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RECURRENCE RELATIONS FOR MOMENTS OF PROGRESSIVELY TYPE-II RIGHT CENSORED ORDER STATISTICS FROM HALF LOGISTIC DISTRIBUTION

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Abstract

In this paper some recurrence relations between the single and product moments of progressively Type-II right censored order statistics from half logistic distribution have been established. These relations would enable one to compute all the single and product moments, and hence all the means, variances and covariances of half logistic progressively Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, ..., R_m)$, $m \leq n$, in a simple recursive manner. The results presented in this paper generalize the results given by Balakrishnan (1985) for the single and product moments of usual order statistics from the half logistic distribution.

Keywords and Phrases: Single moments; Product moments; Progressively Type-II right censored order statistics; Recurrence relations; half logistic distribution.
AMS 2000 Subject Classification : 62G30

1. Introduction

A random variable X is said to have a half logistic distribution if the p.d.f. is of the form

$$f(x)=\begin{cases}\dfrac{2e^{-x}}{(1+e^{-x})^2}, & 0\le x<\infty,\\ 0, & \text{elsewhere}\end{cases} \tag{1}$$

and the c.d.f. is of the form

$$F(x)=\frac{1-e^{-x}}{1+e^{-x}},\quad 0\le x<\infty. \tag{2}$$

Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables representing failure times of n identical units placed on a life-test. Under the progressive Type-II right censoring scheme, at the time of i-th failure $(i=1,2,...,m,$ where $m\le n)$, R_i surviving items are removed at random from the experiment, where $R_1, R_2, ..., R_m$ are fixed integers. Thus, in this type of sampling, we observe in all m failures and $\sum_{i=1}^{m} R_i$ items are progressively censored so that $n=m+\sum_{i=1}^{m} R_i$. The withdrawal of elements may be seen as a model describing drop-outs of units due to failures, which have causes other than the specific one under study. In this sense, progressive censoring schemes are applied in clinical trials as well. The drop-outs of patients may be caused, e.g., by personal or ethical decisions, and they are regarded as random withdrawals.

We shall denote the m ordered observed failure times by $X_{1:m:n}^{(R_1,R_2,...,R_m)}$, $X_{2:m:n}^{(R_1,R_2,...,R_m)}, ..., X_{m:m:n}^{(R_1,R_2,...,R_m)}$, and call them the progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, ..., R_m)$, $m\le n$.

Progressive censoring sampling scheme is very useful in reliability and life time studies. This scheme reduces the experimenter cost while still allowing for the observation of some extreme data. Inferential issues based on this scheme have been extensively studied in the literature for a number of distributions by several authors including Cohen (1963, 1966, 1975, 1976, 1991); Mann (1969, 1971); Cohen and Whitten (1988); Viveros and Balakrishnan (1994) and Balakrishnan and Sandhu (1995). Aggarwala and Balakrishnan (1996) and Balakrishnan and Aggarwala (2000, Chapter 4) have derived recurrence relations for single and product moments of progressively Type-II right censored order statistics from exponential, Pareto and power function distributions and their truncated forms. Also, Saran and Pushkarna (2001) have obtained several recurrence relations for the single and product moments of progressively Type-II right censored order statistics from doubly truncated Burr distribution.

In the present paper, by assuming the underlying distribution of failure times as half logistic, we shall establish some recurrence relations for the single and product moments of the corresponding progressively Type-II right censored order statistics which would allow for the recursive computation of these moments for all sample sizes and all possible censoring schemes $(R_1, R_2, ..., R_m)$, $m \le n$. The results obtained in this paper generalize the results given by Balakrishnan (1985) for the usual order statistics from the half logistic distribution.

We shall denote the single and product moments of progressively Type-II right censored order statistics $X_{i:m:n}^{(R_1,R_2,...,R_m)}$, $1 \le i \le m$, as follows:

$$\mu_{i:m:n}^{(R_1,R_2,...,R_m)^{(k)}} = E\left[X_{i:m:n}^{(R_1,R_2,...,R_m)}\right]^k, \quad 1 \le i \le m \le n,\ k \ge 0,$$

$$\mu_{i:m:n}^{(R_1,R_2,...,R_m)^{(1)}} \equiv \mu_{i:m:n}^{(R_1,R_2,...,R_m)},$$

$$\mu_{i,j:m:n}^{(R_1,R_2,...,R_m)} = E\left[X_{i:m:n}^{(R_1,R_2,...,R_m)} X_{j:m:n}^{(R_1,R_2,...,R_m)}\right], \quad 1 \le i < j \le m \le n,$$

$$\mu_{i,i:m:n}^{(R_1,R_2,...,R_m)} \equiv \mu_{i:m:n}^{(R_1,R_2,...,R_m)^{(2)}}, \quad 1 \le i \le m \le n. \tag{3}$$

2. Recurrence Relations for Single Moments

From equations (1) and (2), we note that the characterizing differential equation for the half logistic distribution is given by

$$f(x) = [1 - F(x)] - \frac{1}{2}[1 - F(x)]^2, \tag{4}$$

which will be utilized for deriving recurrence relations for the single moments of progressively Type-II right censored order statistics from half logistic distribution.

Let $X_{1:m:n}^{(R_1,R_2,...,R_m)} < X_{2:m:n}^{(R_1,R_2,...,R_m)} < ... < X_{m:m:n}^{(R_1,R_2,...,R_m)}$ be the m ordered observed failure times in a sample of size n from the half logistic distribution (1) under progressive Type-II right censoring scheme $(R_1, R_2, ..., R_m)$. Then the joint p.d.f. of $X_{1:m:n}^{(R_1,R_2,...,R_m)}, X_{2:m:n}^{(R_1,R_2,...,R_m)}, ..., X_{m:m:n}^{(R_1,R_2,...,R_m)}$ is given by (Balakrishnan and Sandhu, 1995)

$$f_{1,2,...,m:m:n}(x_1, x_2, ..., x_m) = A(n, m-1)\prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},$$

$$0 < x_1 < x_2 < ... < x_m < \infty, \tag{5}$$

where

$$A(n,m-1)=n(n-R_1-1)(n-R_1-R_2-2)...(n-R_1-R_2-...-R_{m-1}-m+1),$$

and $f(x)$ and $F(x)$ are as given in (1) and (2), respectively. Here, note that all the factors in $A(n,m-1)$ are positive integers. Also it may be observed that the different factors in $A(n,m-1)$ represent the number of units still on test immediately preceding the first, second,…, m-th observed failures, respectively.

Similarly, for convenience in notation, let us define

$$A(p,q)=p(p-R_1-1)(p-R_1-R_2-2)...(p-R_1-R_2-...-R_q-q), \tag{6}$$

for $q=0,1,...,p-1$, with all the factors being positive integers. Thus

$$\mu_{i:m:n}^{(R_1,R_2,...,R_m)^{(k)}}=A(n,m-1)\int\int_{0<x_1<...<x_m<\infty}...\int x_i^k\prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}\,dx_t\,. \tag{7}$$

Theorem 2.1. For $2\le m\le n$ and $k\ge 0$,

$$\mu_{1:m:n+1}^{(R_1+1,R_2,...,R_m)^{(k+1)}}=\frac{2(n+1)}{n(R_1+2)}\Big[(n-R_1-1)\,\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,...,R_m)^{(k+1)}}+(R_1+1)\,\mu_{1:m:n}^{(R_1,R_2,...,R_m)^{(k+1)}}$$
$$-\frac{n}{2}\frac{(n-R_1-1)}{(n+1)}\,\mu_{1:m-1:n+1}^{(R_1+R_2+2,R_3,...,R_m)^{(k+1)}}-(k+1)\,\mu_{1:m:n}^{(R_1,...,R_m)^{(k)}}\Big], \tag{8}$$

and for $m=1,\ n=1,2,...$ and $k\ge 0$,

$$\mu_{1:1:n+1}^{(n)^{(k+1)}}=2\left[\mu_{1:1:n}^{(n-1)^{(k+1)}}-\frac{k+1}{n}\mu_{1:1:n}^{(n-1)^{(k)}}\right]. \tag{9}$$

Proof. Let us consider (7) with $i=1$, i.e.,

$$\mu_{1:m:n}^{(R_1,...,R_m)^{(k)}}=A(n,m-1)\int\int_{0<x_2<x_3\,...\,<x_m<\infty}...\int\left[\int_0^{x_2}x_1^k f(x_1)[1-F(x_1)]^{R_1}dx_1\right]$$
$$\times\ f(x_2)[1-F(x_2)]^{R_2}...f(x_m)[1-F(x_m)]^{R_m}dx_2...dx_m$$
$$=A(n,m-1)\int\int_{0<x_2<x_3<...<x_m<\infty}...\int I(x_2)\prod_{t=2}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t, \tag{10}$$

where

$$I(x_2)=\int_0^{x_2}x_1^k f(x_1)[1-F(x_1)]^{R_1}dx_1\,. \tag{11}$$

Making use of the relation in (4) by replacing therein x by x_1, we have

$$I(x_2)=I_1(x_2)-\frac{1}{2}I_2(x_2), \tag{12}$$

where

$$I_a(x_2) = \int_0^{x_2} x_1^k [1-F(x_1)]^{R_1+a} dx_1 , \quad a = 1, 2 .$$

Integration by parts yields,

$$I_a(x_2) = \frac{1}{(k+1)}\left[x_2^{k+1}[1-F(x_2)]^{R_1+a} + (R_1+a)\int_0^{x_2} x_1^{k+1}[1-F(x_1)]^{R_1+a-1} f(x_1)dx_1 \right]. \qquad (13)$$

Upon substituting for $I_1(x_2)$ and $I_2(x_2)$ from (13) in (12) and then substituting the resultant expression for $I(x_2)$ in equation (10), we get

$$\begin{aligned}
\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} &= \frac{A(n,m-1)}{(k+1)} \int\int_{0<x_2<x_3\,\dots\,<x_m<\infty}\dots\int x_2^{k+1}[1-F(x_2)]^{R_1+R_2+1} f(x_2)dx_2 \prod_{t=3}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t \\
&+ \frac{A(n,m-1)(R_1+1)}{(k+1)} \int\int_{0<x_1<x_2<\dots<x_m<\infty}\dots\int x_1^{k+1} \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t \\
&- \frac{1}{2}\frac{A(n,m-1)}{(k+1)} \int\int_{0<x_2<x_3<\dots<x_m<\infty}\dots\int x_2^{k+1} f(x_2)[1-F(x_2)]^{R_1+R_2+2} dx_2 \prod_{t=3}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t \qquad \text{wh} \\
&- \frac{1}{2}\frac{A(n,m-1)(R_1+2)}{(k+1)} \int\int_{0<x_1<x_2<\dots<x_m<\infty}\dots\int x_1^{k+1} f(x_1)[1-F(x_1)]^{R_1+1} dx_1 \prod_{t=2}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t \\
&= \frac{(n-R_1-1)}{(k+1)} \mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(k+1)}} + \frac{(R_1+1)}{(k+1)} \mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+1)}} - \frac{n(n-R_1-1)}{2(n+1)(k+1)} \mu_{1:m-1:n+1}^{(R_1+R_2+2,R_3,\dots,R_m)^{(k+1)}} \\
&- \frac{n(R_1+2)}{2(n+1)(k+1)} \mu_{1:m:n+1}^{(R_1+1,R_2,\dots,R_m)^{(k+1)}} ,
\end{aligned}$$

ich on rearranging the terms leads to (8).

To prove the relation in (9), we take $i=1$, $m=1$ in (7) and then using (4) with *x* replaced by x_1, we get

$$\mu_{1:1:n}^{(R_1)^{(k)}} = A(n,0)\left[\int_0^{\infty} x_1^k [1-F(x_1)]^{R_1+1} dx_1 - \frac{1}{2}\int_0^{\infty} x_1^k [1-F(x_1)]^{R_1+2} dx_1 \right].$$

Integrating the right hand side integrals by parts and noting that $R_1 = n-1$, since the equation $R_1 + R_2 + \dots + R_m + m = n$ must be satisfied, we have

$$\begin{aligned}
\mu_{1:1:n}^{(n-1)^{(k)}} &= \frac{n}{(k+1)}\left[n\int_0^{\infty} x_1^{k+1}[1-F(x_1)]^{n-1} f(x_1)dx_1 - \frac{(n+1)}{2}\int_0^{\infty} x_1^{k+1}[1-F(x_1)]^{n} f(x_1)dx_1 \right] \\
&= \frac{n}{(k+1)}\left[\mu_{1:1:n}^{(n-1)^{(k+1)}} - \frac{1}{2}\mu_{1:1:n+1}^{(n)^{(k+1)}} \right],
\end{aligned}$$

which leads to (9).

Remark 2.1. It may be noted that the first progressively Type-II right censored order statistic $X_{1:m:n}^{(R_1,R_2,\dots,R_m)}$ is the same as the first usual order statistic from a sample of size *n*, regardless of the censoring scheme employed. This is because no censoring has taken place before this time. We could, therefore, use Balakrishnan's (1985) result on moments of usual order statistics for obtaining the moments of the first progressively censored order statistic from the half logistic distribution. However, Theorem 2.1 has been included here in order to establish completeness of the recurrence relations even without this knowledge.

Theorem 2.2. For $2 \le i \le m-1$, $m \le n$ and $k \ge 0$,

$$\mu_{i:m:n+1}^{(R_1,\ldots,R_{i-1},R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}} = \frac{2A(n+1,i-1)}{(R_i+2)A(n,i-1)} \times \Bigg[(n-R_1-R_2-\ldots-R_i-i)\,\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}} - (n-R_1-R_2-\ldots-R_{i-1}-i+1)\,\mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}} + (R_i+1)\,\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} - \frac{1}{2}\frac{A(n,i)}{A(n+1,i-1)}\,\mu_{i:m-1:n+1}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+2,R_{i+2},\ldots,R_m)^{(k+1)}} + \frac{1}{2}\frac{A(n,i-1)}{A(n+1,i-2)}\,\mu_{i-1:m-1:n+1}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+2,R_{i+1},\ldots,R_m)^{(k+1)}} - (k+1)\,\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} \Bigg]. \quad (14)$$

Proof. From (7), we have for $2 \le i \le m-1$,

$$\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = A(n,m-1) \underset{0<x_1<\ldots<x_{i-1}<x_{i+1}<\ldots<x_m<\infty}{\int \ldots \int \ldots \int} J(x_{i-1},x_{i+1}) \prod_{\substack{t=1\\ t\neq i}}^{m} f(x_t)[1-F(x_t)]^{R_t}\,dx_t, \quad (15)$$

where

$$J(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{R_i}\,dx_i. \quad (16)$$

Making use of the relation in (4) and splitting the integral accordingly into two, we have

$$J(x_{i-1},x_{i+1}) = J_1(x_{i-1},x_{i+1}) - \frac{1}{2} J_2(x_{i-1},x_{i+1}), \quad (17)$$

where

$$J_a(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^k [1-F(x_i)]^{R_i+a}\,dx_i, \quad a = 1,2\,.$$

Integration by parts yields,

$$J_a(x_{i-1},x_{i+1}) = \frac{1}{(k+1)} \Bigg[x_{i+1}^{k+1}[1-F(x_{i+1})]^{R_i+a} - x_{i-1}^{k+1}[1-F(x_{i-1})]^{R_i+a} + (R_i+a)\int_{x_{i-1}}^{x_{i+1}} x_i^{k+1} f(x_i)[1-F(x_i)]^{R_i+a-1}\,dx_i \Bigg]. \quad (18)$$

Upon substituting for $J_1(x_{i-1}, x_{i+1})$ and $J_2(x_{i-1}, x_{i+1})$ from (18) in (17) and then substituting the resultant expression for $J(x_{i-1}, x_{i+1})$ in (15) and simplifying, on using (7), it leads to (14).

Likewise, the following recurrence relation can also be established.

Theorem 2.3. For $2 \le m \le n$ and $k \ge 0$,

$$\mu_{m:m:n+1}^{(R_1,\ldots,R_{m-1},R_m+1)^{(k+1)}} = \frac{A(n+1,m-1)}{A(n,m-1)} \frac{1}{(R_m+2)} \Bigg[\frac{A(n,m-1)}{A(n+1,m-2)} \mu_{m-1:m-1:n+1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+2)^{(k+1)}} - 2(n-R_1-R_2-\ldots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} + 2(R_m+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} - 2(k+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} \Bigg]. \tag{19}$$

Remark 2.2. Using the recurrence relations established in Section 2, we can obtain all the single moments of all progressively Type-II right censored order statistics for all sample sizes and all censoring schemes $(R_1, R_2, \ldots, R_m)$ in a simple recursive way.

3. Recurrence Relations for Product Moments

In this section, we shall again exploit the relation in (4) to obtain the recurrence relations for the product moments of progressively Type-II right censored order statistics from half logistic distribution.

Using (5), we have

$$\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)} = A(n,m-1) \int_{0<x_1<x_2<\ldots<x_m<\infty} \!\!\!\ldots \int \ldots \int x_i x_j \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t . \tag{20}$$

Theorem 3.1. For $1 \le i < j \le m-1$ and $m \le n$

$$\mu_{i,j:m:n+1}^{(R_1,\ldots,R_{j-1},R_j+1,R_{j+1},\ldots,R_m)} = \frac{A(n+1,j-1)}{A(n,j-1)} \frac{1}{(R_j+2)} \Bigg[\frac{A(n,j-1)}{A(n+1,j-2)} \mu_{i,j-1:m-1:n+1}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+2,R_{j+1},\ldots,R_m)} - \frac{A(n,j)}{A(n+1,j-1)} \mu_{i,j:m-1:n+1}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+2,R_{j+2},\ldots,R_m)} + 2(R_j+1)\mu_{i,j:m:n}^{(R_1,\ldots,R_m)} + 2(n-R_1-R_2-\ldots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)} - 2(n-R_1-R_2-\ldots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)} - 2\mu_{i:m:n}^{(R_1,\ldots,R_m)} \Bigg]. \tag{21}$$

Proof. For $1 \le i < j \le m-1$, we have from (7), with $k = 1$,

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)} = A(n,m-1) \int_{0<x_1<x_2<\ldots<x_{j-1}<x_{j+1}<\ldots<x_m<\infty} \cdots \int \cdots \int x_i \left[\int_{x_{j-1}}^{x_{j+1}} f(x_j)[1-F(x_j)]^{Rj} dx_j \right]$$

$$\times \prod_{\substack{t=1 \\ t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t$$

$$= A(n,m-1) \int_{0<x_1<\ldots<x_{j-1}<x_{j+1}<\ldots<x_m<\infty} \cdots \int \cdots \int x_i \, T(x_{j-1}, x_{j+1}) \prod_{\substack{t=1 \\ t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t, \qquad (22)$$

where

$$T(x_{j-1}, x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} f(x_j)[1-F(x_j)]^{R_j} dx_j \,. \qquad (23)$$

Making use of the relation in (4) with x replaced by x_j, we have

$$T(x_{j-1}, x_{j+1}) = T_1(x_{j-1}, x_{j+1}) - \frac{1}{2} T_2(x_{j-1}, x_{j+1}), \qquad (24)$$

where

$$T_a(x_{j-1}, x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} [1-F(x_j)]^{R_j+a} dx_j, \quad a = 1, 2 \,.$$

Integration by parts yields,

$$T_a(x_{j-1}, x_{j+1}) = x_{j+1}[1-F(x_{j+1})]^{R_j+a} - x_{j-1}[1-F(x_{j-1})]^{R_j+a}$$

$$+ (R_j + a) \int_{x_{j-1}}^{x_{j+1}} x_j f(x_j)[1-F(x_j)]^{R_j+a-1} dx_j \,. \qquad (25)$$

Upon substituting for $T_1(x_{j-1}, x_{j+1})$ and $T_2(x_{j-1}, x_{j+1})$ from (25) in (24) and then substituting the resulting expression for $T(x_{j-1}, x_{j+1})$ in (22) and simplifying on using (20), it leads to (21).

Remark 3.1. It may be noted that Theorem 3.1 holds even for $j = i+1$, without altering the proof, provided we realize that $\mu_{i,i:m:n}^{(R_1,R_2,\ldots,R_m)} = \mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)(2)}$, as mentioned in (3).

Theorem 3.2. For $1 \le i \le m-1$ and $m \le n$,

$$\mu_{i,m:m:n+1}^{(R_1,\ldots,R_{m-1},R_m+1)} = \frac{A(n+1,m-1)}{A(n,m-1)} \frac{1}{(R_m+2)} \Bigg[\frac{A(n,m-1)}{A(n+1,m-2)} \mu_{i,m-1:m-1:n+1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+2)}$$

$$- 2(n - R_1 - R_2 - \ldots - R_{m-1} - m + 1) \mu_{i,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)}$$

$$+ 2(R_m+1) \mu_{i,m:m:n}^{(R_1,R_2,\ldots,R_m)} - 2\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)} \Bigg]. \qquad (26)$$

Proof. The relation in (26) may be proved by following exactly the same steps as those used in proving Theorem 3.1.

Remark 3.2. Using these recurrence relations, we can obtain all the product moments of progressively Type-II right censored order statistics for all sample sizes and all censoring schemes $(R_1, R_2, ..., R_m)$.

Remark 3.3. It may be mentioned that if $R_1 = R_2 = ... = R_{k-1} = 0$, i.e., there is no censoring before the time of the k-th failure, then the first k progressively Type-II right censored order statistics are simply the first k usual order statistics. Thus for the special case $R_1 = R_2 = ... = R_m = 0$ so that $m = n$ in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, ..., X_{n:n}$, whose single moments are denoted by $\mu_{i:n}^{(k)}$ for $1 \le i \le n$ and product moments are denoted by $\mu_{i,j:n}$ for $1 \le i < j \le n$, the recurrence relations established in Sections 2 and 3 reduce to the following [cf. Remark 5 in Aggarwala and Balakrishnan, 1996, pp. 762-763]:

(i) $\mu_{1:n+1}^{(k+1)} = 2\left[\mu_{1:n}^{(k+1)} - \frac{(k+1)}{n}\mu_{1:n}^{(k)}\right]$, $n \ge 1$ and $k \ge 0$,

(ii) $\mu_{i+1:n+1}^{(k+1)} = \frac{1}{i}\left[\frac{(n+1)(k+1)}{(n-i+1)}\mu_{i:n}^{(k)} + \frac{(n+1)}{2}\mu_{i-1:n}^{(k+1)} - \frac{(n-2i+1)}{2}\mu_{i:n+1}^{(k+1)}\right]$, $2 \le i \le n-1$ and $k \ge 0$,

(iii) $\mu_{n+1:n+1}^{(k+1)} = \frac{1}{n}\left[(n+1)(k+1)\mu_{n:n}^{(k)} + \frac{(n+1)}{2}\mu_{n-1:n}^{(k+1)} + \frac{(n-1)}{2}\mu_{n:n+1}^{(k+1)}\right]$, $n \ge 2$ and $k \ge 0$,

(iv) $\mu_{i,j:n+1} = \mu_{i,j-1:n+1} + \frac{2(n+1)}{(n-j+2)}\left[\mu_{i,j:n} - \mu_{i,j-1:n} - \frac{\mu_{i:n}}{(n-j+1)}\right]$, $1 \le i < j \le n-1$,

(v) $\mu_{i,i+1:n+1} = \mu_{i:n+1}^{(2)} + \frac{2(n+1)}{(n-i+1)}\left[\mu_{i,i+1:n} - \mu_{i:n}^{(2)} - \frac{\mu_{i:n}}{(n-i)}\right]$, $1 \le i < n-1$, and

(vi) $\mu_{i,n:n+1} = \mu_{i:n-1:n+1} + (n+1)\left[\mu_{i,n:n} - \mu_{i,n-1:n} - \mu_{i:n}\right]$, $1 \le i < n-1$.

These recurrence relations agree with the recurrence relations given by Balakrishnan (1985) for the usual order statistics from the half logistic distribution.

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