+1} \sum_{A_2=A_1+1}^{m-k+2} ... \sum_{A_k=A_{k-1}+1}^{m}.
\]

Let \( n - X_{(m)} = j \Rightarrow n = X_{(m)} + j \).

Consider

\[
\sum_{n=x_{(m)}}^{\infty} \prod_{i=1}^{m} C(n, x_{(i)}) q^{mn-T} (\frac{q_1}{q})^{(nk)} = \sum_{j=0}^{\infty} \prod_{i=1}^{m} C(x_{(m)} + j, x_{(i)}) (q^{m-k} \frac{q_1}{q})^{j} q^{(m-k)x_{(m)}-T} q_1^k,
\]

\[
= \sum_{j=0}^{\infty} \left( \frac{\Gamma(x_{(m)} + j + 1)}{\Gamma(x_{(m)} + 1)} \right) \prod_{i=1}^{m-1} \frac{\Gamma(x_{(m)} + 1)}{\Gamma(x_{(i)} + 1) \Gamma(x_{(m)} - x_{(i)} + 1)} \left( q^{m-k} \frac{q_1}{q} \right)^{j} \frac{\Gamma(x_{(m)} + j + 1)}{j!},
\]
\[ m \prod_{i=1}^{m} C(x_m, x_i) mF_{m-1} \left( (x_m + 1)m; x_m - x_1 + 1, \ldots, x_m - x_{(m-1)} + 1; (q^{m-k} q_1^k) \right), \quad (7) \]

where \( (x_m + 1)m = x_m + 1, \ldots, x_m + 1 \), and \( mF_{m-1}(.) \) is a hypergeometric function.

\[
f(x, p) = [C(m, k)]^{-1} p^T q^{(m-k)x_m} q_1^{x_m k} \sum_{i=1}^{m} \frac{\alpha_{qi}^X}{i!} \prod_{i=1}^{m-1} C(x_m, x_i) \times mF_{m-1} \left( (x_m + 1)m; x_m - x_1 + 1, \ldots, x_m - x_{(m-1)} + 1; (q^{m-k} q_1^k) \right) \quad (8) \]

\[
E(n) = x_m + \sum_{m=x_m}^{\infty} \frac{m!}{(n-x_m-1)!} p^T q^{mn} q_1^{nk} \sum_{i=1}^{m} \frac{\alpha_{qi}^X}{i!} \frac{C(n, x_i)}{f(x, p)} \sum_{m=x_m}^{\infty} \frac{\alpha_{qi}^X}{i!} \frac{C(n, x_i)}{f(x, p)}.
\]

Let \( n - x_m - 1 = j \Rightarrow n = x_m + j + 1, \)
\[
= x_m + \sum_{j=0}^{\infty} \frac{\alpha_{qi}^X}{j!} \frac{C(x_m, x_i)}{f(x, p)} \sum_{j=0}^{\infty} \frac{\alpha_{qi}^X}{j!} \frac{C(x_m, x_i)}{f(x, p)} \times mF_{m-1} \left( (x_m + 1)m; x_m - x_1 + 1, \ldots, x_m - x_{(m-1)} + 1; (q^{m-k} q_1^k) \right).
\]

Consider
\[
\sum_{j=0}^{\infty} \frac{\alpha_{qi}^X}{j!} \frac{C(x_m, x_i)}{f(x, p)} \sum_{j=0}^{\infty} \frac{\alpha_{qi}^X}{j!} \frac{C(x_m, x_i)}{f(x, p)} \times \prod_{i=1}^{m} \frac{\Gamma(x_m + j + 2)}{\Gamma(x_m + 2)} \prod_{i=1}^{m-1} \left( \frac{\Gamma(x_m - x_i + 2)}{\Gamma(x_m - x_i + j + 2)} \right)
\]

\[
\hat{n}_B = x_m + (x_m + 1)q^{m-k} q_1^k \prod_{i=1}^{m} \left( \frac{x_m + 1}{x_m - x_i + 1} \right) \times mF_{m-1} \left( (x_m + 2)m; x_m - x_1 + 2, \ldots, x_m - x_{(m-1)} + 2; (q^{m-k} q_1^k) \right) \quad (9)
\]
Binomial Distribution in the Presence of Outliers

For homogeneous case consider $\alpha = 1$ in (9)

$$\hat{n}_B = x_{(m)} + (x_{(m)} + 1)q^m \prod_{i=1}^{m-1} \left( \frac{x_{(m)} + 1}{x_{(m)} - x_{(i)} + 1} \right)^{m-1}$$

$$\times \frac{mF_{m-1}((x_{(m)} + 2); x_{(m)} - x_{(1)} + 2, \ldots, x_{(m)} - x_{(m-1)} + 2; q^m)}{mF_{m-1}((x_{(m)} + 1); x_{(m)} - x_{(1)} + 1, \ldots, x_{(m)} - x_{(m-1)} + 1; q^m)}.$$  (10)

When $m = 1$ in (10),

$$\hat{n}_B = \frac{x_1 + q}{p}.$$  (11)

The expression in (11) is given Hamedani and Walter(1988)(Page:1832).

**Case(1) $p$ is known and $\alpha = 1$**

We can estimate $\hat{n}_B$ from (10).

**Case(2) $p$ is unknown and $\alpha = 1$**

By using moment estimate of $p$,

$$\hat{p} = 1 - \frac{m'_2 -(m'_1)^2}{m'_1},$$  (12)

where $m'_i = m^{-1} \sum^m_{j=1} x_{ji}; \quad i = 1, 2$

Then we can estimate $\hat{n}_B$ from (10).

**Case(3) $p$ and $\alpha$ are known**

We can estimate $\hat{n}_B$ from (9).

**Case(4) $p$ and $\alpha$ are unknown**

From (3)

$$\ln L(p, \alpha) \simeq T \ln p + (mn - T) \ln q + nk[\ln q_1 - \ln q] + \ln \sum^* \left( \frac{\alpha q}{q_1} \right)^X,$$  (13)

Let

$$\sum^* \left( \frac{\alpha q}{q_1} \right)^X = S, \quad \sum^* X \left( \frac{\alpha q}{q_1} \right)^X = S_x, \text{ and } \sum^* X^2 \left( \frac{\alpha q}{q_1} \right)^X = S_{xx}.$$

The likelihood Equations are as follows:

$$\frac{T}{p} - \frac{mn - T}{q} + nk\left[\frac{-\alpha}{q_1} + \frac{1}{q} \right] + \frac{\alpha - 1}{qq_1} S_x = 0,$$  (14)

$$\frac{-nk p}{q_1} + \frac{1}{q q_1} S_x = 0.$$  (15)
From (14) and (15) we get one more equation

\[ m'_1 = np(b\alpha + \bar{b}), \quad (16) \]

write (15) as

\[ nk\alpha - \frac{S_x}{S} = 0. \quad (17) \]

Let

\[ t_1(p, \alpha) = \bar{x} - np(b\alpha + \bar{b}), \quad (18) \]

and

\[ t_2(p, \alpha) = \frac{S_x}{S} - nk\alpha. \quad (19) \]

Expanding the equations (18) and (19) by Taylor series around \((p_0, \alpha_0)\),

\[ t_1(p, \alpha) = t_1(p_0, \alpha_0) + \Delta pt'_1(p_0, \alpha_0) + \Delta \alpha t'_{1\alpha}(p_0, \alpha_0) + o(m^{-1}), \quad (20) \]

and

\[ t_2(p, \alpha) = t_2(p_0, \alpha_0) + \Delta pt'_2(p_0, \alpha_0) + \Delta \alpha t'_{2\alpha}(p_0, \alpha_0) + o(m^{-1}), \quad (21) \]

where \(\Delta \alpha = \alpha - \alpha_0\), \(\Delta p = p - p_0\), \(t'_1(p, \alpha)\) and \(t'_{1\alpha}(p, \alpha)\) are the first order derivative of \(t_1(p, \alpha)\) with respect to \(p\) and \(\alpha\) respectively. Similarly \(t'_2(p, \alpha)\) and \(t'_{2\alpha}(p, \alpha)\) are the first order derivative of \(t_2(p, \alpha)\) with respect to \(p\) and \(\alpha\) respectively.

\[ t'_1 = -n(b\alpha + \bar{b}), \]

\[ t'_{1\alpha} = -np, \]

\[ t'_2 = \frac{\alpha - 1}{qq_1}\left[ \frac{S_{xx}}{S} - \left( \frac{S_x}{S} \right)^2 \right] - nk\alpha, \]

and

\[ t'_{2\alpha} = \frac{1}{\alpha q_1}\left[ \frac{S_{xx}}{S} - \left( \frac{S_x}{S} \right)^2 \right] - nk\alpha. \]

Write (18) and (19) as follows

\[ t_1(p, \alpha) = c_1 + g\Delta p + h\Delta \alpha = 0, \quad (22) \]
and

\[ t_2(p, \alpha) = c_2 + e \triangle p + f \triangle \alpha = 0. \] (23)

Solving (22) and (23)

\[ \triangle \alpha = \frac{-c_2 g + c_1 e}{fg - eh}, \]

and

\[ \triangle p = \frac{-c_2 h + c_1 f}{eh - fg}. \]

The corrections \( \triangle p \) and \( \triangle \alpha \) are obtained such that we may expect

\[ p^* = \triangle p + p_0, \]

\[ \alpha^* = \triangle \alpha + \alpha_0. \]

Select the initial solutions \( p_0 \) and \( \alpha_0 \) then from (9) calculate \( \hat{n}_B \) and call it as \( n_0 \). After iterations we will get \( p^* \) and \( \alpha^* \). Kale(1962) had considered the multiparameter case and shown that this method is justified. From that we will get \( \hat{n}^*_B \). For details, see Dixit and Kelkar(2011).

### 3 Prior of \( n \) is Poisson(\( \lambda \))

Let

\[ g(n) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2, ..., \quad \lambda > 0. \] (24)

The joint probability distribution of \((X_1, X_2, ..., X_m)\) is

\[ f(x_1, x_2, ..., x_m, n, p) = [C(m, k)]^{-1} \prod_{i=1}^{m} C(n, x_i) p^T q^{mn-T} \left( \frac{q_1}{q} \right)^{(nk)} \sum_{q} \left( \frac{\alpha q}{q_1} \right) X e^{-\lambda} \frac{n^X}{n!}, \] (25)

\[ f(x_1, x_2, ..., x_m, p) = \sum_{n=x(m)}^{\infty} [C(m, k)]^{-1} \prod_{i=1}^{m} C(n, x_i) p^T q^{mn-T} \left( \frac{q_1}{q} \right)^{(nk)} \sum_{q} \left( \frac{\alpha q}{q_1} \right) X e^{-\lambda} \frac{n^X}{n!}, \] (26)
After simplification
\[
f(x,p) = \left[C(m,k)\right]^{-1} \left(\frac{p}{q}T_e^{-\lambda q^m(x)}\right)^k \sum \sum C(x(m),x(i)) \prod \left[C(x(m),x(i))]^{-1}
\]
\[
\times m^{-1} F_{m-1} \left((x(m) + 1)m - x(1) + 1, ..., x(m) - x(m-1) + 1; (q^{m-k}q^k)\right).
\]

Bayes estimate of \(n\) in squared error loss function is
\[
\hat{n}_B = x(m) + q^{(m-k)}q^k \prod \left(\frac{x(m) + 1}{x(m) - x(i) + 1}\right)
\]
\[
\times m^{-1} F_{m-1} \left((x(m) + 2)m - x(1) + 2, ..., x(m) - x(m-1) + 2; q^{m-k}q^k\right).
\]

For homogeneous case \(\alpha = 1\),
\[
\hat{n}_B = x(m) + q^{m} \prod \left(\frac{x(m) + 1}{x(m) - x(i) + 1}\right)
\]
\[
\times m^{-1} F_{m-1} \left((x(m) + 2)m - x(1) + 2, ..., x(m) - x(m-1) + 2; q^{m}\right).
\]

When \(m = 1\),
\[
\hat{n}_B = x_1 + q\lambda.
\]

The expression in (30) is given Hamedani and Walter(1988)(Page:1834).

Case(1) \(p, \lambda\) are known and \(\alpha = 1\)
We can estimate \(\hat{n}_B\) from (29).

Case(2) \(p\) is unknown, \(\lambda\) is known and \(\alpha = 1\)
By using moment estimate of \(p\) from (12) and then estimate \(\hat{n}_B\) from (29).

Case(3) \(p, \alpha\) and \(\lambda\) are known
We can estimate \(\hat{n}_B\) from (28).

Case(4) \(p\) and \(\alpha\) are unknown, \(\lambda\) is known
By using method of mle expressions from (13) to (23), we can estimate \(\hat{n}_B\).

4 Prior of \(n\) is Negative Binomial(\(\theta, r\))

\[
g(n) = \frac{\Gamma(r+n)}{\Gamma(r)\Gamma(n+1)} (1-\theta)^r \theta^n, \quad n = 0, 1, 2, ..., r = 1, 2, 3, ..., \quad \theta > 0.
\]
The joint probability distribution of \((X_1, X_2, \ldots, X_m)\) is

\[
f(x_1, x_2, \ldots, x_m, n, p) = [C(m, k)]^{-1} \prod_{i=1}^{m} C(n, x_i)p^T q^{mn-T} \left( \frac{q_1}{q} \right)^{(nk)\sum^*} \left( \frac{\alpha q}{q_1} \right)^X \times \frac{\Gamma(r + n)}{\Gamma(r)\Gamma(n + 1)}(1 - \theta)^r \theta^n,
\]

After simplification

\[
f(z, p) = [C(m, k)]^{-1}C(r + x(m) - 1, x(m)) \prod_{i=1}^{m} C(x(m), x_i) \left( \frac{p}{q} \right)^T (1 - \theta)^r (q^{m-k} q_1^k \theta)^{x(m)} \sum^* \left( \frac{\alpha q}{q_1} \right)^X \times mF_{m-1} \left( (x(m) + 1)_{m-1}, r + x(m); x(m) - x(1) + 1, \ldots, x(m) - x(m-1) + 1; (q^{m-k} q_1^k \theta) \right).
\]

Bayes estimate of \(n\) in squared error loss function is

\[
\hat{n}_B = x(m) + q^{(m-k)} q_1^k \theta \left( \prod_{i=1}^{m-1} \frac{x(m) + 1}{x(m) - x(i) + 1} \right) \left( \frac{x(m) + r}{r} \right) \times mF_{m-1} \left( (x(m) + 2)_{m-1}, r + x(m) + 1; x(m) - x(1) + 2, \ldots, x(m) - x(m-1) + 2; q^{m-k} q_1^k \theta \right)
\]

For homogeneous case, \(\alpha = 1\)

\[
\hat{n}_B = x(m) + q^n \theta \left( \prod_{i=1}^{m-1} \frac{x(m) + 1}{x(m) - x(i) + 1} \right) \left( \frac{x(m) + r}{r} \right) \times mF_{m-1} \left( (x(m) + 2)_{m-1}, r + x(m) + 1; x(m) - x(1) + 2, \ldots, x(m) - x(m-1) + 2; q^n \theta \right)
\]

**Case(1) \(p, \theta, r\) are known and \(\alpha = 1\)**

We can estimate \(\hat{n}_B\) from (36).

**Case(2) \(p\) is unknown, \(\theta, r\) are known and \(\alpha = 1\)**

By using moment estimate of \(p\) from (12) and then estimate \(\hat{n}_B\) from (36).

**Case(3) \(p, \alpha, \theta\) and \(r\) are known**

We can estimate \(\hat{n}_B\) from (35).

**Case(4) \(p\) and \(\alpha\) are unknown, \(\theta\) and \(r\) are known**

By using method of mle expressions from (13) to (23), we can estimate \(\hat{n}_B\).
5 Numerical Study

Our all results of estimation of \( n \) are same as Draper and Guttman(1971) and Hamedani and Walter(1988) in case of improper and poisson prior (\( \lambda=13 \)) of \( n \). i.e. In homogeneous case when improper and poisson priors are used the estimator of \( n \) is 13 for \( m=2, x_1=10, x_2=12 \) and \( p=0.8 \).

Moreover we also have used Negative Binomial prior for \( n \) and the estimate of \( n \) is 12 where \( m=2, x_1=10, x_2=12, p=0.8, \theta=0.6 \) and \( r=3 \).

In order to get the idea of selecting prior of \( n \), we have generated one thousand samples of size \( m = 10(20)50 \) from the Binomial distribution with \( n = 10, p=.4,.5,.6 \) and \( .8, \alpha=.4, .5, .6 \) and \( .8 \) and \( k=0,1,2,3 \) for the following three prior distributions of \( n \),

(i)\( g_1(n) = 1 \) for all \( n \)
(ii)\( g_2(n) = \frac{e^{-\lambda \alpha}}{\lambda^n n!} \) \( n = 0, 1, 2, ..., \lambda = 2 \) and \( 3 \)
(iii)\( g_3(n) = \frac{\Gamma(r+n)}{\Gamma(r+1)}(1-\theta)^r \theta^n \) \( n = 0, 1, 2, ..., r = 1, 2, 3, \theta = .4 \) and \( .6 \)

We would like to comment on bias of \( \hat{n}, \hat{\alpha} \) and \( \hat{p} \).

A)Homogeneous case \((k = 0)\):
1.Bias of \( \hat{n}, \hat{\alpha} \) need not be positive for \( m = 10(10)50 \).
2.Bias of \( \hat{p} \) is more for \( m=10, 20 \) for an improper prior distribution of \( n \) as compared to the other prior distributions of \( n \).
3.Bias of \( \hat{p} \) is almost same for \( m=30 \) for all prior distributions of \( n \).
4.Bias of \( \hat{p} \) is less for \( m=40, 50 \) for an improper prior distribution of \( n \) as compared to the other prior distributions of \( n \).
5.We can not draw any such conclusions in case of bias of \( \hat{n} \) for all prior distributions of \( n \).
6.For \( m > 50 \) the bias of \( \hat{n} \) is less than 1 and bias of \( \hat{p} \) tends to zero.

B)Outliers case \((k > 0)\):
We have plotted graphs for various values of \((p, \alpha)\) for \( n = 10 \).
1.Bias of \( \hat{p} \) is always positive for all prior distributions of \( n \).
2.Bias of \( \hat{n} \) and \( \hat{\alpha} \) need not be positive for all prior distributions of \( n \).
3.Bias of \( \hat{n}, \hat{p} \) and \( \hat{\alpha} \) is more in case of improper prior distribution of \( n \) as compared to the Poison and Negative Binomial prior distributions of \( n \).
4. Interestingly for $k=3$, bias of $\hat{\alpha}$ is almost same for $m=10(10)50$.

5. For $m > 50$ bias of $\hat{n}$ is less than 1, bias of $\hat{p}$ and bias of $\hat{\alpha}$ tends to zero.

In case of multiparametric case we have to compare the efficiencies of estimators on the basis of the determinant of variance-covariance matrix.

Here we have calculated the determinant of variance-covariance matrix of $(\hat{p}, \hat{\alpha}, \hat{n})$ under $g_1$, $g_2$ and $g_3$.

In homogeneous case i.e. $k = 0$ and $\alpha = 1$, we have presented determinant in Tables 1-4 for $m = 10(20)50$ with $n = 10$, $p = .4, .5, .6$ and .8. Determinant of variance-covariance matrix under $g_1$ is more as compared to under $g_2$ and $g_3$. We can conjecture that estimates of $n$ and $p$ under $g_2$ and $g_3$ are better than under $g_1$. But there is no significant difference in the determinant under $g_2$ and $g_3$.

For outlier case i.e. $k=1$ and 3, we have plotted the generalized variance-covariance matrix. Determinant under $g_1$ is significantly less than the determinant under $g_2$ and $g_3$. In this case estimates of $p$, $\alpha$ and $n$ are better under $g_1$ than the estimates of $p$, $\alpha$ and $n$ under $g_2$ and $g_3$. Here also determinant does not change significantly under $g_2$ and $g_3$. 
Appendix

Determinant of \( \hat{p} \) and \( \hat{n} \)

**Table 1.** \( \lambda = 2 \) \( \theta = 0.4 \) \( r=1 \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>Prior</th>
<th>( (p = 0.4) )</th>
<th>( (p = 0.5) )</th>
<th>( (p = 0.6) )</th>
<th>( (p = 0.8) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Improper</td>
<td>2.19999739</td>
<td>0.99312454</td>
<td>0.0917377</td>
<td>0.00040044</td>
</tr>
<tr>
<td>10</td>
<td>Poisson</td>
<td>0.00433595</td>
<td>0.00328884</td>
<td>0.00205026</td>
<td>0.00033842</td>
</tr>
<tr>
<td>10</td>
<td>NegBin</td>
<td>0.00433285</td>
<td>0.00359194</td>
<td>0.00216831</td>
<td>0.00036458</td>
</tr>
<tr>
<td>30</td>
<td>Improper</td>
<td>0.52781872</td>
<td>0.10576992</td>
<td>0.00077724</td>
<td>0.00008396</td>
</tr>
<tr>
<td>30</td>
<td>Poisson</td>
<td>0.00106245</td>
<td>0.00077132</td>
<td>0.00048884</td>
<td>0.00001434</td>
</tr>
<tr>
<td>30</td>
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<td>0.00100415</td>
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</tr>
<tr>
<td>50</td>
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<td>0.30105318</td>
<td>0.00074582</td>
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<td>0.0000253</td>
</tr>
<tr>
<td>50</td>
<td>Poisson</td>
<td>0.0007507</td>
<td>0.00061755</td>
<td>0.00041965</td>
<td>0.0000258</td>
</tr>
<tr>
<td>50</td>
<td>NegBin</td>
<td>0.00070377</td>
<td>0.00066386</td>
<td>0.00050307</td>
<td>0.0000574</td>
</tr>
</tbody>
</table>

**Table 2.** \( \lambda = 2 \) \( \theta = 0.4 \) \( r=3 \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>Prior</th>
<th>( (p = 0.4) )</th>
<th>( (p = 0.5) )</th>
<th>( (p = 0.6) )</th>
<th>( (p = 0.8) )</th>
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<tbody>
<tr>
<td>10</td>
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<td>2.19999739</td>
<td>0.99312454</td>
<td>0.0917377</td>
<td>0.00040044</td>
</tr>
<tr>
<td>10</td>
<td>Poisson</td>
<td>0.00433595</td>
<td>0.00328884</td>
<td>0.00205026</td>
<td>0.00033842</td>
</tr>
<tr>
<td>10</td>
<td>NegBin</td>
<td>0.00393732</td>
<td>0.00306371</td>
<td>0.00194653</td>
<td>0.0003351</td>
</tr>
<tr>
<td>30</td>
<td>Improper</td>
<td>0.52781872</td>
<td>0.10576992</td>
<td>0.00077724</td>
<td>0.00008396</td>
</tr>
<tr>
<td>30</td>
<td>Poisson</td>
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<td>0.00077132</td>
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<td>0.00001434</td>
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<tr>
<td>30</td>
<td>NegBin</td>
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<tr>
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<td>0.0000253</td>
</tr>
<tr>
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<td>Poisson</td>
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<td>0.00061755</td>
<td>0.00041965</td>
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<tr>
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<td>0.00018024</td>
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Determinant of $\hat{p}$ and $\hat{n}$

**Table 3.** $\lambda = 3 \theta = 0.6 \ r=1$

<table>
<thead>
<tr>
<th>$m$</th>
<th>Prior</th>
<th>$(p = 0.4)$</th>
<th>$(p = 0.5)$</th>
<th>$(p = 0.6)$</th>
<th>$(p = 0.8)$</th>
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</thead>
<tbody>
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<td>0.99312454</td>
<td>0.0917377</td>
<td>0.00040044</td>
</tr>
<tr>
<td>10</td>
<td>Poisson</td>
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<td>0.00366246</td>
<td>0.00220827</td>
<td>0.00034856</td>
</tr>
<tr>
<td>10</td>
<td>NegBin</td>
<td>0.00403551</td>
<td>0.00311399</td>
<td>0.00229634</td>
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</tr>
<tr>
<td>30</td>
<td>Improper</td>
<td>0.52781872</td>
<td>0.10576992</td>
<td>0.00077724</td>
<td>0.00008396</td>
</tr>
<tr>
<td>30</td>
<td>Poisson</td>
<td>0.00114861</td>
<td>0.00076347</td>
<td>0.0005312</td>
<td>0.00002073</td>
</tr>
<tr>
<td>30</td>
<td>NegBin</td>
<td>0.0010768</td>
<td>0.00086622</td>
<td>0.00070313</td>
<td>0.00003545</td>
</tr>
<tr>
<td>50</td>
<td>Improper</td>
<td>0.30105318</td>
<td>0.00074582</td>
<td>0.00064234</td>
<td>0.0000253</td>
</tr>
<tr>
<td>50</td>
<td>Poisson</td>
<td>0.00074576</td>
<td>0.00065308</td>
<td>0.00045689</td>
<td>0.00000339</td>
</tr>
<tr>
<td>50</td>
<td>NegBin</td>
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<td>0.00059875</td>
<td>0.00000951</td>
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**Table 4.** $\lambda = 3 \theta = 0.6 \ r=3$

<table>
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<tr>
<th>$m$</th>
<th>Prior</th>
<th>$(p = 0.4)$</th>
<th>$(p = 0.5)$</th>
<th>$(p = 0.6)$</th>
<th>$(p = 0.8)$</th>
</tr>
</thead>
<tbody>
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<td>10</td>
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<td>0.99312454</td>
<td>0.0917377</td>
<td>0.00040044</td>
</tr>
<tr>
<td>10</td>
<td>Poisson</td>
<td>0.005034</td>
<td>0.00366246</td>
<td>0.00220827</td>
<td>0.00034856</td>
</tr>
<tr>
<td>10</td>
<td>NegBin</td>
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<td>0.00315474</td>
<td>0.00200292</td>
<td>0.00034295</td>
</tr>
<tr>
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<td>Improper</td>
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<td>0.10576992</td>
<td>0.00077724</td>
<td>0.00008396</td>
</tr>
<tr>
<td>30</td>
<td>Poisson</td>
<td>0.00114861</td>
<td>0.00076347</td>
<td>0.0005312</td>
<td>0.00002073</td>
</tr>
<tr>
<td>30</td>
<td>NegBin</td>
<td>0.00099152</td>
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</tr>
<tr>
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<td>0.0000253</td>
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<tr>
<td>50</td>
<td>Poisson</td>
<td>0.00074576</td>
<td>0.00065308</td>
<td>0.00045689</td>
<td>0.00000339</td>
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<tr>
<td>50</td>
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<td>0.00033741</td>
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<td>0.00000122</td>
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</tbody>
</table>
Bias of $\hat{n}$

For $n=10$, $k=1$, $\lambda=2$, $\theta=0.4$, and $r=3$:

- $m=10$
  - Improper
  - Poisson
  - NegBin

For $n=10$, $k=3$, $\lambda=2$, $\theta=0.4$, and $r=3$:

- $m=10$
  - Improper
  - Poisson
  - NegBin

For $n=30$, $k=1$, $\lambda=2$, $\theta=0.4$, and $r=3$:

- $m=30$
  - Improper
  - Poisson
  - NegBin

For $n=30$, $k=3$, $\lambda=2$, $\theta=0.4$, and $r=3$:

- $m=30$
  - Improper
  - Poisson
  - NegBin

For $n=50$, $k=1$, $\lambda=2$, $\theta=0.4$, and $r=3$:

- $m=50$
  - Improper
  - Poisson
  - NegBin

For $n=50$, $k=3$, $\lambda=2$, $\theta=0.4$, and $r=3$:

- $m=50$
  - Improper
  - Poisson
  - NegBin
Bias of $\hat{p}$

$n=10$  $k=1$  $\lambda=2$  $\theta=0.4$  $r=3$

$n=10$  $k=3$  $\lambda=2$  $\theta=0.4$  $r=3$
Bias of $\hat{\alpha}$

- $n=10$, $k=1$, $\lambda=2$, $\theta=0.6$, $r=3$
- $n=10$, $k=3$, $\lambda=2$, $\theta=0.6$, $r=3$
Generalized variance of $\hat{p}$, $\hat{\alpha}$ and $\hat{n}$

$n=10$  $k=1$  $\lambda=2$  $r=1$  $\theta=0.4$

$n=10$  $k=1$  $\lambda=2$  $r=3$  $\theta=0.4$
Generalized variance of $\hat{\rho}$, $\hat{\alpha}$ and $\hat{\beta}$
Generalized variance of $\hat{p}$, $\hat{\alpha}$, and $\hat{\mu}$

$n=10 \quad k=3 \quad \lambda=2 \quad r=1 \quad \theta=0.4$

$n=10 \quad k=3 \quad \lambda=2 \quad r=3 \quad \theta=0.4$
### Generalized variance of $\hat{p}$, $\hat{\alpha}$ and $\hat{n}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
<th>$\lambda$</th>
<th>$r$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
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<td>3</td>
<td>1</td>
<td>0.6</td>
</tr>
</tbody>
</table>

![Graphs for different values of $m$](image1)

- $m=10$
- Improper
- Poisson
- NegBin

![Graphs for different values of $m$](image2)

- $m=30$
- Improper
- Poisson
- NegBin

![Graphs for different values of $m$](image3)

- $m=50$
- Improper
- Poisson
- NegBin
Acknowledgment

The authors are thankful to the editors and the referees for their valuable comments.

References


