ON SOME PROPERTIES OF STOCHASTIC CONDITIONAL DURATION MODELS

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Abstract

Recently Bauwens and Galli (2009) [Computational Statistics & Data Analysis, 53(6), 1974-1992] studied estimation of Stochastic Conditional Duration (SCD) models for high frequency data. In this paper, quadratic and long memory versions of the SCD model are proposed and their moments are derived. For the proposed models, maximum likelihood estimation turns out to be intractable and the moments formulas derived in this paper can be used to estimate the model parameters.

Key Words: Quadratic stochastic, conditional duration, autocorrelation, long memory, high frequency

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1 Introduction

There is growing interest in the analysis of intra-day financial data such as transaction and quote data. Such data have increasingly been made available by many stock exchanges. Unlike closing prices which are measured daily, monthly or yearly, intra-day data or high frequency data tend to be irregularly spaced. Furthermore, the durations between events themselves are random variables. The Autoregressive Conditional Duration (ACD) process due to Engle and Russell (1998) had been proposed to model such durations.

Bauwens and Veredas (2004) had proposed the Stochastic Conditional Duration (SCD) models for inter-event duration processes where the conditional duration is modelled with a latent variable (See also Bauwens and Galli (2009)). The motivation for using a latent variable is that it captures the random flow of information that, in the case of financial markets, is very difficult to observe directly. Such information however influences the probability of a quote revision and hence the inter-quote durations over time. Since its introduction, the SCD model has become a leading tool in modeling the behavior of high frequency data, opening the door to both theoretical and empirical developments. In this paper, we first study the quadratic SCD models followed by long memory SCD models and then derive the moments.

The plan of this paper is as follows. In Section 2, the results on moments of SCD models are extended to long memory SCD models. Then a new class of quadratic SCD models are introduced and the moments are derived. Section 3 concludes the paper.

2 Stochastic Conditional Durations (SCD) Models

Duration models are concerned with time intervals between trades. Longer durations indicate lack of trading activities, which imply a period of no new information and also low volatility. The time-varying behavior of durations therefore contains useful information about intraday market activities. Using concepts similar to the Autoregressive Conditional Heteroscedastic (ARCH) models for volatility, Engle and Russell (1998) propose an ACD model to describe the evolution of time durations for (heavily traded) stocks. In contrast to the ARCH model, the Stochastic Volatility (SV) model due to Taylor (1986) captures additional uncertainty in volatility through an underlying latent process. The SCD models for duration can be viewed
as the analog of the stochastic volatility models. In this section, we introduce some simple extensions of SCD models.

Consider the time series of intraday durations \( \{D_i\} \), \( i = 1, \ldots, n \) of a stock. Intraday transactions tend to exhibit some sort of a diurnal pattern (p. 225, Tsay 2005). Typically, trading tends to be heavy when the market opens but decreases around the lunch hour and tends to increase thereafter until closing time. In what follows, assume \( \{D_i\} \) is a time-adjusted series of durations.

We propose the class of SCD models for a sequence of durations \( \{D_i\} \) of the form:

\[
D_i = M_i \varepsilon_i \quad (1)
\]
\[
M_i = \exp(x_i) \quad (2)
\]
\[
x_i - \mu = \sum_{j=0}^{\infty} \psi_j a_{i-j} \quad (3)
\]

where the moving average representation guarantees the stationarity of the latent process provided the variance is finite, a sufficient condition for finite variance is \( \sum \psi_j^2 < \infty \). Let \( F_{i-1}^D \) be the information set up to and including \( D_{i-1} \) and the past values of the latent process \( x_i \). We make the following distributional assumptions:

(i) \( a_i | F_{i-1}^D \sim N(0, \sigma_a^2) \) (independent),

(ii) \( \varepsilon_i | F_{i-1}^D \) follows a distribution with positive support,

(iii) \( a_i \) is independent of \( \varepsilon_j | F_{i-1}^D \), \( \forall i, j \).

The uncentered moments of \( \varepsilon_i \) are assumed to exist and are denoted by \( m_p = E(\varepsilon_i^p), p = 1, 2, \ldots \)

The SCD model of Bauwens and Veredas (2004) turns out to be a special case when the linear process \( \{x_i\} \) is an autoregressive process of order one. The following theorem provides the form of the autocorrelation function of the proposed generalized SCD model.

**Theorem 2.1** For the duration process \( \{D_i\} \) and the latent process \( \{M_i\} \) given by (1)-(3), the moments and the autocorrelation function \( \rho_k^D \) are

(i) \( \mu_M = e^{\mu_x + \frac{1}{2} \sigma_x^2} \), where \( \sigma_x^2 = \sigma_u^2 \sum_{j=0}^{\infty} \psi_j^2 \),

(ii) \( \mu_D = m_1 \mu_M \).
\( \sigma^2_M = \mu^2_M (e^{\sigma^2_a} - 1) \),

\( \sigma^2_D = \mu^2_D \left( \frac{m_2}{m_1} e^{\sigma^2_a} - 1 \right) \),

\( \rho^D_k = \frac{e^{\gamma x_k} - 1}{m_2 e^{\gamma x_k} / m_1^2 - 1}, \) \( k \geq 1 \).

**Proof.** The autocorrelation function of the latent process \( x_i \) is given by

\[
\rho^x_k = \frac{\sum_{j=0}^{\infty} \psi^j \psi^{j+k}}{\sum_{j=0}^{\infty} \psi^j}.
\]

Hence the autocovariance function of \( x_i \) is given by

\[
\gamma^x_k = \rho^x_k \sigma^2_a \sum_{j=0}^{\infty} \psi^j.
\]

\[
Cov(M_i, M_{i+k}) = E(e^{x_i + x_{i+k}}) - E^2(e^{x_i}) \quad \text{and using the fact that}
\]

\[
x_i + x_{i+k} \sim N(2\mu_x, 2\sigma^2_x + 2\sigma^2_x \rho^x_k),
\]

we have

\[
Cov(M_i, M_{i+k}) = e^{2\mu_x + \frac{1}{2}[2\sigma^2_x (1 + \rho^x_k)]} - e^{2\mu_x + \sigma^2_x}.
\]

Hence the autocorrelation function of the duration process is given by

\[
\rho^D_k = \frac{E(D_i D_{i+k}) - \mu^2_D}{\sigma^2_D} = \frac{m_2 E(M_i M_{i+k}) - m_1^2 \mu^2_M}{m_2 E(M_i^2) - m_1^2 \mu^2_M} = \frac{\exp(\gamma^x_k) - 1}{m_2 e^{\sigma^2_a} / m_1^2 - 1}.
\]

The following corollary provides the first two moments of an SCD model with an autoregressive process of order one as a latent process as in Theorem 2.1.

**Corollary 2.1** For an autoregressive process of order one given by \( x_i - \mu = \phi x_{i-1} + a_i \), where \( a_i \sim NID(0, \sigma^2_a) \) as a latent process, as in Bauwens and Veredas (2004), the moments are

\[
(i) \ \mu_M = \exp \left( \frac{\mu}{1 - \phi} + \frac{1}{2} \frac{\sigma^2_a}{1 - \phi^2} \right)
\]

\[
(ii) \ \mu_x = \frac{\mu}{1 - \phi},
\]

\[
(iii) \ \sigma^2_x = \frac{\sigma^2_a}{1 - \phi^2},
\]

\[
(iv) \ \mu_D = m_1 \mu_M,
\]

\[
(v) \ \sigma^2_M = e^{2\mu_x + \sigma^2_a} [e^{\sigma^2_x} - 1],
\]

\[
(vi) \ \sigma^2_D = m_2 e^{2(\mu_x + \sigma^2_x)} - \mu^2_D.
\]
The moments turn out to be the moments given in Bauwens and Veredas (2004) for an SCD models with an autoregressive process of order one as the latent process.

Evidence of long memory as displayed by slowly decaying hyperbolic autocorrelation function has been reported for durations data such as IBM trade durations (see Engle and Russell (1998), Jasiak(1998), Bauwens et al. (2004)). Thus, Jasiak (1998) had been motivated to introduce the Fractionally Integrated Autoregressive Conditional Duration (FIACD) model.

In analogy with the FIACD model, we propose a long-memory stochastic conditional duration (LMSCD) model and derive its moment properties using Theorem 2.1. Consider the stochastic duration process \( \{D_t\} \) given by:

\[
D_t = M_t \varepsilon_t \quad (4)
\]

\[
M_t = \exp(u_t) \quad (5)
\]

\[
u_t = (1 - B)^{-d} a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}. \quad (6)
\]

where \( d \in (0, 0.5) \), \( a_t \sim NID(0, \sigma_a^2) \), \( \varepsilon_t | F_{t-1}^D \) follows a distribution whose support is positive,

\[
\psi_k = \frac{\Gamma(k + d)}{\Gamma(d) \Gamma(k + 1)},
\]

and \( \Gamma(\cdot) \) is the gamma function such that \( \psi_k \) converges hyperbolically to zero. Now by using the variance of \( u_t \) as

\[
\sigma_u^2 = \sigma_a^2 \frac{\Gamma(1 - 2d)}{\Gamma(1-d)^2} \quad (8)
\]

and the auto-covariance of \( u_t \) as

\[
\gamma_k^u = \sigma_u^2 \frac{\Gamma(1 - 2d) \Gamma(k + d)}{\Gamma(d) \Gamma(1 - d) \Gamma(k + 1 - d)} \quad (9)
\]

in Theorem 2.1, we can obtain the moments of the long memory SCD model. Moreover, the following corollary provides the form of the autocorrelation for the LMSCD model.

**Corollary 2.2** For a long memory stochastic conditional duration process given by equations (4)-(6), the autocorrelation function is

\[
\rho_k^D = \frac{e^{\gamma_k^u} - 1}{m_t^2 e^{\sigma_u^2} - 1},
\]
2.1 Quadratic Stochastic Conditional Duration (QSCD)

In volatility modelling literature, non-normality of the error term has been modelled through flexible functional specifications of volatility. For example, Kawakatsu (2007) following Meddahi’s (2001) specification of nonlinear terms in volatility had proposed the class of log-Quadratic stochastic volatility (LQSV) models. The LQSV model is capable of capturing the kurtosis present in log-returns even when the error distribution is assumed to be Gaussian.

In duration modelling, Engle and Russell (1998) had initially used the standard exponential distribution as the error distribution. For greater flexibility, the Burr-ACD due to Grammig and Maurer (2000) had been proposed; it contains the ACD model with standardized exponential, Weibull and log-logistic error distributions special cases. Motivated by Kawakatsu (2007), we propose a duration model with a non-linear specification of conditional duration.

Consider the quadratic SCD model for a duration process \( \{D_i\} \) given by the following:

\[
D_i = M_i \varepsilon_i \tag{10}
\]

\[
M_i = \exp(ax_i + bx_i^2) \tag{11}
\]

\[
x_i = \sum_{k=0}^{\infty} \psi_k a_{i-k}, \tag{12}
\]

where \( \sum_{k=0}^{\infty} \psi_k^2 < \infty \) and \( a_i \sim NID(0, \sigma_a^2) \) and \( \varepsilon_i | F_{t-1} \) follows a distribution whose support is positive. The following lemma on the moment generating function of a quadratic function of a normal random variable will be used to derive the moments of the quadratic SCD process.

**Lemma 2.1** For a normal random variable \( Y \) having mean zero and variance \( \sigma_Y^2 \) and constants \( r, a \) and \( b \) such that \( b < 1/(2r) \), the \( r \)th moment of a quadratic function of \( Y \) is given by

\[
E[e^{r(aY + bY^2)}] = \frac{1}{\sqrt{1 - 2r b \sigma_Y^2}} \exp \left[ \frac{r^2 a^2 \sigma_Y^2}{2(1 - 2r b \sigma_Y^2)} \right].
\]

**Theorem 2.2** For the QSCD process \( \{D_i\} \), and the latent process \( \{M_i\} \) given by (10)-(12), have the following moments respectively.

(a) \( E(M_i) = \frac{1}{\sqrt{1 - 2b \sigma_x^2}} \exp \left[ \frac{a^2 \sigma_x^2}{2(1 - 2b \sigma_x^2)} \right] \),

(b) \( E(M_i^2) = E[e^{2(ax_i + bx_i^2)}] = \frac{1}{\sqrt{1 - 4b \sigma_x^2}} \exp \left[ \frac{2a^2 \sigma_x^2}{(1 - 4b \sigma_x^2)} \right] \),
(c) \( \text{Var}(M_i) = E(M_i^2) - E^2(M_i) \).

The proof follows by applying the Lemma 2.1 and is omitted. The following lemma on linear representation of a stationary linear process will be used to derive the correlation of a QSCD.

**Lemma 2.2** For any stationary process of the form \( x_i = \sum_{j=0}^{\infty} \psi_j a_{i-j} \), \( \psi_0 = 1 \) and where \( \sum_{j=0}^{\infty} \psi_j^2 < \infty \), the \( l \)-steps ahead forecast based on observations \( x_1, \ldots, x_n \) is given by

\[
x_{n+l} = x_n(l) + e_n(l),
\]

where the \( x_n(l) = E(x_{n+l}|F_n^x) \) is the minimum mean square error forecast of \( x_{n+l} \) and \( e_n(l) \) is the \( l \)-steps ahead forecast error:

\[
e_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}.
\]

In particular, for popular linear time series, the \( \psi \)-weights are given by the following:

(a) For the autoregressive process of the form \( x_i = \phi x_{i-1} + a_i \), \( \psi_j = \phi^j \), \( j \geq 0 \).

(b) For the autoregressive moving average process of the form \( x_i = \phi x_{i-1} + a_i - \theta a_{i-1} \), \( \psi_0 = 1 \), \( \psi_j = (\phi - \theta)\phi^{j-1} \) for \( j \geq 1 \).

(c) For the long-memory process given by \( (1-B)^d x_i = a_i \), where \( d \in (0,0.5) \), \( B \) is the back-shift operator and \( a_i \sim \text{NID}(0, \sigma^2_a) \), \( \psi_j = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \).

For the durations \( (D_i) \) and latent process \( (M_i) \) in (10)-(12), the following theorem gives the autocorrelation functions.

**Theorem 2.3** For the durations \( (D_i) \) and latent process \( (M_i) \) in (10)-(12), the autocorrelation functions \( \rho_D^l \) and \( \rho_M^l \) respectively, of \( D_i \) and \( M_i \) are the following:

(a) \[
\rho_M^l = \frac{\gamma_M^l}{\sigma_M^2}, \quad \text{where} \quad \gamma_M^l = \frac{1}{\sqrt{1-2b\sigma_e^2}} \exp \left[ \frac{(a + 2b x_n(l))^2}{2(1-2b\sigma_e^2)} \right] E \left[ e^{ax_n+bx_n^2+ax_{n+l}+bx_{n+l}^2} \right].
\]

(b) \[
\rho_D^l = \frac{\sigma^2_d m^2_1 \rho_M^l - m_2 E(M_i^2)}{m_2 E(M_i^2) - m_1^2 E^2(M_i)}.
\]

**Proof.** \( E[M_n M_{n+l}] = E[e^{ax_n+bx_n^2+ax_{n+l}+bx_{n+l}^2}] \). We can write \( x_{n+l} = E[x_{n+l}|F_n^x] + e_n(l) \). Then substituting the identity for \( x_{n+l} \) into the expression for \( E[M_n M_{n+l}] \) and using the law of
iterated expectations, we obtain the auto-covariance function of $M_i$ as
\begin{align*}
\gamma_i^M &= E \left[ e^{a x_n + a x_{n+1} + b x_n^2 + b x_{n+1}^2} \right] - E^2(M_i) \\
&= EE \left[ e^{a x_n + a x_n(l) + a x_{n+1}(l) + b x_n(l) + b x_{n+1}(l)} \right] \bigg| F_n^x - E^2(M_i) \\
&= E \left[ e^{a x_n + b x_n^2 + a x_{n+1} + b x_{n+1}^2} \right] E \left( e^{a + 2 b x_n(l) + b x_{n+1}(l)} \right) F_n^x - E^2(M_i).
\end{align*}

Using Lemma 1, $E \left( e^{a + 2 b x_n(l) + b x_{n+1}(l)} \big| F_n^x \right) = \frac{1}{\sqrt{1 - 2 b \sigma_e^2}} \exp \left[ \frac{(a + 2 b x_n(l))^2}{2(1 - 2 b \sigma_e^2)} \right]$, where $\sigma_e^2 = \sigma_n^2 \sum_{j=0}^{l-1} \psi_j^2$. The following example illustrates Theorem 2.3

**Example 1** When the latent process is an autoregressive process of the form $x_i = \phi x_{i-1} + a_i$, where $a_i \sim N(0, \sigma_a^2)$, $E \left[ e^{a x_n + b x_n^2 + a x_{n+1} + b x_{n+1}^2} \right] = E \left[ e^{a(1+\phi) x_n + b(1+\phi) x_n^2} \right]$. The auto-covariance function of $M_i$ is given by

\begin{align*}
\gamma_i^M &= \frac{1}{\sqrt{1 - 2 b \sigma_a^2/(1 - \phi^2)}} \exp \left[ \frac{(a + 2 b \phi x_n)^2}{2(1 - 2 b \sigma_a^2/(1 - \phi^2))} + \frac{a^2(1 + \phi^2) \sigma_n^2}{2(1 - 2 b \sigma_a^2/(1 + \phi^2))} \right].
\end{align*}

Then $E(M_i) = \frac{1}{\sqrt{1 - 2 b \sigma_a^2/(1 - \phi^2)}} \exp \left[ \frac{a^2 \sigma_n^2}{2(1 - 2 b \sigma_a^2/(1 - \phi^2))} \right]$ and $E(M_i^2) = \frac{1}{\sqrt{1 - 4 b \sigma_a^2/(1 - \phi^2)}} \exp \left[ \frac{2 a^2 \sigma_n^2}{1 - 4 b \sigma_a^2/(1 - \phi^2)} \right]$. When $\varepsilon_i \big| F_{i-1}^D$ follows a standard exponential distribution, then $m_1 = 1$ and $m_2 = 2$ and $\sigma_M^2 = E(M_i^2) - E^2(M_i)$ is given by

\begin{align*}
\sigma_M^2 &= \frac{1}{1 - 4 b \sigma_a^2/(1 - \phi^2)} \exp \left[ \frac{2 a^2 \sigma_n^2}{1 - 4 b \sigma_a^2/(1 - \phi^2)} \right] \left[ \frac{a^2 \sigma_n^2}{1 - 2 b \sigma_a^2/(1 - \phi^2)} \right] \exp \left[ \frac{a^2 \sigma_n^2}{1 - 2 b \sigma_a^2/(1 - \phi^2)} \right],
\end{align*}

and $\rho_l^D = \frac{\sigma_M^2 \rho_l^M - 2 E(M_i^2)}{2 E(M_i^2) - E^2(M_i)}$, for $l \geq 1$.

## 3 Conclusions

In this paper, following Bauwens and Veredas (2004) a new class of quadratic SCD models and a class of long memory SCD models are defined. The moment properties of the extended models have been studied in some detail. Generalized Method of Moments (GMM) estimation of model parameters for the models discussed is likely to be feasible. Recently, Hu (2010) had estimated
the model parameters of the QSCD model using GMM based on a working paper version of our manuscript.

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