Journal of Statistical Theory and Applications ISSN 1538-7887
Volume 11, Number 1, 2012, pp.47-61

E-OPTIMAL SEMI-REGULAR GRAPH DESIGNS AND PARTIALLY EFFICIENCY BALANCED DESIGNS

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Abstract

Jacroux (1985) extended the definition of Regular graph designs of Mitchell and John (1976) to Semi-regular graph (SRG) designs, and studied the type 1 optimality of block designs. Here, the construction and optimality of some more SRG designs are discussed. Moreover, in this investigation, it is established that a class of two associate partially efficiency balanced design is, in fact SRG designs.

Key words: Semi-regular graph (SRG) design, Symmetrical balanced incomplete block (SBIB) design, Partially efficiency balanced (PEB) design with two associate classes, E−optimality.

1. INTRODUCTION

Block designs are extensively used in many fields of research activities. A wide range of balanced incomplete block (BIB) designs and partially balanced incomplete block (PBIB) designs are available in the literature. However, these designs are restricted to equi-replicate and equi-block sizes. Another new class of incomplete block designs, termed as partially efficiency balanced (PEB) designs was developed by Puri and Nigam (1977), which can be made available in varying replications and unequal block sizes.

Let d be a block design having v treatments arranged in b blocks of size k($v > k$) whose incidence matrix is N_d where entries n_{dij} give the number of times the i^{th} treatment occurs in the j^{th} block. When n_{dij}= 1 or 0 for all i, j , the design is said to be binary. The i^{th} row of N_d is denoted by r_{di} and represents the number of times treatment i is replicated in the design. The matrix $N_d N'_d$ where N'_d is the transpose of N_d is referred as the concurrence matrix of d, and its entries are denoted by λ_{dij}.

The mathematical model which is usually used to analyze the data obtained from d is the two-way additive model. This model specifies that all observations y_{ij}(the observation obtained after applying the i^{th} treatment to a unit occurring in the j^{th} block) are uncorrelated, have constant variance, and have expectation $\alpha_i + \beta_j$, where α_i and β_j are unknown parameters representing the effects of the i^{th} treatment and j^{th} block, respectively. Let T_d and B_d denote vectors of treatment and block totals respectively, then the reduced normal equation for estimating the treatment effects in d can be written in matrix form as

$$C_d \alpha = T_d - N_d K^{-1} B_d$$

where $C_d = \text{diag}(r_{d1}, ..., r_{dv}) - N_d \,\text{diag}(1/k_1, 1/k_2, ..., 1/k_b) N'_d$, $\alpha' = (\alpha_1, ..., \alpha_v)$ and $\text{diag}(r_{d1}, ..., r_{dv})$ denotes a v × v diagonal matrix. The matrix C_d is called the information matrix or C-matrix of d, and is positive semi-definite with zero row sums and zero column sum. In the subsequent sections, we need to find the M_o-matrix, where

$$M_o = M - J(r_{d1}, ..., r_{dv})'/n$$

and $M = \text{diag}(1/r_1, .., 1/r_v)\,(N_d \,\text{diag}(1/k_1, 1/k_2, ..., 1/k_b)\, N'_d)$. Under the two-way additive model given above for d, it is well known that a necessary condition for a linear combination

$\sum_{i=1}^{v} C_i \ \alpha_i$ of the treatment effects to be estimable is that $\sum_{i=1}^{v} C_i$ =0. Such a linear combination of the treatment effects is called a treatment contrast. A contrast of the form $\alpha_i - \alpha_j$ is called a treatment difference. A design is said to be connected provided all possible treatment differences are estimable. Alternatively, a design d is connected if and only if its C-matrix has rank v − 1. Since connectedness is a desirable property for most block designs to have, only such designs are considered in this investigation. Let D(v, b, k) denotes the class of all connected block designs having v treatments arranged in b blocks of size k.

In this paper, two different methods for the construction of semi-regular graph designs are carried out in section 3. In the first method, the SRG design is constructed by augmenting a new treatment in each of the b blocks of the BIB design, and then augmenting one more block which contains all the (v + 1) treatments. In the second method, the SRG design is constructed by augmenting a new treatment in each of the b blocks of the BIB design, and then augmenting (r − λ + 1) blocks which contain all the v treatments of the BIB design only. Further, it is verified that these semi-regular graph (SRG) designs belong to a particular class of partially efficiency balanced (PEB) designs with two efficiency classes. This shows that a particular class of PEB designs with two efficiency classes is also semi-regular graph designs.

Definition 1.1. If in a design all the diagonal elements and off-diagonal elements of the concurrence matrix are differing by at most one, then it is called a semi-regular graph design.
This definition is due to Jacroux (1985).

Definition 1.2. A design d(v, b, k, r) is said to be a PEB design with m-efficiency classes if
(i) there exists a set of (v−1) linearly independent contrasts s_i, i = 1, 2, ..., m, such that ρ_i of them satisfy the equation

$$M_o = \mu_i s_{ij}, i = 1,2, \dots, m \ ; j = 1,2, \dots, \rho_i$$

so that the efficiency factor associated with every contrast of the i^{th} class is $(1 - \mu_i)$ where μ_i (i = 1, 2, ..., m) are eigen values of M_o with multiplicities ρ_i ($\Sigma\rho_i = v - 1$), and
(ii) there exists mutually orthogonal idempotent matrices L_i (i = 1, 2, ..., m) of ranks ρ_i such that $M_o = \sum_{i=1}^{m} \mu_i L_i$, and $\sum_{i=1}^{m} L_i = I - Jr'/n$, where, $M_o = R^{-1} P - J r'/n$, $P = N K^{-1} N'$, and N is the v × b incidence matrix, r is the v × 1 vector of treatment replications, k is the b × 1 vector of block sizes, R and K denote the diagonal matrices with diagonal elements as r and k, and R^{-1} and

K^{-1} are their inverses, and n denotes the total number of units.

The parameters of PEB design with *m*-efficiency classes may now be written as v, b, r, k, μ_i, ρ_i, L_i(i = 1, 2, ..., m). This definition was given by Puri and Nigam (1977).

2. PRELIMINARY RESULTS

For establishing the E-optimality of designs in sections 4.1 and 4.2, the following results are needed.

Theorem 2.1. *Let $d \in D\ (r_1, ..., r_v; b; k)$ have C-matrix C_d and let m equal to the smallest off-diagonal entry occurring in $N_d N'_d = ((\lambda_{dij}))$. Then*

$$vm / k \le Z_{d1} \le r_p (k-1)\, v / \{(v-1)k\}$$

with strict inequality on the right-hand side whenever $\lambda_{dpq} \neq r_p (k-1)/(v-1)$ for some $q \neq p$, where Z_{d1} is assumed as the smallest nonzero eigenvalue of C_d matrix.

Theorem 2.2. *Let $d \in D\ (r_1, ..., r_v; b; k)$ have information matrix C_d. If the entries of $N_d N'_d = ((\lambda_{dij}))$ satisfy the condition that $\lambda_{dij} \ge r_p (k-1)/(v-1)$ for all $i \ne j$ then $Z_{d1} = r_p (k-1)v/\{(v-1)k\}$ and d is E-optimal in $D(r_1, ..., r_v; b; k)$.*

Theorems 2.1 and 2.2 are due to Jacroux (1980).

3. METHODS OF CONSTRUCTION OF SEMI-REGULAR GRAPH DESIGNS

In this section, two methods of construction of semi-regular graph designs, which are obtained using BIB designs, are discussed. These SRG designs are also happened to be PEB designs with two efficiency classes.

3.1. Construction using BIB designs with b = r + 1

Theorem 3.1.1. *Let $d(v{=}b, r{=}k, \lambda)$ be a symmetrical BIB design and let its parameters b and r satisfy the relation b = r + 1, where b be the number of blocks and r the number of replications of the treatments and $\lambda = r - 1$. Let N_d be the incidence matrix of the BIB design d, and N_{d*} be the incidence matrix of the augmented design d^*. The incidence matrix N_{d*} defined as:*

$$N_{d*} = \begin{bmatrix} N_{d\, v\times b} & | & J_{v\times 1} \\ --- & --- & --- \\ J_{1\times b} & | & J_{1\times 1} \end{bmatrix}$$

gives a semi-regular graph (SRG) design with parameters $v^* = v+1$, $b^* = b+1$, $r^* = (r+1, \ldots, r+1, b+1)$, $k^* = (k+1, \ldots, k+1, v+1)$, $\lambda^*_1 = \lambda+1$, $\lambda^*_2 = r+1$.

Proof. Let us consider a symmetrical BIBD with parameters v =b, r = k, and λ provided b =r+1. Using these parameters we obtain the concurrence matrix of the augmented design d_+ as

$$N_{d*}N'_{d*} = \begin{bmatrix} (r-\lambda)I_v + (\lambda+1)J_{vv} & | & (r+1)\,J_{v\times 1} \\ ------------ & - & --- \\ (r+1)\,J_{1\times v} & | & (b+1) \end{bmatrix}$$

For this concurrence matrix we observed the following:

(i) The differences of the diagonal elements are:

(b+1)-(r+1) = b-r = 1 (since b = r+1)

(ii) The differences of the off-diagonal elements are:

(r+1)- (λ+1) = r- λ = 1 (since for a symmetrical BIBD with parameters v, b= r+1 and k=r and then using the relation λ(v – 1) = r(k – 1), we get λ = r-1).

That is, the diagonal elements and the off-diagonal elements of the concurrence matrix are differing by at most one. Hence the design d_+ is a semi-regular graph (SRG) design. This completes the proof.

Remark: The BIBD, discussed in section 3.3, can easily be obtained from a reduced BIBD with parameters v =b = v, r = k = v – 1and λ = v – 2.

Corollary 3.1.1. The class of semi-regular graph (SRG) designs constructed using Theorem 3.1.1 is in fact PEB designs with two associate classes.

Proof. This Corollary is proved using the M_o matrix of the design d_+. The M_o matrix of the design d_+ is obtained from the expression:

M_o=diag($1/r_1$, $1/r_2$,..,$1/r_{v+1}$) (N_{d*} diag ($1/k_1$, $1/k_2$,…,$1/k_{b+1}$)) N_{d*}')–J (r_1, r_2, ...,r_{v+1})'/n (3.1)

where J is a column vector of one’s of order v+1 and n is the total number of units. Using the parameters of symmetrical BIBD discussed in Theorem 3.1.1, we obtain the M_0 matrix of design d^* as

M_0 =

$$\begin{bmatrix} \frac{r-\lambda}{(r+1)(k+1)} I_v + \left[\frac{1}{r+1}\left(\frac{\lambda}{k+1}+\frac{1}{v+1}\right)-\frac{r+1}{m}\right] J_{vv} & \Big| & \left[\left(\frac{1}{r+1}\right)\left(\frac{r}{k+1}+\frac{1}{v+1}\right)-\frac{b+1}{m}\right] J_{v\times 1} \\ ------------ & | & ------------ \\ \left[\left(\frac{1}{b+1}\right)\left(\frac{r}{k+1}+\frac{1}{v+1}\right)-\frac{r+1}{m}\right] J_{1\times v} & \Big| & \frac{1}{b+1}\left(\frac{b}{k+1}+\frac{1}{b+1}\right)-\frac{b+1}{m} \end{bmatrix}$$

where m = (r + 1)v + (b + 1). We found that the eigen values of M_o are $\mu_1 = \frac{r-\lambda}{(r+1)(k+1)}$ with multiplicity (v-1), and $\mu_2 = \text{trace}(M_o) - (v-1)\mu_1$ with multiplicity one respectively. So the design d^* is partially efficiency balanced (PEB) with efficiency factor $1-\mu_1$ with multiplicity (v-1), and $(1-\mu_2)$ with multiplicity one. This completes the proof.

Theorem 3.1.2. *Let d(v= b, k= r, λ) be a symmetrical BIB design and its parameters satisfy the relation b = r + 1. Let N_d be the incidence matrix of the BIB design d, and N_{d^*} be the incidence matrix of the augmented design d^*. The incidence matrix N_{d^*} defined as:*

$$N_{d^*} = \begin{bmatrix} N_{v\times b} & | & J_{v\times(r-\lambda+1)} \\ --- & --- & ----- \\ J_{1\times b} & | & O_{1\times(r-\lambda+1)} \end{bmatrix}$$

gives a semi-regular graph (SRG) with parameters $v^* = v+1$, $b^* = b+(r-\lambda+1)$, $r^* = (2r-\lambda+1, \ldots, 2r-\lambda+1, b)$, $k^* = (k+1, \ldots, k+1, v)$, $\lambda^*_1 = r+1$, $\lambda^*_2 = r$.

Proof. Let us consider a symmetrical BIB design with parameters v= b, k= r, λ and we also observed that for this series of BIB design b = 2r – λ. The concurrence matrix of the augmented design d^* be

$$N_{d^*}N'_{d^*} = \begin{pmatrix} (r-\lambda)I_v + (r+1)J_{vv} & | & rJ_{v\times 1} \\ ---------- & | & -- \\ rJ_{1\times v} & | & b \end{pmatrix}$$

For this concurrence matrix, we observed the following:

(i) The differences of the diagonal elements of the concurrence matrix $N_{d^*}N'_{d^*}$ are

$(r-\lambda)+(r+1)-b = 2r-\lambda+1-b = 1$ (since $b = 2r-\lambda$)

(ii) The differences of the off-diagonal elements are (r+1)-r = 1.

That is, the diagonal elements and the off-diagonal elements of the concurrence matrix $N_{d*}N'_{d*}$ are differing by at most one. Hence the design d^* is a semi-regular graph (SRG) design. This completes the proof.

Corollary 3.1.2. The class of SRG designs constructed using Theorem 3.1.2 is in fact partially efficiency balanced (PEB) designs with two efficiency classes.

Proof. This Corollary is proved using the M_o-matrix of the design d_+. The M_o-matrix of the design d_+ is given by the following expression:

M_o =diag $(1/r_1, 1/r_2,..,1/r_{v+1})$ $(N_{d*}$ diag $(1/k_1, 1/k_2,\ldots,1/k_{b+(r-\lambda+1)})$ N_{d*}') –J $(r_1, r_2, \ldots, r_{v+1})'/n$

where J is a column vector of one's of order v+1 and n is the total number of units. Further M_0 is simplified as

M_o =

$$\left[\begin{array}{c|c} \frac{r-\lambda}{(k+1)m}I_v+\left[\frac{1}{m}\left(\frac{\lambda}{k+1}+\frac{r-\lambda+1}{v}\right)-\frac{m}{vm+b}\right]J_{vv} & \left[\left(\frac{r}{k+1}\right)\left(\frac{1}{m}\right)-\frac{b}{vm+b}\right]J_{v\times 1} \\ \hline \left(\frac{r}{b(k+1)}-\frac{m}{vm+b}\right)J_{1\times v} & \left(\frac{1}{k+1}\right)-\frac{b}{vm+b} \end{array}\right]$$

where m = 2r-λ+1. Here eigenvalues of M_o are $\mu_1=\frac{r-\lambda}{(k+1)(2r-\lambda+1)}$ with multiplicity (v-1),

and $\mu_2=\text{trace}(M_o)-(v-1)\mu_1$ with multiplicity one. So the design d^* is partially efficiency balanced (PEB) with efficiency factor $(1-\mu_1)$ with multiplicity (v-1), and $(1-\mu_2)$ with multiplicity one. This completes the proof.

3.2. Numerical Examples

As an application of the above two theorems, two numerical examples are given in this section.

Example 3.2.1. Consider the symmetrical BIB design d(4, 4, 3, 3, 2) such that b = r + 1, and blocks (as column) of this design are shown below:

$$d=\begin{pmatrix}1&2&1&1\\2&3&3&2\\3&4&4&4\end{pmatrix}$$

Using Theorem 3.1.1, the augmented design d_+ with parameters $v^*= 5$, $b^*= 5$, $r^*= (4, ..., 4, 5)$, $k^*= (4, ..., 4, 5)$, $\lambda^*_1= 3$ and $\lambda^*_2= 4$ is obtained, whose incidence matrix N_{d*} and the concurrence matrix $N_{d*} N_{d*'}$ of the design d^* is given by

$$N_{d*} = \begin{bmatrix} 1 & 0 & 1 & 1 & | & 1 \\ 1 & 1 & 0 & 1 & | & 1 \\ 1 & 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & 1 & | & 1 \\ - & - & - & - & -- & - \\ 1 & 1 & 1 & 1 & | & 1 \end{bmatrix} \qquad N_{d*}N_{d*}' = \begin{bmatrix} 4 & 3 & 3 & 3 & 4 \\ 3 & 4 & 3 & 3 & 4 \\ 3 & 3 & 4 & 3 & 4 \\ 3 & 3 & 3 & 4 & 4 \\ 4 & 4 & 4 & 4 & 5 \end{bmatrix}$$

This class of designs is Semi regular graph designs, because one can find that all the diagonal and the off-diagonal elements of the concurrence matrix $N_{d*}N'_{d*}$ are differing by at most one.
The M_o-matrix of the design d^* is obtained as follows:

$$Mo = \begin{bmatrix} 0.047 & -.0155 & -.0155 & -.0155 & -.0006 \\ -.0155 & 0.047 & -.0155 & -.0155 & -.0006 \\ -.0155 & -.0155 & 0.047 & -.0155 & -.0006 \\ -.0155 & -.0155 & -.0155 & 0.047 & -.0006 \\ -.0005 & -.0005 & -.0005 & -.0005 & 0.0019 \end{bmatrix}$$

The eigen values of M_o are $\mu_1 = 0.0625$ with multiplicity 3 (= v-1) and $\mu_2 = 0.0025$ with multiplicity one. So the design d^* is partially efficiency balanced (PEB) with efficiency factors: (i) $1 - \mu_1 = 0 .9375$ with multiplicity (v-1) = 3 and (ii) $1 - \mu_2 = 0 .9975$ with multiplicity 1.
The following Table 3.2.1 gives the examples of unequi-replicated SRG designs with unequal block sizes constructed using BIB designs.

Table 3.2.1 showing the parameters of BIB designs and SRG Designs

Parameters of BIB design					Parameters of resulting Semi regular Graph designs					
v	b	r	k	λ	v^*	b^*	r^*	k^*	λ^*_1	λ^*_2
4	4	3	3	2	5	5	4,…,4, 5	4,…,4, 5	3	4
5	5	4	4	3	6	6	5,…,5, 6	5,…,5, 6	4	5
6	6	5	5	4	7	7	6,…,6, 7	6,…,6, 7	5	6
7	7	6	6	5	8	8	7,…,7, 8	7,…,7, 8	6	7
8	8	7	7	6	9	9	8,…,8, 9	8,…,8, 9	7	8
9	9	8	8	7	10	10	9,…,9, 10	9,…,9, 10	8	9
10	10	9	9	8	11	11	10,…,10, 11	10,…,10, 11	9	10
11	11	10	10	9	12	12	11,…,11, 12	11,…,11, 12	10	11

Further, it is found that all the above SRG designs belong to a particular class of PEB designs with two efficiency classes.

Example.3.2.2. Consider the symmetrical BIB design d(4, 4, 3, 3, 2) such that b = r+1. Blocks of this design are shown below:

$$d = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{pmatrix}$$

Using Theorem 3.1.2, the augmented design d+ with parameters $v^* = 5$, $b^* = 6$, $r^* = (5, ..., 5, 4)$,

$k^* = 4$, $\lambda^* 1 = 3$ and $\lambda^* 2 = 4$ is obtained, and its incidence matrix Nd* is expressed as:

$$N_{d*} = \begin{bmatrix} 1 & 1 & 1 & 0 & | & 1 & 1 \\ 1 & 1 & 0 & 1 & | & 1 & 1 \\ 1 & 0 & 1 & 1 & | & 1 & 1 \\ 0 & 1 & 1 & 1 & | & 1 & 1 \\ - & - & - & - & -- & - & - \\ 1 & 1 & 1 & 1 & | & 0 & 0 \end{bmatrix}$$

Moreover, the concurrence matrix of the design d^* is given by the expression:

$$N_{d*}N'_{d*} = \begin{bmatrix} 5 & 4 & 4 & 4 & 3 \\ 4 & 5 & 4 & 4 & 3 \\ 4 & 4 & 5 & 4 & 3 \\ 4 & 4 & 4 & 5 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{bmatrix}$$

This class of designs is Semi regular graph designs, because one can find that all the diagonal and the off-diagonal elements of the concurrence matrix Nd* Nd* are differing by at most one.

The Mo -matrix of the design d^* is obtained as follows:

$$M_o = \begin{bmatrix} 0.0417 & -.0083 & -.0083 & -.0083 & -.0167 \\ -.0083 & 0.0417 & -.0083 & -.0083 & -.0167 \\ -.0083 & -.0083 & 0.0417 & -.0083 & -.0167 \\ -.0083 & -.0083 & -.0083 & 0.0417 & -.0167 \\ -.0208 & -.0208 & -.0208 & -.0208 & 0.0833 \end{bmatrix}$$

The eigen values of Mo are µ1 = 0.05 with multiplicity 3 (= v − 1) and µ2 = 0.10 with multiplicity one. So, the design d^* is partially efficiency balanced (PEB) with efficiency factors:

(i) 1 - µ1 = 0.95 with multiplicity (v - 1) = 3, and (ii) 1 - µ2 = 0.90 with multiplicity 1.

The following Table 3.2.2 gives the examples of unequal-replicated and proper SRG designs constructed using BIB designs.

Table 3.2.2 showing the parameters of BIB designs and SRG Designs

Parameters of BIB design					Parameters of resulting Semi regular Graph designs					
v	b	r	k	λ	v^*	b^*	r^*	k^*	λ^*_1	λ^*_2
4	4	3	3	2	5	6	5,…,5, 4	4	3	4
5	5	4	4	3	6	7	6,…,6, 5	5	4	5
6	6	5	5	4	7	8	7,…,7, 6	6	5	6
7	7	6	6	5	8	9	8,…,8, 7	7	6	7
8	8	7	7	6	9	10	9,…,9, 8	8	7	8
9	9	8	8	7	10	11	10,…,10, 9	9	8	9
10	10	9	9	8	11	12	11,…,11, 10	10	9	10
11	11	10	10	9	12	13	12,…,12, 11	11	10	11

Further, it is found that all the above SRG designs belong to a particular class of PEB designs with two efficiency classes.

4. E-OPTIMALITY OF SRG DESIGNS

A design d^* is E-optimal if and only if the maximal variance among all best linear unbiased estimators of normalized linear contrasts is minimal under d^*. That is, a design $d^* \in \Omega v,b,k$ is called E-optimal if $Z_{d*1} \geq Z_{d1}$ for all designs $d \in \Omega v,b,k$, where Z_{d1} is the least nonzero eigen value of the information matrix C_d. Jacroux (1980) obtained characterization of E-optimal block designs in the subclasses of proper designs, with unequal number of replications of the treatments. Using the results of Jacroux (1980), we obtained the E-optimality of semi regular graph design with unequal-replicated and unequal block sizes.

In this section, the E-optimality of unequal-replicated designs with unequal block sizes and the E-optimality of unequal-replicated proper designs are considered.

4.1. E-optimality of unequi-replicated designs with unequal block sizes

Now, the results of Jacroux (1980) is extended, and the E-optimality of unequi-replicated designs with unequal block sizes is considered.

Let $d \in D(rp , v - 1; ks , b - 1; n)$ with incidence matrix Nd and information matrix Cd ; where $rp = \min(r1 , r2 , ..., rv)$ and $ks = \min(k1 , k2 , ..., kb)$ and $C_d = Rd - Nd\, K_d^{-1} N_d' = R_d - P_d$ and let $P_d = (\delta ii')$. Now, consider the matrix

$$T_{xd} = ks\, C_d - x\{v(v - 1)^{-1} I - (v - 1)^{-1} J\} \qquad (4.1)$$

where x is any real number, I is the v × v identity matrix and J is the v × v matrix of ones. The eigenvalues of T_{xd} are $0 < ks\, Z_{d1} - xv/(v - 1) \leq ... \leq k_s\, Z_{dv-1} - xv/(v - 1)$.

Now, consider the following theorems:

Theorem 4.1.1. *For any design, $d \in D(rp , v - 1; ks , b - 1; n)$ and $m = \min_{i \neq i'} (ks\, \delta ii')$ then*

$$mv/ks \leq Zd1 \leq rp\, (v)(ks - 1)/\{(v - 1)ks\}$$

Further, if $m = rp\, (ks - 1)/(v - 1)$, then $Zd1 = rp\, (v)(ks - 1)/\{(v - 1)ks\}$ and then d is E-optimal in $D(rp , v - 1; ks , b - 1; n)$.

Proof. Considering x = rp (ks − 1), the proof follows on the lines of the proof of Theorem 3.1 of Jacroux (1980) by taking Txd as defined in (4.1).

Theorem 4.1.2. *The class of SRG designs constructed using Theorem 3.1.1, with parameters* $v^* = v + 1$, $b^* = b + 1$, $r^* = (r + 1, ..., r + 1, b + 1)$, $k^* = (k + 1, .., k + 1, v + 1)$, $\lambda^* 1 = \lambda + 1$ *and* $\lambda^* 2 = r + 1$ *having the incidence matrix* N_{d*} *is in fact an E-optimal in* $D(rp^*, v^* - 1; ks^*, b^* - 1; n^*)$.

Proof. *Let,*

$$Pd^* = Nd\, K_d^{-1} N_d'$$

After simplification, we obtain P_{d*} as

$$P_{d*} = \begin{pmatrix} \frac{r-\lambda}{k+1} I_v + \left(\frac{\lambda}{k+1} + \frac{1}{v+1}\right) J_{vv} & | & \left(\frac{r}{k+1} + \frac{1}{v+1}\right) J_{v\times 1} \\ ---------------- & | & --------- \\ \left(\frac{r}{k+1} + \frac{1}{v+1}\right) J_{1\times v} & | & \frac{b}{k+1} + \frac{1}{b+1} \end{pmatrix}$$

Here the off-diagonal elements are $\left(\frac{\lambda}{k+1} + \frac{1}{v+1}\right)$ and $\left(\frac{r}{k+1} + \frac{1}{v+1}\right)$ respectively.

Also, $\min_{i \neq i'} (\delta_{ii'*}) = \left(\frac{\lambda}{k+1} + \frac{1}{v+1}\right)$ (Since, λ = r − 1).

Since k < v, and $k^* = (k + 1)$ or $v^* = v + 1$, the minimum block size ks* = (k + 1);

and, $\min_{i \neq i'} (k_{s*} \delta_{ii'*}) = (k+1)\left(\frac{\lambda}{k+1} + \frac{1}{v+1}\right) \approx$ r, since λ = r − 1 and k < v.

Also, rp* = r + 1 = b (since b = r + 1). So, rp* (ks* − 1)/(v^*− 1) = bk/v = r. That is, $\min_{i \neq i'} (k_{s*} \delta_{ii'*})$ = rp* (ks* − 1)/(v* − 1). Therefore, using the preliminary results of Theorems 2.1 and 2.2 , we observed that

$$Z_{d1} = rp^* (v^*)(ks^* - 1)/\{(v^* - 1)ks^*\}$$

and hence the above class of designs is E-optimal in D(rp* , v* − 1; ks* , b* −1; n^*).

Example 4.1.1. Consider the SRG design d^* with parameters v* = 5, b* = 5, r* = (4, ..., 4, 5), k* = (4, ..., 4, 5), λ* 1 = 3 and λ* 2 = 4 given in Example 3.3.1. Now, $Pd^* = Nd\, K_d^{-1} N_d'$ which after solving, we have

$$P_{d*} = \begin{pmatrix} 0.95 & 0.70 & 0.70 & 0.70 & 0.95 \\ 0.70 & 0.95 & 0.70 & 0.70 & 0.95 \\ 0.70 & 0.70 & 0.95 & 0.70 & 0.95 \\ 0.70 & 0.70 & 0.70 & 0.95 & 0.95 \\ 0.95 & 0.95 & 0.95 & 0.95 & 1.20 \end{pmatrix}$$

For this design the minimum block size, ks* = 4, rp* = 4 and by using Theorems 4.1.1 and 4.1.2 , we get m = $\min_{i \neq i'}(k_{s*}\delta_{ii'*})$ = 2.8 ≈ 3 (m an integer). Also, rp* (ks* − 1)/(v* − 1) = 4(4 - 1)/(5 - 1) = 3.0 i.e., m = rp* (ks* − 1)/(v * − 1), then the minimum eigenvalue of the information matrix Cd* , is given by Z_{d1}= rp* (v*)(ks* − 1)/{(v *− 1)ks* }= 4(5)(4 − 1)/{(5 − 1)4} = 3.75. Hence d^* is E-optimal in D(4, 4; 4, 4; 21).

4.2. E-optimality of unequal-replicated proper designs

Now, the results of Jacroux (1980) is extended, and the E-optimality of unequal-replicated proper designs is considered.

Let d ϵ D(r1 , r2 , ..., rv ; b; k) have incidence matrix Nd and information matrix Cd and rp = min(r1 , r2 , ..., rv). Now, consider the matrix

$$Txd = kC_d - x\{v(v-1)-1\ I - (v-1)-1\ J\} \qquad (4.2)$$

where x is any real number, I is the v × v identity matrix and J is the v × v matrix of ones. The eigenvalues of Txd are $0 < kZ_{z1} - xv/(v-1) \leq ... \leq kZ_{dv-1} - xv/(v-1)$

Now, we consider the following theorems:

Theorem 4.2.1. *Let d ϵ D(r1 , r2 , ..., rv ; b; k) have C−matrix C_d and let m equal to the smallest off-diagonal entry occurring in Nd N'd = ((λdij)). Then* $mv/k \leq Z_{d1} \leq rp\,(v)(k-1)/\{(v-1)k\}$ *Further, if λdij ≥ rp (k − 1)/(v − 1) for all i ≠ j then Zd1 = rp (v)(k − 1)/{(v −1)k} and then d is E-optimal in d ϵ D(r1 , r2 , ..., rv ; b; k).*

Proof. Consider x = rp (k − 1). Next we can easily prove this theorem on similar lines of the proof of Theorem 3.1 of Jacroux (1980 by taking Txd as defined in (4.2).

Theorem 4.2.2. *The class of SRG designs constructed using Theorem 3.2.1, having incidence* *matrix Nd+ and parameters* $v^* = v + 1$, $b^* = b + (r - \lambda + 1)$, $r^* = (2r - \lambda + 1, ..., 2r - \lambda + 1, b)$, $k^* = (k + 1, k + 1, ..., k + 1, v, ..., v)$, $\lambda^* 1 = r + 1$, *and* $\lambda^* 2 = r$ *is E-optimal in* $D(r_{1*}, r_{2*}, ..., r_{v*}; b^*; k^*)$.

Proof. Here we obtain the concurrence matrix of the augmented design d^* as

$$N_{d*}N'_{d*} = \begin{pmatrix} (r-\lambda)I_v + (r+1)J_{vv} & | & rJ_{v\times 1} \\ ----------- & | & -- \\ rJ_{1\times v} & | & b \end{pmatrix}$$

Here, the off-diagonal elements of the concurrence matrix are (r + 1) and r, That is, $\lambda d*ij = r + 1$, for all $i \neq j = 1,2,\ldots,v$ and $\lambda d*ij = r$, for all $I = 1,2,\ldots,v$ and $j = v+1$. Also, $rp+ (k+ - 1)/(v+ - 1) = bk/v = k = r$, (since the design is a symmetrical BIB design). So, $\lambda d*ij \geq rp* (k* - 1)/(v^* - 1)$ for all $i \neq j$. Therefore, using the preliminary results of Theorems 2.1 and 2.2, we observed that

$$Z_{d1*} = rp^* (v^*)(k^* - 1)/\{(v^* - 1)k^*\}$$

and hence the class of SRG design is E-optimal in $D(r_{1*}, r_{2*}, ..., r_{v*}; b^*; k^*)$.

Example 4.2.1. Consider the SRG design d^* with parameters $v^* = 5$, $b^* = 6$, $r^* =(5,\ldots, 5, 4)$, $k^* = 4$, $\lambda^* 1 = 3$ and $\lambda^* 2 = 4$ given in Example 3.3.2.
The concurrence matrix of the design d^* of example 4.2.1 is given by the expression

$$N_{d*}N'_{d*} = \begin{bmatrix} 5 & 4 & 4 & 4 & 3 \\ 4 & 5 & 4 & 4 & 3 \\ 4 & 4 & 5 & 4 & 3 \\ 4 & 4 & 4 & 5 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{bmatrix}$$

Here, the off-diagonal elements of the concurrence matrix are 4 and 3, that is, $\lambda d*ij = 4$ for all $i \neq j = 1,2,\ldots,4$ and $\lambda d*ij = 3$ for all $i = 1,2,\ldots,4$ and $j = 5$. Also, $rp* (k^* - 1)/(v^* - 1) = 4(3)/4 = 3$. Therefore, $\lambda d*ij \geq rp* (k* - 1)/(v* - 1)$ for all $i \neq j$. Hence, using the Theorems 2.1 and 2.2 , we have $Z_{d1*} = rp* (v*)(k* - 1)/\{(v* - 1)k*\} = 4(3)(5)/4(4) = 3.75$. Hence we say that the design d^* is E-optimal in D(5,...,5, 4; 6; 4).

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