The semigroups B_2 and B_0 are inherently nonfinitely based, as restriction semigroups.

Peter R. Jones

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Abstract

The five-element Brandt semigroup B_2 and its four-element subsemigroup B_0, obtained by omitting one nonidempotent, have played key roles in the study of varieties of semigroups. Regarded in that fashion, they have long been known to be finitely based. The semigroup B_2 carries the natural structure of an inverse semigroup. Regarded as such, in the signature $\{\cdot, ^{-1}\}$, it is also finitely based. It is perhaps surprising, then, that in the intermediate signature of restriction semigroups – essentially, 'forgetting' the inverse operation $x \to x^{-1}$ and retaining the induced operations $x \to x^+ = xx^{-1}$ and $x \to x^* = x^{-1}x$ – it is not only nonfinitely based but inherently so (every locally finite variety that contains it is also nonfinitely based). The essence of the nonfinite behaviour is actually exhibited in B_0, which carries the natural structure of a restriction semigroup, inherited from B_2. It is again inherently nonfinitely based, regarded in that fashion. It follows that any finite restriction semigroup on which the two unary operations do not coincide is nonfinitely based. Therefore for finite restriction semigroups, the existence of a finite basis is decidable 'modulo monoids'.

These results are consequences of – and discovered as a result of – an analysis of varieties of 'strict' restriction semigroups, namely those generated by Brandt semigroups and, more generally, of varieties of 'completely r-semisimple' restriction semigroups: those semigroups in which no comparable projections are related under the generalized Green relation $\mathbb{D}$. For example, explicit bases of identities are found for the varieties generated by B_0 and B_2.

Completely 0-simple semigroups have played a central role in semigroup theory from the very beginnings of its history. So it is naturally of great interest to study the varieties that they generate, together with their subvarieties. These so-called Rees-Sushkevich varieties have received considerable attention in recent years. (For example [19, 21, 22, 27, 28].) Regarded instead as unary semigroups, the inverse semigroups that are completely 0-simple – the Brandt semigroups – likewise generate varieties of inverse semigroups, though in this context the entire situation was clarified some decades ago [26, XII.4].

We take here an intermediate path that quite naturally lies in the realm of varieties of restriction semigroups, which are biunary semigroups in the signature $(\cdot, ^+, ^*)$. Although such semigroups are known to arise in several contexts (for a survey, see [11]), for the purposes of this paper we need only view them as follows. Any inverse semigroup $(S, \cdot, ^{-1})$ may be regarded as a restriction semigroup under the induced operations $x \to x^+ = xx^{-1}$ and $x \to x^* = x^{-1}x$, forgetting the inverse operation entirely. The restriction semigroups are the members of the variety generated by the semigroups induced from inverse semigroups in this way. The 'distinguished' idempotents x^+ and x^* are here termed projections. We emphasize that restriction semigroups actually have a long history, though in their most general form a somewhat shorter one, but we refer the reader to the appendix for more details.

The study of the lattice of varieties of restriction semigroups was initiated by the author in [16] and, from that perspective, the current paper is essentially a sequel. A key point to note is that the role that groups play in the theory of inverse semigroups is now played by monoids (restriction semigroups containing just one projection).

The five-element, combinatorial Brandt semigroup $B_2 = \{a, b, ab, ba, 0\}$ and its full subsemigroup $B_0 = \{a, ab, ba, 0\}$ play pivotal roles in the study of Rees-Sushkevich varieties (see the papers cited above, among others). The former is an inverse semigroup and, not surprisingly, it also plays a pivotal role in the study of varieties of inverse semigroups. For instance, a variety of inverse semigroups contains B_2 if and only if it does not consist of semilattices of groups.

The semigroup B_0 is in a very natural way a restriction semigroup that is already known to play an important role in the study of varieties of such semigroups [16]. In this paper we show that it embodies a remarkable complexity for a semigroup of only four elements, most obviously through the property that it is inherently nonfinitely based, regarded as a restriction semigroup: it is nonfinitely based and, further, any locally finite variety (of restriction semigroups) that contains it is also nonfinitely based. Of course this implies that the same is true for B_2, again regarded as a restriction semigroup. Thus while all semigroups with five or fewer elements are known to be finitely based, as semigroups, the five-element semigroup B_2 and the four-element semigroup B_0 are not, if viewed as restriction semigroups. Yet, viewed as an inverse semigroup, B_2 is again known to be finitely based.

These results are consequences of the existence of a 'series of critical semigroups', in the terminology of Volkov [34]. (See the last part of Section 1 for a general discussion.) In Section 4, we construct a sequence Ψ_k of restriction semigroups, for positive, even integers k, each of which is infinite, generated, as restriction semigroups, by k elements and has the property that any restriction subsemigroup that is generated by fewer than k elements belongs to the variety generated by B_0. It follows that any variety that contains B_0 but none of the Ψ_k's is nonfinitely based. Since the semigroups Ψ_k are not locally finite, that B_0 is inherently nonfinitely based immediately follows. In the same section, we state the pertinent theorem, Theorem 4.1, and outline the steps needed to complete the proof.

As remarked in the abstract, this theorem is but one consequence of a much more widely-ranging study of varieties of restriction semigroups, and was found only as a result of that study. For instance, what *are* the varieties that contain B_0 but no Ψ_k? We show (Theorem 11.10) that they are the varieties of completely r-semisimple semigroups: those whose principal r-factors are Brandt semigroups or monoids (here monoids play the role that is played by groups in the study of inverse semigroups and the definitions are in terms of the 'generalized Green relations' $\mathbb{D} = \mathbb{L} \vee \mathbb{R}$ and $\mathbb{J}$, which are studied in Section 5).

That raises the question of what is the variety that is *generated* by Brandt semigroups? This turns out (Theorem 8.1) to be the variety $\mathbf{B}$ of strict restriction semigroups, namely the join of the variety generated by B_2 and all monoids. In the same theorem, we provide an infinite basis of identities for $\mathbf{B}$. (The identities are defined and studied in Section 7.) An essential ingredient is a structural characterization of the members of $\mathbf{B}$, by means of '$\mathbb{D}$-majorization' (introduced and studied in Section 6).

In Sections 9 and 10, we quickly deduce infinite bases of identities for the varieties $\mathbf{B}_2$ and $\mathbf{B}_0$ of restriction semigroups that are generated, respectively, by B_2 and B_0. Structurally, we show that $\mathbf{B}_2$ comprises the '$\mathbb{H}$-combinatorial' members of $\mathbf{B}$; and $\mathbf{B}_0$ comprises the members of $\mathbf{B}_2$ whose only regular elements are projections.

A characterization of $\mathbf{B}$ in terms of 'forbidden semigroups' Λ_k, analogous to that above for the completely r-semisimple varieties, is also obtained (Theorem 12.2).

Finally, readers may be aware that the one-sided restriction semigroups (where, in the language used above, one retains from inverse semigroups only one of the induced unary operations) have also received considerable attention. In the paper [16] cited above, the author also initiated the study of the lattice of varieties of left restriction semigroups. It would be of interest to pursue a study of the varieties of left restriction semigroups generated by B_2 and by any of its left restriction subsemigroups.

1 Background

We first introduce restriction semigroups more formally, along with their basic properties and related definitions. Some of the material is repeated, for convenience, from the 'prequel' [16]. For the purposes of this work, it is appropriate to define these semigroups by means of their identities. A *restriction* semigroup is a biunary semigroup $(S, \cdot, {}^+, {}^*)$ that satisfies the 'left restriction' identities

$$x^+x = x; \qquad (x^+y)^+ = x^+y^+; \qquad x^+y^+ = y^+x^+; \qquad xy^+ = (xy)^+x,$$

the 'dual' identities, obtained by replacing $^+$ by * and reversing the order of each expression,

$$xx^* = x; \qquad (yx^*)^* = y^*x^*; \qquad x^*y^* = y^*x^*; \qquad y^*x = x(yx)^*,$$

along with $(x^+)^* = x^+$ and $(x^*)^+ = x^*$.

(We take this particular definition from [12], where it is attributed to Jackson and Stokes [15].) The last of each set of four identities are often termed the 'ample' identities.

From the identities it follows that for all $x \in S$, x^+ is idempotent and $(x^+)^+ = x^+$. We term these 'distinguished' idempotents the *projections* of S. Denote the set of projections by P_S and the set of all idempotents by E_S. Although, by the third identity, P_S is a semilattice, this need by no means be true of E_S. In the usual way, E_S is partially ordered by $e \leq f$ if $e = ef = fe$.

The following consequence of the identities is well known.

LEMMA 1.1 *Let S be a restriction semigroup. Then S satisfies $x^+ \geq (xy)^+$ and $(xy)^+ = (xy^+)^+$ and their duals, namely $x^* \geq (xy)^*$ and $(xy)^* = (x^*y)^*$.*

The term 'restriction' is relatively recent, deriving from its use in the one-sided case by Cockett and Lack [3], in one of the several sources of these semigroups (and categories, in their paper). Until quite recently, the term 'weakly E-ample' was used, providing evidence of a succession of generalizations – by the so-called York school – of Fountain's 'ample semigroups' (though yet again different terminology was used in the original papers [6, 7]).

When expressed in the language of varieties and identities, many definitions (for example the generalized Green relations) have very simple formulations that require no knowledge of their historical development. Moreover, in only one proof (that of Proposition 5.8) are any substantive theorems quoted from the literature.

Nevertheless, in an appendix we provide a brief summary of the interpretation of this background in the language of the York school. Much fuller exposition of this material may be found in the work of Gould [11] and Hollings [13], for instance, and in the thesis of Cornock [4].

We refer the reader to standard texts such as [14] for general semigroup theory and [26] for background on inverse semigroups and their varieties.

As noted earlier, an inverse semigroup $(S, \cdot, ^{-1})$ may be regarded as a restriction semigroup by setting $x^+ = xx^{-1}$ and $x^* = x^{-1}x$ and 'forgetting' the inverse operation. In that case, $P_S = E_S$. This source of examples may be expanded upon by noting that any subsemigroup that is full (that is, contains all its idempotents) again induces such a restriction subsemigroup. Each such semigroup is, in fact, ample (see Section 13).

For the purposes of this paper, the relevant generalized Green relations may be defined as follows. In any restriction semigroup, $\mathbb{R} = \{(a, b) : a^+ = b^+\}$, $\mathbb{L} = \{(a, b) : a^* = b^*\}$, $\mathbb{H} = \mathbb{L} \cap \mathbb{R}$ and $\mathbb{D} = \mathbb{L} \vee \mathbb{R}$. It follows easily from Lemma 1.1 that each contains the corresponding usual Green relations, that $\mathbb{R}$ is a left congruence and that $\mathbb{L}$ is a right congruence. In the case of a restriction semigroup that is induced from a (full subsemigroup of an) inverse semigroup, each actually coincides with the restriction of the usual Green relation. That $\mathbb{D}$ is not simply $\mathbb{L} \circ \mathbb{R}$, in contrast to the usual Green relations, is at the heart of the main results of this paper. These relations, together with the generalized Green relation $\mathbb{J}$, will be explored in depth in Section 5.

By analogy with the term 'combinatorial' for inverse semigroups in which $\mathcal{H}$ is the identical relation ι, a restriction semigroup will be called $\mathbb{H}$-*combinatorial* if $\mathbb{H} = \iota$. On any restriction semigroup S, μ denotes the greatest congruence contained in $\mathbb{H}$, equivalently, the greatest congruence that separates P_S.

In general, the terms 'homomorphism', 'congruence' and 'division' will be used appropriate to context; that is, they should respect both unary operations for restriction semigroups (and the inverse operation in an inverse semigroup). Thus, for instance, a restriction semigroup divides a restriction semigroup S if it is a (biunary) homomorphic image of a (biunary) subsemigroup of S.

In the standard terminology, restriction semigroups S with $|P_S| = 1$ are termed *reduced*. Since, in essence, they are just monoids, regarded as restriction semigroups by setting $a^+ = a^* = 1$ for all a, we will generally use the latter term, except in case of possible ambiguity. A *submonoid* of a restriction semigroup S is a restriction subsemigroup that is a monoid, that is, contains a unique projection of S. It is well known that the maximal submonoids are precisely the $\mathbb{H}$-classes $\mathbb{H}_e$, where $e \in P_S$.

Munn semigroups will be the source of several examples in the sequel, and also used in a key proof, so we briefly review their definition and properties (see [14, Chapter 5]). Let E be a semilattice. Then T_E is the inverse subsemigroup of the symmetric inverse semigroup on E that consists of the isomorphisms between principal ideals of E. Its semilattice of idempotents comprises the identity maps on principal ideals and so is isomorphic with E. Idempotents in T_E are $\mathcal{D}$-related if they generate isomorphic principal ideals. Thus if E is *uniform* – that is, all its principal ideals are isomorphic – then T_E is bisimple; if E has a zero and is 0-*uniform* – all its nonzero principal ideals are isomorphic – then it is 0-bisimple.

An elementary, but important, example in the sequel will be the Munn semigroup of an antichain with zero, which is a combinatorial Brandt semigroup.

An ω-*chain* is a semilattice isomorphic to the nonnegative integers under the reverse of their usual order. If Y is such a semilattice, then T_Y is the *bicyclic monoid*, which may be presented by $\langle c \mid cc^{-1} \geq c^{-1}c\rangle$, as an inverse semigroup, or by $C = \langle c, d \mid cd = 1\rangle$, or as a (plain) monoid.

It was shown in [10] that Munn's idempotent-separating representation of any inverse semigroup S in T_{E_S} generalizes to restriction semigroups: for any such semigroup S, there is a P-separating representation of S in T_{P_S}, which induces the congruence μ on S.

We turn now to universal algebraic background, referring the reader to [1] for generalities. If A is an algebra, the variety $\mathbf{V}(A)$ that it generates consists of all algebras that divide a power of A, equivalently, all the algebras that satisfy all the identities satisfied in A. A variety is termed *finitely generated* if it can be generated by some finite algebra. Every finitely generated variety is *locally finite*, in that each of its finitely generated members is finite. Within the context of a given variety of algebras, an algebra A is *finitely based* if $\mathbf{V}(A)$ has a finite basis for its identities, that is, there is a finite set of identities from which all identities satisfied in A are consequences. Otherwise it is *nonfinitely based*. A finite algebra A is *inherently nonfinitely based* if every locally finite variety that contains A is also nonfinitely based. In that event, every finite algebra B such that $A \in \mathbf{V}(B)$ is also (inherently) nonfinitely based.

The article by Volkov [34] gives a useful overview of finite and nonfinite basability for semigroups, inverse semigroups and monoids. In contrast to the situation for groups, whereby every finite group is finitely based, in each case no algorithm is known (at the time of writing) that will decide, given a finite semigroup of the appropriate type, whether it is finitely based.

Before moving to specifics, we outline a common method for proving that an algebra A of a certain type is nonfinitely based, which Volkov terms a *series of critical algebras* for A. It is implicit here, and in the following, that all the statements are relative to some global variety. Such a series constitutes a sequence of algebras A_n with the following properties: A_n is n-generated, and any subalgebra generated by fewer than n elements belongs to $\mathbf{V}(A)$. Although the argument that justifies the following application is elementary (see [34, Section 4.2]), we include it both for completeness' sake and because a little thought will reveal that the formal definition may be tweaked somewhat to achieve the same effect, as we need to do in Section 4.

RESULT 1.2 *If the sequence A_n, $n \geq 1$, is a series of critical algebras for an algebra A, then any variety of algebras that (a) contains A and (b) contains no A_n, is nonfinitely based.*

If A is finite and all the members of the series are infinite, then A is inherently nonfinitely based.

Proof. Suppose $\mathbf{V}$ is such a variety but is finitely based. Then it is defined by identities in at most N variables, for some N. Let $n > N$: then the contradiction $A_n \in \mathbf{V}$ is obtained, because in that algebra each of those identities is evaluated within an N-generated subalgebra which, by assumption, belongs to $\mathbf{V}(A)$ and therefore to $\mathbf{V}$.

If all the algebras in the series are *infinite*, then no locally finite variety whatsoever contains any A_n and so the hypotheses of the first statement are satisfied whenever the variety contains A itself. $\square$

If A is finite and all the algebras A_n in such a series are finite, then A is also nonfinitely based within the class of finite members (of the given global variety). According to [34], within the variety of semigroups per se, the two concepts of finite basability coincide [30], although they do not do so in general.

Some historical background is in order. Perkins [25] showed that the six-element monoid B_2^1 is nonfinitely based, as a 'plain' semigroup. However, all semigroups of five or fewer elements are finitely based (work by various authors), standing in stark contrast to the main results of our paper. In fact B_2^1 is inherently nonfinitely based. However there exist finite, inherently nonfinitely based semigroups that do not include B_2^1 in the variety they generate.

For inverse semigroups, regarded as such, Kleiman [17] showed that B_2^1 is nonfinitely based in this context, too. In contrast with the situation for plain semigroups, any finite inverse semigroup that is nonfinitely based must include B_2^1 in the variety it generates. Equivalently, any finite strict inverse semigroup – see the next section for the definition – is finitely based. This result also stands in stark contrast to the main results of our paper. It is still an open question whether the converse is true. Sapir [29] showed, however, that no finite inverse semigroup is inherently nonfinitely based.

Volkov [34] observes that a finite monoid M is finitely based as a monoid if and only if it is finitely based as a semigroup. Thus B_2^1 is nonfinitely based as a monoid.

A related concept is that of *limit variety*: a variety that is infinitely based, all of whose proper subvarieties are finitely based. The first limit variety of semigroups was found by Volkov [33]. Limit varieties of semigroups are plentiful (see [23]) but have yet to be classified. By the result of Kleiman cited above, other than group varieties the only limit variety of inverse semigroups is that generated by B_2^1. We shall investigate this topic for varieties of restriction semigroups in Section 10.

2 The semigroups B_2, B_0, B^+ and B^-

This entire paper revolves around B_0 and, to a lesser degree B_2. (The semigroups B^+ and B^- act as foils, in a sense.) We first provide some background on B_2. As a 'plain' semigroup, it is presented by $\langle a, b \mid aba = a, bab = b, a^2 = b^2 = 0\rangle$. It consists of the elements $\{a, b, ab, ba, 0\}$. It is a combinatorial, completely 0-simple inverse semigroup.

RESULT 2.1 *As a 'plain' semigroup, B_2 is finitely based: the variety it generates can be defined by the identities*

$$x^3 = x^2, \quad xyx = xyxyx, \quad x^2y^2 = y^2x^2.$$

The first basis was first found by Trahtman [32]; according to Reilly [27], there was a small lacuna in the proof, which the latter closed in the cited paper. Not only is B_2 finitely based, it is hereditarily finitely based, that is, every subvariety of the variety it generates is finitely based [20, Corollary 3.8].

Regarded as an inverse semigroup, $B_2 = \{a, a^{-1}, aa^{-1}, a^{-1}a, 0\}$ and is again finitely based (see Result 2.3).

Being a completely 0-simple inverse semigroup, B_2 is an instance of a Brandt semigroup. Following [26, II.3], Brandt semigroups are semigroups representable in the form $B(G, I)$, where G is a group and I a nonempty set: $B(G, I) = (I \times G \times I) \cup \{0\}$, where $(i, g, j)(j, h, \ell) = (i, gh, \ell)$ and all other products are zero. (Of course, this is a specialization of the Rees matrix semigroup construction.) In this terminology, the semigroup with parameters $|G| = 1$ and $|I| = n$ is usually denoted B_n.

As defined in [26, II.4], an inverse semigroup is *strict* if it is a subdirect product of Brandt semigroups and groups.

Denote by $\mathbf{I}$ the variety of inverse semigroups, in the signature $(\cdot, ^{-1})$. The subvariety of groups will be denoted $\mathbf{G}$ and the subvariety of semilattices will be denoted by $\mathbf{SL}$. Let $\mathbf{SI}$ be the variety of inverse semigroups generated by the Brandt semigroups (that is, by the completely 0-simple inverse semigroups); let $\mathbf{CSI}$ be the variety generated by the combinatorial Brandt semigroups.

RESULT 2.2 *[26, Theorem II.4.5, Proposition XII.4.6] The following are equivalent for an inverse semigroup S:*

(1) $S \in \mathbf{SI}$;

(2) S is strict;

(3) S satisfies the identity $exe \in G$;

(4) S satisfies $\mathcal{D}$-majorization: if $e, f, g \in E_S$, $e > f, g$ and $f \mathcal{D} g$, then $f = g$;

(5) the local monoids eSe, $e \in E_S$, are semilattices of groups.

The expression $exe \in G$ in (3) is shorthand for 'exe belongs to a subgroup – that is, $(exe)(exe)^{-1} = (exe)^{-1}(exe)$ – for every idempotent term $e = yy^{-1}$ (or $e = y^{-1}y$)'. The actual statement in the cited theorem is $yxy^{-1} \in G$, but this is equivalent to $(y^{-1}y)x(y^{-1}y) \in G$.

RESULT 2.3 *[26, Proposition XII.4.8] The following are equivalent for an inverse semigroup S:*

(1) $S \in \mathbf{CSI}$*;*

(2) S *is combinatorial and strict;*

(3) S *satisfies the identity* $yxy^{-1} = (yxy^{-1})^2$*;*

(4) S *belongs to the variety of inverse semigroups generated by* B_2*;*

(5) the local monoids of S *are semilattices.*

RESULT 2.4 $\mathbf{SI} = \mathbf{CSI} \vee \mathbf{G}$.

RESULT 2.5 *[26, Proposition XII.4.13] Let* $\mathbf{V}$ *be a variety of inverse semigroups. Then* $\mathbf{V}$ *consists of semilattices of groups if and only if it does not contain* B_2*. It consists of strict inverse semigroups if and only if it does not contain* B_2^1*.*

It is appropriate at this point to consider the wider class of *completely semisimple* inverse semigroups (see [26, IX.7]): those whose principal factors are Brandt semigroups or groups. If an inverse semigroup is not completely semisimple, there exist $\mathcal{D}$-related idempotents $e > f$ and therefore an element c such that $cc^{-1} = e$, $c^{-1}c = f$. Then the classic theorem of O. Anderson [2, Theorem 2.54] implies that c generates, as an inverse semigroup, the bicyclic semigroup C, with identity e.

RESULT 2.6 *An inverse semigroup is completely semisimple if and only if it contains no bicyclic subsemigroup. A variety of inverse semigroups consists of completely semisimple semigroups if and only if it satisfies* $x^n = x^m$ *for some* $m \neq n$*.*

Regarded as a restriction semigroup, $B_2 = \{a, b, a^+ = b^*, a^* = b^+, 0\}$ and P_{B_2} consists of the incomparable projections a^+ and a^*, together with 0.

Turning now to B_0, regarded as a plain semigroup it is the full subsemigroup $\{a, ab, ba, 0\}$ of B_2. (We follow the notation of [20]. In [5] it was denoted $S(4, 21)$.)

RESULT 2.7 *[5] As a 'plain' semigroup,* B_0 *is finitely based: the variety it generates can be defined by the identities*

$$x^2 = x^3, \quad xyx = yxy = (xy)^2 = x^2y^2$$

Just as B_2 has the natural structure of an inverse semigroup, B_0 has that of a restriction semigroup: that generated in B_2 by a. Thus $B_0 = \{a, a^+, a^*, 0\}$. Observe that the only pairwise products that do not yield 0 are a^+a^+, a^*a^*, a^+a and aa^*.

Finally, the restriction semigroups B^+ and B^-, the *semibicyclic semigroups*, are the restriction subsemigroups of the bicyclic monoid C (see the previous section) that are generated respectively by c and by d. These were studied in depth in [16]. The elements of B^+ can be uniquely represented in the form $(c^m)^*c^k$, where $m, k \geq 0$. Its semilattice of projections is the ω-chain $1 = h_0 > h_1 > h_2 > \cdots$, where $h_i = (c^i)^*$ (with the convention that $c^0 = 1$). Then $((c^m)^*c^k)^+ = h_m$ and $((c^m)^*c^k)^* = h_{m+k}$. Any results about B^- will be obtained by duality.

The author neglected to state the following result separately in [16]: its proof is essentially the proof of Theorem 2.12(1) therein.

RESULT 2.8 *Let S be a restriction semigroup and $a \in S$. Either (i) $a^+ > a^*$, in which case a generates a semigroup isomorphic to B^+, or (ii) $a^+ < a^*$, in which case a generates a semigroup isomorphic to B^-, or (iii) $a^+ \parallel a^*$, in which case B_0 divides S, or (iv) $a^+ = a^*$, in which case a belongs to a submonoid of S.*

The investigation in Section 10 of limit varieties of restriction semigroups devolves to the interesting question of whether or not the restriction semigroups B^+ and B^- are finitely based. (The bicyclic semigroup is nonfinitely based, both as a plain semigroup (see [31]) and as an inverse semigroup [17]. The inverse semigroup variety it generates is not a limit variety, because it contains B_2^1.)

3 Varieties of restriction semigroups

Denote by $\mathbf{R}$ the variety of restriction semigroups. If $\mathbf{V}$ is any variety of restriction semigroups, then $\mathcal{L}(\mathbf{V})$ will denote its lattice of subvarieties. The varieties of restriction semigroups consisting of trivial semigroups, monoids, and semilattices, respectively, will be denoted $\mathbf{T}$, $\mathbf{M}$ and $\mathbf{SL}$. Other varieties will be introduced as needed. As a subvariety of $\mathbf{R}$, $\mathbf{M}$ may be defined by the identity $x^+ = y^+$. Note that subvarieties of $\mathbf{M}$ are essentially varieties of monoids, and we shall treat them as such (although care must be taken to replace any 1 in a true monoid identity by x^+ when regarding it as an identity of restriction semigroups). The variety $\mathbf{SL}$ may be defined by the identity $x = x^+$.

Given a variety $\mathbf{V}$ of restriction semigroups, it is easily verified that the class $\mathbf{loc}(\mathbf{V}) = \{S \in \mathbf{R} : eSe \in \mathbf{V}\ \forall e \in P_S\}$ is again a variety. We say that its members are 'locally' in $\mathbf{V}$.

In a related vein, given a variety $\mathbf{N}$ of monoids, regarded as restriction semigroups, let $\mathbf{mon}(\mathbf{N})$ consist of the restriction semigroups S all of whose (maximal) submonoids $\mathbb{H}_e$, $e \in P_S$, belong to $\mathbf{N}$. For instance, $\mathbf{mon}(\mathbf{T})$ consists of the restriction semigroups all of whose submonoids are trivial. Any class $\mathbf{mon}(\mathbf{N})$ is closed under products and (biunary) subsemigroups, but not in general under homomorphic images (since the free restriction semigroups are $\mathbb{H}$-combinatorial [9] and so belong to $\mathbf{mon}(\mathbf{T})$).

PROPOSITION 3.1 *A restriction semigroup, all of whose submonoids are trivial, need not be $\mathbb{H}$-combinatorial.*

Proof. Our example, which we denote TR_2, is no doubt folklore and could be defined in various ways. Intuitively, it is the result of amalgamating two copies of B_0 over the common semilattice $\{ab, ba, 0\}$. It will useful to represent it more concretely as a restriction subsemigroup of the Brandt semigroup $B(Z_2, I)$, where $I = \{1, 2\}$ and $Z_2 = \{1, g\}$. Let $TR_2 = \{(1, 1, 1), (1, 1, 2), (1, g, 2), (2, 1, 2), 0\}$. That it is a full subsemigroup of $B(Z_2, I)$ and, therefore, a restriction semigroup is straightforwardly verified. The projections are $(1, 1, 1)$, $(2, 1, 2)$ and 0. The relation $\mathbb{H}$ is just the restriction of the usual Green relation $\mathcal{H}$. Thus $\{(1, 1, 2), (1, g, 2)\}$ is a nontrivial $\mathbb{H}$-class and the only such. $\square$

The inverse semigroups, when regarded as restriction semigroups, do not form a variety since they are not closed under taking restriction subsemigroups. However, they play an important role, since $\mathbf{R}$ is *generated* by the inverse semigroups (considered as restriction semigroups). This follows from the description of the free restriction semigroups [9].

Observe that since the class of inverse semigroups is closed under direct products and ordinary homomorphic images, if $\mathbf{V}$ is a variety of inverse semigroups then the variety of restriction semigroups that it generates consists of the restriction semigroups that divide members of $\mathbf{V}$, when the latter are regarded as restriction semigroups. For example, the variety of restriction semigroups generated by the variety $\mathbf{G}$ of inverse semigroups consisting of groups is the variety $\mathbf{M}$ of monoids, since free monoids are embeddable in the corresponding free groups.

Also observe that if S is an inverse semigroup, then the variety of *restriction* semigroups that S generates is the same as the variety of restriction semigroups that is generated by the *inverse semigroup* variety that S generates. In particular, as a consequence of Result 2.2, the variety of restriction semigroups generated by the Brandt semigroups is that generated by the strict inverse semigroups; and as a consequence of Result 2.3, the variety of restriction semigroups generated by B_2 is that generated by the combinatorial strict inverse semigroups.

This section is concluded by a summary of relevant results from the precursor to this paper, [16], which focused on the 'bottom' of the lattice of varieties of restriction semigroups (and the analogues for left restriction semigroups).

RESULT 3.2 *[16, Theorem 2.1] If $\mathbf{V} \in \mathcal{L}(\mathbf{R})$, then $\mathbf{V} \vee \mathbf{M} = \{S \in \mathbf{R} : S/\mu \in \mathbf{V}\}$. Hence the map $\mathbf{V} \longrightarrow \mathbf{V} \vee \mathbf{M}$ is a complete lattice homomorphism. If $\mathbf{V}$ is defined by the identities $u_i = v_i$, $i \in I$, then $\mathbf{V} \vee \mathbf{M}$ is defined by the identities $(u_i x)^+ = (v_i x)^+$, $i \in I$, where x is a letter distinct from any in the original set of identities.*

In the context of restriction semigroups, the term *S is a semilattice Y of monoids S_α* specifically requires that $Y \in \mathbf{SL}$, the map $S \longrightarrow Y$ is a biunary homomorphism and that each S_α belongs to $\mathbf{M}$. The following result contains a selection from the many equivalent properties proved in the cited paper.

RESULT 3.3 *[16, Theorem 2.6] The following are equivalent for a restriction semigroup S:*

(i) $S \in \mathbf{SL} \vee \mathbf{M}$;

(ii) S satisfies $x^+ = x^*$;

(iii) S is a semilattice of monoids.

In view of this result, we denote $\mathbf{SL} \vee \mathbf{M}$ by $\mathbf{SM}$.

RESULT 3.4 *[16, Theorem 2.8] The sublattice $\mathcal{L}$ $(\mathbf{SM})$ of $\mathcal{L}$ $(\mathbf{R})$ is isomorphic to the direct product of the two-element lattice $\mathcal{L}(\mathbf{SL})$ and the lattice $\mathcal{L}(\mathbf{M})$, under the map $\mathbf{V} \mapsto (\mathbf{V} \cap \mathbf{SL}) \vee (\mathbf{V} \cap \mathbf{M})$. If $\mathbf{V}$ is not simply a variety of monoids, then it consists of all semilattices of monoids from $\mathbf{V} \cap \mathbf{M}$.*

We may extend the notation $\mathbf{SM}$ to $\mathbf{SN} = \mathbf{SL} \vee \mathbf{N}$, for any variety $\mathbf{N}$ of monoids. Defining identities for $\mathbf{SN}$, within $\mathbf{SM}$, can be derived from those for $\mathbf{N}$ using the following result. (See the earlier remark regarding identities in restriction monoids.)

PROPOSITION 3.5 *Let $S \in \mathbf{SM}$. Then its submonoids satisfy a (restriction) monoid identity $u = v$ if and only if S satisfies $v^+u = u^+v$.*

Proof. Say the identity $u = v$ defines the monoid variety $\mathbf{N}$. Then both $\mathbf{SL}$ and $\mathbf{N}$ satisfy $v^+u = u^+v$, since $u = u^+$ and $v = v^+$ in the former and $u^+ = v^+$ in the latter; so the identity is satisfied in $\mathbf{SL} \vee \mathbf{N} = \mathbf{SN}$. Conversely, if S satisfies $v^+u = u^+v$ (and is a semilattice of monoids), then each submonoid satisfies this identity, which reduces there to $u = v$. So $S \in \mathbf{SN}$. $\square$

Denote by $\mathbf{B}^+$ and $\mathbf{B}^-$ the varieties of restriction semigroups that are generated by the semibicyclic semigroups B^+ and B^-, respectively, which were defined in the previous section.

The first statement of the next result is the analogue of the first statement of Result 2.5. The second and third statements were not included in [16], but will be useful in the sequel. We include proofs of (1) – (3), for completeness.

RESULT 3.6 *[16, Theorems 2.12, 2.13]*

(1) Any variety of restriction semigroups that does not consist of semilattices of monoids contains either B^+, B^- or B_0.

(2) Any locally finite variety that does not consist of semilattices of monoids contains B_0.

(3) Any variety of restriction semigroups that does not contain B_0 is contained in either the variety defined by $x^+ \geq x^*$ or the variety defined by $x^+ \leq x^*$.

(4) $\mathbf{B}_0$ covers $\mathbf{SL}$.

Proof. (1) is an immediate consequence of Result 2.8. (2) follows from the fact that the semibicyclic semigroups are not locally finite. To prove (3), we also need [16, Proposition 2.14(2)], which states that $B_0 \in \mathbf{B}^+ \vee \mathbf{B}^-$. It follows that if $B_0 \notin \mathbf{V}$, then either $a^+ \geq a^*$ for all $a \in S$ and for all $S \in \mathbf{V}$, or the dual relation holds. $\square$

4 The semigroups Ψ_k and an outline of the proof that B_0 is inherently nonfinitely based.

In the language introduced near the end of Section 1, we construct a *series of critical restriction semigroups*[1] Ψ_k for B_0. (In fact, the sequence Ψ_k is defined only for positive, even integers k, but this suffices, in view of the proof of Result 1.2.)

[1]The author thanks Mikhail Volkov for pointing out to the author the method of critical series and asking whether preliminary results on finite basability – see Sections 8 to 10 – could be adapted to this method.

By Result 1.2 any variety of restriction semigroups that (a) contains B_0 and (b) contains no Ψ_k, is nonfinitely based. Now our sequence Ψ_k has the additional property that its members are infinite. Therefore any locally finite variety that contains B_0 is nonfinitely based, that is, B_0 is inherently nonfinitely based.

By Results 3.6 and 3.3, a locally finite variety does *not* contain B_0 if and only if it satisfies $x^+ = x^*$, equivalently it consists of semilattices of monoids.

Specializing to finitely generated varieties, if S is a finite restriction semigroup for which $B_0 \notin \mathbf{V}(S)$, then (either by the proof of Result 3.3 or by Proposition 6.3 herein) $\mathbf{V}(S) = \mathbf{SL} \vee \mathbf{V}(M)$, where M may be taken to be the product of its submonoids, for example. By Proposition 3.5, S is finitely based if and only if M is finitely based, as a monoid.

We summarize as follows.

THEOREM 4.1 *The infinite restriction semigroups Ψ_k comprise a series of critical semigroups for B_0 and so any variety of restriction semigroups that contains B_0 but no Ψ_k is finitely based. Hence B_0 is inherently nonfinitely based. Any finite restriction semigroup for which the two unary operations do not coincide is nonfinitely based. Decidability of finite basability for finite restriction semigroups reduces to the corresponding problem for finite monoids.*

We now construct the semigroups Ψ_k and outline the proof of this theorem. In following the construction, it may be helpful to refer to Figure 1, which provides a graphical representation of Ψ_4 and exhibits all the salient features of the general case.

Let k be a positive, even integer. Partially order the set $\{e_n : n \geq 1\}$ by $e_n < e_m$ if and only if $m \equiv n \mod k$ and $m < n$; this poset is the cardinal sum of the ω-chains $e_r > e_{k+r} > e_{2k+r} > \cdots$, $1 \leq r \leq k$. Let Y be the semilattice obtained by adjoining an element 0 that is the meet of each pair of elements from distinct chains.

The Munn semigroup T_Y has the following properties. Each nonzero principal ideal is an ω-chain, with zero adjoined. Therefore, given any two nonzero principal ideals, there is a unique isomorphism from one to the other. Thus [26, IV.2] Y is 0-uniform and T_Y is a combinatorial, 0-bisimple inverse semigroup. For each n, denote the identity map 1_{Ye_n} by ϵ_n. (Recall that $E_{T_Y} \cong Y$ under the map $\epsilon_n \mapsto e_n$.)

Let $\Psi_k = \{\alpha_n : n \geq 1\} \cup E_{T_Y}$, where $\alpha_n \in T_Y$ is defined as follows, for $n = 1, 2, \ldots$:

$$\alpha_n : \begin{cases} Ye_n \longrightarrow Ye_{n+1} & \text{if } n \text{ is odd,} \\ Ye_{n+1} \longrightarrow Ye_n & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

We show here that Ψ_k is a restriction semigroup and prove some its properties, leaving others to be proven when the necessary machinery has been developed and, in some cases, the necessary terms defined.

Observe first that, for any n, the domain of α_n is generated by an idempotent with odd index and its range is generated by an idempotent with even index. Therefore for any m, n, possibly equal, the range of α_m intersects the domain of α_n only at 0, that is, $\alpha_m\alpha_n = 0$. So, together with 0, the set $\{\alpha_n : n \geq 1\}$ forms a null semigroup.

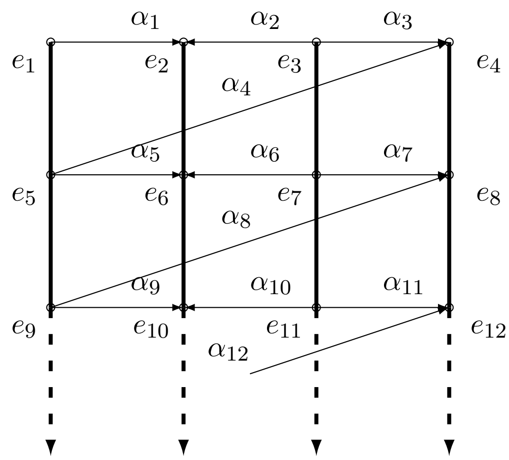

Figure 1: Ψ_4 as a semigroup of mappings.

By similar reasoning, the only nonzero products of projections with non-projections are the following:

$$\begin{cases} \epsilon_m\alpha_n = \alpha_{\max(m,n)} = \alpha_n\epsilon_{m+1} & \text{if } m \equiv n \mod k, \text{ for } m \text{ odd} \\ \epsilon_{m+1}\alpha_n = \alpha_{\max(m,n)} = \alpha_n\epsilon_m & \text{if } m \equiv n \mod k, \text{ for } m \text{ even.} \end{cases} \tag{2}$$

Therefore Ψ_k is a full subsemigroup of T_Y and thus a restriction semigroup, with semilattice of projections $P_{\Psi_k} = \{\epsilon_n : n \geq 1\} \cup \{0\}$. From the definition,

$$\alpha_n^+ = \epsilon_n \text{ and } \alpha_n^* = \epsilon_{n+1}, \text{ if } n \text{ is odd;} \quad \alpha_n^+ = \epsilon_{n+1} \text{ and } \alpha_n^* = \epsilon_n, \text{ if } n \text{ is even.} \tag{3}$$

Note that since $\{\alpha_n : n \geq 1\} \cup \{0\}$ is a null semigroup, Ψ_k contains no regular elements, other than projections; and since $\alpha_n^2 = 0$ for all n, Ψ_k satisfies the identity $x^3 = x^2$.

Next we prove that Ψ_k is generated, as a restriction semigroup, by $\{\alpha_1, \ldots, \alpha_k\}$. Temporarily, denote by S the subsemigroup so generated. A particular instance of (2) is the following, for $n \geq 1$:

$$\alpha_n = \epsilon_n\alpha_r, \text{ if } n \text{ is odd, and } \alpha_n = \alpha_r\epsilon_n \text{ if } n \text{ is even, where } n \equiv r \mod k, \ 1 \leq r \leq k. \tag{4}$$

In view of these equations, it suffices to show that S contains all the projections. It is clear that $\epsilon_1, \ldots, \epsilon_{k+1}$ belong to S. Now assume that $\epsilon_n \in S$, where $n \geq k+1$. Let $n \equiv r \mod k$, where $r \in \{1, \ldots, k\}$. If n is odd, then $\epsilon_{n+1} = \alpha_n^* = (\epsilon_n\alpha_r)^* \in S$; if n is even, then $\epsilon_{n+1} = \alpha_n^+ = (\alpha_r\epsilon_n)^+ \in S$.

Further properties make use of the generalized Green relations, which were defined in Section 1, though all that is needed here is the fact, noted there, that in this situation $\mathbb{R}$ and $\mathbb{L}$ are simply the restrictions of the usual relations $\mathcal{R}$ and $\mathcal{L}$. Therefore $\alpha_n \ \mathbb{L} \ \alpha_{n+1}$, for n odd, and $\alpha_n \ \mathbb{R} \ \alpha_{n+1}$, for n even, and so all nonzero elements of Ψ_k are $\mathbb{D}$-related, that is, in the terminology of the next section, Ψ_k is 0-$\mathbb{D}$-bisimple. Similarly, since $\mathcal{H} = \iota$ on T_Y, $\mathbb{H} = \iota$ on Ψ_k.

Again, in terminology to be introduced more formally in the next section, Ψ_k is not completely 0-r-simple and not completely r-semisimple: it contains distinct, comparable $\mathbb{D}$-related projections. We summarize the properties of Ψ_k so far obtained.

PROPOSITION 4.2 *For any positive, even integer k, Ψ_k is an infinite restriction semigroup that is generated, as such, by $\{\alpha_1, \ldots, \alpha_k\}$ and satisfies $x^3 = x^2$. It is $\mathbb{H}$-combinatorial and 0-$\mathbb{D}$-bisimple, but not completely 0-r-simple and so not completely r-semisimple. Its only regular elements are projections.*

The proof of Theorem 4.1 will be completed by Proposition 11.1, which shows that each Ψ_k has the property that any restriction subsemigroup that is generated by fewer than k elements belongs to $\mathbf{B}_0$. The study of $\mathbf{B}_0$ and of, more generally, $\mathbf{B}_2$, the variety generated by B_2, and $\mathbf{B}$, the variety of strict restriction semigroups (generated by Brandt semigroups in general) forms the core of this paper. However, by no means is all of that material needed to prove the proposition.

The key fact is that such a subsemigroup, T say, satisfies $\mathbb{D}$-*majorization*: no projection is above two distinct $\mathbb{D}$-related projections. This property will be studied in depth in Sections 6 and 7. (As shown in Section 5, $\mathbb{D}$-majorization implies complete r-semisimplicity.) The essence of the proof is that any distinct, comparable projections in Ψ_k need to be 'connected' by a sequence of consecutive α_n's of length at least k, which requires the existence of a generator in each of the classes α_n for $n \equiv r \mod k$, $r = 1, \ldots, k$. (See Figure 1.)

By Proposition 6.3, it will then follow that T is a subdirect product of its 'r-principal factors', which are either monoids or 'primitive'. By Proposition 5.8, those factors will then belong to $\mathbf{B}$. In fact, by the $\mathbb{H}$-combinatorial property, they must belong to $\mathbf{B}_2$ (see Corollary 6.4). Finally, since Ψ_k has no regular elements, other than projections, the characterization of $\mathbf{B}_0$ in Theorem 10.3 yields $T \in \mathbf{B}_0$.

The analysis of the varieties that arise during this proof, and of completely r-semisimple varieties, was actually the origin of the nonfinite basability results of the paper. The techniques yield considerably more information, such as bases of identities for $\mathbf{B}$, $\mathbf{B}_2$ and $\mathbf{B}_0$, and characterizations by 'forbidden' semigroups. The semigroups Ψ_k arise directly from consideration of failure of complete r-semisimplicity (see Section 11), so it is natural to complement the first statement of Theorem 4.1, determining the limit of its applicability to varieties of restriction semigroups in general, with a statement on the limits of its applicability.

THEOREM 4.3 *(Theorem 11.2) A variety of restriction semigroups contains B_0 but no semigroups Ψ_k if and only if it consists of completely r-semisimple semigroups.*

This is an appropriate place to mention another series of critical semigroups for B_0, this time finite, and their role in this work. These semigroups arise directly from consideration of failure of $\mathbb{D}$-majorization, within the context of completely r-semisimple semigroups (cf the remarks on Ψ_k above regarding the failure of the latter). For any positive integer k, let Y instead be the semilattice obtained by adjoining to the antichain $\{e_1, \ldots, e_{k+1}\}$ both a zero element and another element f such that $f > e_1$ and $f > e_{k+1}$.

Let $\Lambda_k = \{\alpha_n : 1 \leq n \leq k\} \cup E_{T_Y}$, where $\alpha_n \in T_Y$ is defined by the same formal rule (1) as for Ψ_k. The nonzero projections are $\epsilon_n = 1_{Ye_n}$, $n = 1, \ldots, k+1$, together with $\phi = 1_{Yf}$. Figure 2 is a graphical representation of Λ_4. The elementary properties of these semigroups are discussed at the end of Section 6.

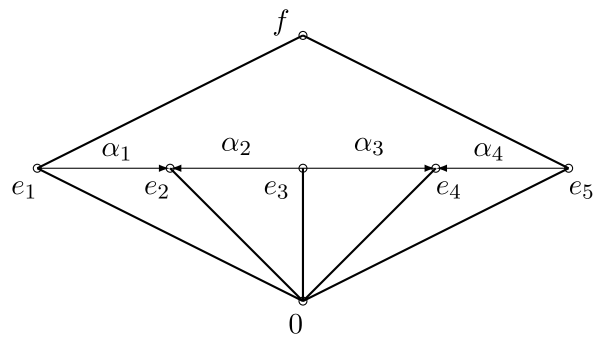

Figure 2: Λ_4 as a semigroup of mappings.

THEOREM 4.4 *(Propositions 6.7 and 10.4) The finite restriction semigroups Λ_k also comprise a series of critical semigroups for B_0. Thus (see following Result 1.2) B_0 is also nonfinitely based within the class of finite restriction semigroups.*

The arguments will be similar to, but somewhat simpler than, those for the Ψ_k's.

Absence of these semigroups characterizes subvarieties of $\mathbf{B}$ in the same way that absence of the Ψ_k's characterizes complete r-semisimplicity in varieties.

THEOREM 4.5 *(Theorem 12.1) A variety of restriction semigroups contains B_0 but no semigroups Λ_k if and only if consists of strict restriction semigroups.*

5 Generalized Green relations and r-ideals

Recall that on a restriction semigroup S, the relations $\mathbb{R}$ and $\mathbb{L}$ are defined by $a \mathbin{\mathbb{R}} b$ if $a^+ = b^+$ and $a \mathbin{\mathbb{L}} b$ if $a^* = b^*$, $\mathbb{H} = \mathbb{L} \cap \mathbb{R}$ and $\mathbb{D} = \mathbb{L} \vee \mathbb{R}$. (We will define $\mathbb{J}$ later in this section.) Although there are some precursors of the results of this section in [8], to the best of our knowledge they are new. While the behaviour of $\mathbb{D}$ is very different from that in inverse semigroups, it turns out that important results on the interplay among Green's relations in inverse semigroups have direct analogues for these generalizations, leading to analogous definitions such as 'completely 0-r-simple' and 'completely r-semisimple' and analogous results concerning them.

The following basic property is well known, but we include a proof because it will be used frequently, and to illustrate the use of the axioms.

LEMMA 5.1 *Let S be a restriction semigroup and $a, b \in S$. Then $a \mathbin{\mathbb{R}} ab$ if and only if $a^* \leq b^+$, and $ab \mathbin{\mathbb{L}} b$ if and only if $a^* \geq b^+$.*

Proof. If $a \mathbb{R} ab$, then $a^+ = (ab)^+$ and so $a = (ab)^+ a = ab^+$, applying one of the ample axioms. So $a^* = (ab^+)^* \leq (b^+)^* = b^+$. Conversely, if $a^* \leq b^+$, then $a = aa^* = aa^*b^+ = ab^+$ and so $a^+ = (ab^+)^+ = (ab)^+$. The second statement is the dual of the first (in the sense specified in Section 1). $\square$

For $k \geq 2$, a $\mathbb{D}$-*zigzag* (or just 'zigzag') of length k is a sequence $a_1, a_2, \ldots, a_k$ of distinct elements of S, linked by alternations of $\mathbb{L}$ and $\mathbb{R}$. We will term a zigzag *standard* if it begins with $\mathbb{L}$. Thus

$$a_1 \mathbb{L} a_2 \mathbb{R} a_3 \cdots a_k,$$

where the last relation is $\mathbb{L}$ if k is even, and $\mathbb{R}$ if k is odd. The *associated sequence of projections* consists of $e_1 = a_1^+$, $e_2 = a_1^* = a_2^*$, $e_3 = a_2^+ = a_3^+$, and so on, with $e_{k+1} = a_k^+$ if k is even, or $e_{k+1} = a_k^*$, if k is odd. A $\mathbb{D}$-zigzag of length 1 consists simply of a_1. Such a zigzag is then *standard* if its associated sequence of projections is defined to be $e_1 = a_1^+, e_2 = a_1^*$.

Figure 3 represents a 'partial eggbox' picture of one such standard zigzag, where the elements in the same row are $\mathbb{R}$-related and those in the same column are $\mathbb{L}$-related. Bold faced elements indicate projections. In general, the picture will not represent entire $\mathbb{R}$- and $\mathbb{L}$-classes; for instance, in an inverse semigroup, where $\mathbb{R} = \mathcal{R}$, $\mathbb{L} = \mathcal{L}$, and so on, the $\mathbb{D}$-classes are the usual $\mathcal{D}$-classes and are therefore 'square'.

If $k \geq 2$, 'dual standard' zigzags begin with $\mathbb{R}$ and their associated sequences of projections are defined dually: $e_1 = a_1^*$, etc. In the case $k = 1$, dual standard means that the sequence is $e_1 = a_1^*, e_2 = a_1^+$. Note that if k is odd, properties of standard zigzags yield those of the duals (by reading the zigzag and its sequence of projections in reverse order). In the even case, the distinction is substantive, as we shall see in Section 7.

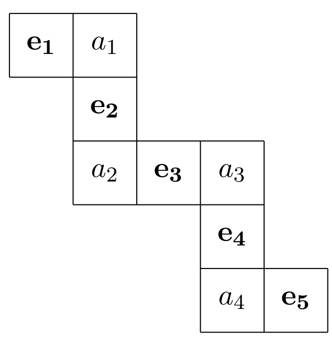

Figure 3: A $\mathbb{D}$-zigzag in the semigroup Δ_4

However, these zigzags can be concretely manifested as entire $\mathbb{D}$-classes in the semigroups Δ_k that we now construct. We represent these semigroups by mappings in a similar way to the representations of Ψ_k and Λ_k in the previous section. (In fact, Δ_k is simply an ideal of the latter.) This viewpoint also demonstrates another way of visualizing $\mathbb{D}$-zigzags, as illustrated in Figure 4. (Alternatively, completing the partial eggbox picture of Δ_k also translates direct into a representation in the combinatorial Brandt semigroup B_{k+1}.)

As alluded to above, Δ_k is obtained from Λ_k by deleting the projection $\phi = 1_{Yf}$. (In terms of the conceptual development of this paper, however, the more natural viewpoint is that the latter is obtained from the former by *adjoining* a projection ϕ that is above only ϵ_1 and ϵ_{k+1}.)

Thus in this context Y is the semilattice obtained by adjoining a zero to the antichain $\{e_1, \ldots, e_{k+1}\}$ and, for $1 \leq n \leq k$, defining $\alpha_n \in T_Y$ by the equations (1) used to define the maps in Ψ_k and Λ_k (Section 4).

Then Δ_k is the (full) subsemigroup of T_Y generated by $\alpha_1, \ldots, \alpha_k$; the generators form a standard $\mathbb{D}$-zigzag. In fact Δ_k consists solely of the generating set, the projections ϵ_i, $i = 1, \ldots, k+1$, and the zero element, and so represents, in a concrete sense, the simplest possible zigzag of such a form. Moreover, this is the unique zigzag from ϵ_1 to ϵ_{k+1}. With the addition of a zero element (and slightly different notation), Figure 3 exhibits Δ_4 in eggbox form. Note that $\Delta_1 \cong B_0$.

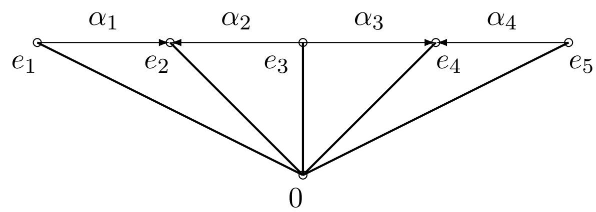

Figure 4: Δ_4 as a semigroup of mappings.

We now turn to the relation $\mathbb{J}$. An *r-ideal* (for 'restriction ideal') in S is an ideal I of S that is also a restriction subsemigroup. It is easily verified that the Rees factor semigroup S/I is again a restriction semigroup. (As usual, for technical reasons it is convenient to allow the empty set to be an r-ideal and, in that case, to put $S/I = S$.) A restriction semigroup S without zero is *r-simple* if S is the only r-ideal; a restriction semigroup with zero is 0-*r-simple* if $\{0\}$ and S are its only r-ideals.

If $A \subseteq S$, denote by $rI(A)$ the r-ideal it generates (and by $I(A)$ the ideal it generates in the usual sense, namely S^1AS^1). Since for any $a \in S$, $a = a^+a = aa^*$, $rI(a) = rI(a^+) = rI(a^*)$. So if either $a \mathbb{L} b$ or $a \mathbb{R} b$, then $rI(a) = rI(b)$, and thus the same holds if $a \mathbb{D} b$. It is then easily seen (cf [7, Lemma 1.7(3)]) that $rI(A)$ is obtained from A as the union of a sequence of subsets, alternately (a) closing under $\mathbb{D}$ and (b) generating an ideal.

To illustrate in Λ_k (or in Δ_k), $rI(\epsilon_1) = \Delta_k$, although the ideal generated by ϵ_1 is just $\{\epsilon_1, \alpha_1, 0\}$. The r-ideals of Λ_k are Λ_k, Δ_k and $\{0\}$. The only r-ideals of Ψ_k are Ψ_k itself and $\{0\}$. Thus Δ_k and Ψ_k are 0-r-simple.

Define $\mathbb{J}$ on S by $a \mathbb{J} b$ if $rI(a) = rI(b)$. The set $rQ(a) = rI(a) \backslash \mathbb{J}_a$ is an r-ideal of $rI(a)$ and the Rees factor semigroup $rI(a)/rQ(a)$ is the r-*principal factor* associated with a. It is 0-r-simple, or r-simple in case $rQ(a)$ is empty. As in the semigroup case, the r-principal factor is essentially the $\mathbb{J}$-class itself, with products that do not lie within the class (if any) sent to zero. A restriction semigroup S without zero is r-simple if and only if it has only one $\mathbb{J}$-class; a restriction semigroup with zero is 0-r-simple if and only if $\{0\}$ and $S \backslash \{0\}$ are its only $\mathbb{J}$-classes.

The three principal r-factors of Λ_k are (isomorphic to) the two-element semilattice (regarded as a restriction semigroup), Δ_k, and the trivial monoid.

As observed above, $\mathbb{D} \subseteq \mathbb{J}$. The following results on the relationship between $\mathbb{D}$ and $\mathbb{J}$ generalize well-known, useful results on inverse semigroups (e.g. [2, Exercise 8.4.3]) and, again to the best of our knowledge, are new.

LEMMA 5.2 *Let S be a restriction semigroup and $e, f, g \in P_S$ be such that $e \,\mathbb{D}\, f$ and $f \geq g$. Then there exists $h \in P_S$ such that $e \geq h$ and $h \,\mathbb{D}\, g$. If $g \neq f$, then h may be chosen to be distinct from e.*

In particular, if $f, g \in P_S$, $f \,\mathbb{D}\, g$ and $f > g$, there exists $h \in P_S$, $g \,\mathbb{D}\, h$ and $g > h$.

Proof. Only the case $g \neq f$ need be considered. It suffices to prove this when $(e, f) \in \mathbb{R} \circ \mathbb{L}$, duality then taking care of the case $(e, f) \in \mathbb{L} \circ \mathbb{R}$ and induction taking care of the general situation. Suppose $e \,\mathbb{R}\, a \,\mathbb{L}\, f$. Put $h = (ag)^+ \leq a^+ = e$. Now $h \,\mathbb{D}\, (ag)^* = (a^*g)^* = fg = g$. Suppose $h = e$. Then $ag = (ag)^+a = a^+a = a$, yielding the contradiction $g = (ag)^* = a^* = f$.

The final statement is simply the special case whereby $e = g$. $\square$

LEMMA 5.3 *Let S be a restriction semigroup and $e \in P_S$, $a \in S$ be such that $a \in S^1eS^1$. Then there exists $h \in P_S$ such that $e \geq h$ and $h \,\mathbb{D}\, a$.*

Proof. There exist $s, t \in S^1$ such that $a = set$. In the following, we let $1^+ = 1$, for convenience. Put $h = (st^+e)^* \leq e$. Then $h \,\mathbb{D}\, (st^+e)^+ = (set^+)^+ = (set)^+ = a^+ \,\mathbb{R}\, a$. $\square$

LEMMA 5.4 *If $e, g \in P_S$, then $g \in rI(e)$ if and only if there exists $h \in P_S$ such that $e \geq h$ and $h \,\mathbb{D}\, g$.*

Proof. Sufficiency is clear. Conversely, there exists a sequence $e = a_0, a_1, \ldots, a_n = g$, of minimum length, where for each $i \geq 1$, either $a_{i+1} \in S^1a_iS^1$ or $a_{i+1} \,\mathbb{D}\, a_i$. On the one hand, if $a_1 \in S^1eS^1$, then by the last lemma, there exists $h \in P_S$, $h \leq e$, $h \,\mathbb{D}\, a_1$. If $n = 1$, this is the required outcome. If $n > 1$, assume the claim is true for all shorter sequences. In fact, the sequence $h, a_2, \ldots, a_n$, beginning with $h \,\mathbb{D}\, a_2$, is shorter and so there is an idempotent $k \leq h \leq e$, $k \,\mathbb{D}\, g$. On the other hand, if $a_1 \,\mathbb{D}\, e$ and $a_2 \in S^1a_1S^1$, then $a_2 \in S^1a_1^+S^1$ so, again by the last lemma, there exists $h \in P_S$, $h \leq a^+$, $h \,\mathbb{D}\, a_2$. But $a^+ \,\mathbb{D}\, e$, so by Lemma 5.2, there exists $\ell \in P_S$, $\ell \leq e$, $\ell \,\mathbb{D}\, h \,\mathbb{D}\, a_2$. This case is then completed in the same fashion as was the first. $\square$

COROLLARY 5.5 *If $e, f \in P_S$, then $e \,\mathbb{J}\, f$ if and only if there exist $h, k \in P_S$ such that $e \geq h \,\mathbb{D}\, f \geq k \,\mathbb{D}\, e$.*

As noted in the previous section, we call a restriction semigroup *completely r-semisimple* if the distinct projections within any $\mathbb{D}$-class are incomparable. For example, the semigroups Λ_k have this property but the Ψ_k's do not.

COROLLARY 5.6 *In any completely r-semisimple semigroup, $\mathbb{J} = \mathbb{D}$.*

Proof. Suppose $e, f \in P_S$ and $e \mathbb{J} f$. Choose h, k as in the previous corollary. Since $h \mathbb{D} f \geq k$, by Lemma 5.2 there exists $\ell \in P_S$ such that $h \geq \ell$ and $\ell \mathbb{D} k$. Then $e \geq h \geq \ell \mathbb{D} k \mathbb{D} e$ and so $e = h = \ell$ and $e \mathbb{D} f$. □

COROLLARY 5.7 *If the projections in any* $\mathbb{D}$*-class of a restriction semigroup* S *satisfy the Descending Chain Condition — in particular, if* S *itself is finite — then* S *is completely r-semisimple.*

Proof. This is immediate from the last statement of Lemma 5.2. □

If S is completely r-semisimple, any r-principal factor of S without zero contains a single projection, in other words it is a monoid. Call a restriction semigroup with zero *primitive* if each of its nonzero projections is minimal. The r-principal factors with zero (if any) in a completely r-semisimple restriction semigroup are primitive and 0-r-simple, in fact 0-$\mathbb{D}$-*simple*, in the sense that their nonzero elements form a single $\mathbb{D}$-class.

We use the term *completely 0-r-simple* for primitive, 0-r-simple restriction semigroups. Thus a restriction semigroup is completely r-semisimple if and only if each r-principal factor is completely 0-r-simple or a monoid. The semigroups Δ_k are completely 0-r-simple. In particular, B_0 has that property.

From Corollary 5.7, any finite 0-r-simple restriction semigroup is completely 0-r-simple. In constrast with the situation for inverse semigroups, the semigroups Ψ_k attest to the fact that a periodic 0-r-simple restriction semigroup need not be completely 0-r-simple. The bicyclic semigroup and its 'upper' and 'lower' triangles, B^+ and B^-, respectively, are examples of restriction semigroups that are $\mathbb{D}$-simple but not completely r-semisimple. In Section 11, complete r-semisimplicity will be characterized at a varietal level by the absence of B^+, B^- and the Ψ_k's.

It is not difficult to provide a structure theorem for completely 0-r-simple restriction semigroups (such a theorem is implicit in the work of [18]), but it is not needed in this paper. The next result, cited in the outline of the proof of Theorem 4.1, suffices. A little background is needed.

The least monoid congruence on a restriction semigroup S is denoted σ. A restriction semigroup is *proper* if $\sigma \cap \mathbb{L} = \sigma \cap \mathbb{R} = \iota$. According to [9, Theorem 7.1], every restriction semigroup S has a *proper cover*, that is, a proper restriction semigroup C and a P-separating homomorphism from C onto S. Refer to Section 1 for the definition and properties of the congruence μ.

PROPOSITION 5.8 *Let* S *be any primitive restriction semigroup with 0. Then* S *divides the direct product of a combinatorial Brandt semigroup and a monoid. If* S *is itself* $\mathbb{H}$*-combinatorial, it embeds into a combinatorial Brandt semigroup.*

Proof. As just noted, there exists a proper restriction semigroup C and a P-separating homomorphism from C onto S. Now $\mu \cap \sigma \subseteq \mathbb{H} \cap \sigma = \iota$ on C, that is, C is a subdirect product of C/μ and C/σ. The congruence μ is induced by the representation of C in the Munn semigroup T_{P_C}. The semilattice $P_C \cong P_S$ is an antichain with zero and so its Munn semigroup is a combinatorial Brandt semigroup, with semilattice of idempotents isomorphic to P_S. Since S/σ is itself a monoid, the first assertion is a consequence.

The second is immediate from the representation of S itself, based on the triviality of μ in that case. $\square$

Note that for the purposes of Theorem 4.1, the covering theorem is not required, only the generalization of the Munn representation.

6 $\mathbb{D}$-majorization and the semigroups Λ_k

By analogy with the use of the term $\mathcal{D}$-majorization for inverse semigroups (e.g. [26]), a restriction semigroup satisfies $\mathbb{D}$-*majorization* if whenever $f, g, h \in P_S$, $f > g, h$ and $g \mathbin{\mathbb{D}} h$, then $g = h$. Schematically, there must be no configuration of the kind exemplified by Figure 5.

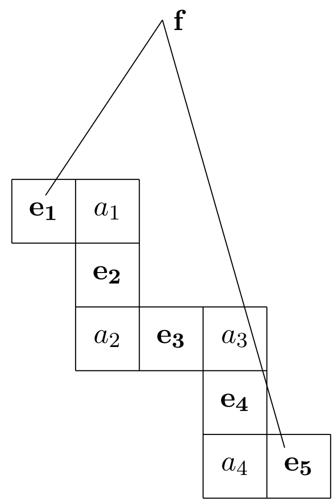

Figure 5: A configuration forbidden by $\mathbb{D}$-majorization.

This figure should not be taken too literally, in that it suggests that e belongs to a different $\mathbb{D}$-class from the zigzag. Failure in that specific manner will be captured by 'Λ_k-configurations' and concretely realized in the semigroups Λ_k that were introduced in Section 4. It turns out that to prove the main results of the next sections, on strict restriction semigroups, this scenario suffices (and Section 12 explains why).

It may be the case, however, that f coincides with the first or last projection associated with the zigzag, that is, $\mathbb{D}$-majorization may fail *within* a $\mathbb{D}$-class. As noted in the last part of the previous section, by virtue of the last statement of Lemma 5.2 this corresponds, in essence, to a failure of complete r-semisimplicity. The 'Ψ_k-configurations' that capture that failure will be the topic of Section 11.

PROPOSITION 6.1 *A restriction semigroup S satisfies $\mathbb{D}$-majorization if and only if whenever $f, g, h \in P_S$, $f \geq g, h$ and $g \,\mathbb{D}\, h$, then $g = h$. If S satisfies $\mathbb{D}$-majorization, then it is completely r-semisimple and, so, $\mathbb{D} = \mathbb{J}$ and each of its r-principal factors is either a monoid or a completely 0-r-simple restriction semigroup.*

Proof. The first assertion follows from the last statement of Lemma 5.2. The second is a consequence; the other consequences of complete r-semisimplicity were derived in the previous section. □

The basic computational tool in Brandt semigroups is the following, which will generally be used without further comment.

LEMMA 6.2 *If a and b are nonzero elements of a Brandt semigroup S, then $ab \neq 0$ if and only if $a^* = b^+$, in which case $a \,\mathbb{R}\, ab \,\mathbb{L}\, b$.*

Proof. Since $(ab)^+ \leq a^+$, if $ab \neq 0$, then by primitivity, $(ab)^+ = a^+$ whence, by Lemma 5.1, $a^* \leq b^+$ and, again by primitivity, $a^* = b^+$. The rest follows from the cited lemma. □

The next result was cited in the outline of the proof of Theorem 4.1.

PROPOSITION 6.3 *If S satisfies $\mathbb{D}$-majorization, then it is a subdirect product of its r-principal factors.*

Proof. The proof is similar to that for inverse semigroups ([26, Theorem II.4.5]). For each $a \in S$, let $K_a = \{x \in S : rI(a) \nsubseteq rI(x)\}$ and $L_a = \{x \in S : rI(a) \not\subset rI(x)\}$. Then it is routinely checked that K_a and L_a are r-ideals of S and that K_a is an r-ideal of L_a. The Rees quotient L_a/K_a is then isomorphic to the r-principal factor associated with a. We may identify the image of $\mathbb{J}_a$ in L_a/K_a with $\mathbb{J}_a$ itself.

Fix $a \in S$ and put $J = \mathbb{J}_a = \mathbb{D}_a$. Let $x \in S$. If $x \in K_a$, define $x\phi = 0$; otherwise, $rI(a) \subseteq rI(x)$ and by Lemma 5.4, there exists $e \in P_S \cap J$ such that $e \leq x^+$. By $\mathbb{D}$-majorization, e is unique with respect to this property. Define $x\phi = ex$.

Now let $x, y \in S$. Since K_a is an r-ideal, if either $x\phi = 0$ or $y\phi = 0$, then $(xy)\phi = 0$. Next suppose both $x\phi, y\phi \neq 0$, so that $x^+ \geq e$, $y^+ \geq f$, say, $e, f \in P_S \cap J$ and $x\phi = ex, y\phi = fy$. Then $(ex)(fy) = (ex)(ex)^*(fy)^+fy = (ex)((ex)^*(fy)^+)y$ and so the product remains in J (that is, the product $x\phi y\phi \neq 0$) if and only if the same is true for the product $(ex)^*(fy)^+$. By Lemma 6.2, the latter occurs if and only if $(ex)^* = (fy)^+$, in which case $x\phi y\phi = (ex)(ex)^*y = exy$.

Thus if $x\phi y\phi \neq 0$, then $xy \notin K_a$ and so $(xy)\phi \neq 0$: $(xy)\phi = gxy$, where $g \in P_S \cap J$ and $g \leq (xy)^+$. But then $g \leq x^+$ and by $\mathbb{D}$-majorization, $g = e$. Therefore $x\phi y\phi = (xy)\phi$.

It remains to be shown that if $(xy)\phi \neq 0$, that is, $xy \notin K_a$, then $x\phi y\phi \neq 0$. It was just observed that $(xy)^+ \geq e$, so that $e = (exy)^+ = (ex(ex)^*y^+y)^+$. It follows that $(ex)^*y^+ \in J$ and so $(ex)^* \leq y^+$. Then by $\mathbb{D}$-majorization $(ex)^* = f$, so $exy = exfy$ and $x\phi y\phi \neq 0$.

For any $x \in J$, $x\phi = x^+x = x$. The family of all such homomorphisms therefore separates the elements of S and yields the stated subdirect product. □

COROLLARY 6.4 *If S satisfies $\mathbb{D}$-majorization, then it belongs to the variety* $\mathbf{B}$ *generated by the Brandt semigroups (equivalently, as noted in Section 3, by the strict inverse semigroups). If, in addition, S is $\mathbb{H}$-combinatorial, it belongs to the variety* $\mathbf{B}_2$*, equivalently, the variety generated by the combinatorial strict inverse semigroups.*

Proof. By Proposition 6.3, S belongs to the variety generated by its r-principal factors, each of which is either a monoid or is primitive, with zero. As noted in Section 3, any monoid belongs to $\mathbf{G}$ and thus to $\mathbf{B}$. Then Proposition 5.8 yields both assertions, in the second case using the fact that $\mathbf{B}_2$ includes all combinatorial Brandt semigroups (Result 2.3). $\square$

We conclude this section by making concrete the failure of $\mathbb{D}$-majorization exhibited in Figure 5 (but see the remarks preceding Proposition 6.1).

A Λ_k*-configuration* in a restriction semigroup S, where k is a positive integer – exemplified for $k = 4$ by Figure 5 – consists of a standard $\mathbb{D}$-zigzag $a_1 \ \mathbb{L} \ a_2 \ \mathbb{R} \ a_3 \cdots a_k$, with associated sequence of projections $e_1, \ldots, e_{k+1}$, together with a projection f, such that $f > e_1$ and $f > e_{k+1}$, but no other proper comparability relations hold among these projections. A dual Λ_k-configuration corresponds to a dual standard zigzag. Note that for odd values of k, Λ_k-configurations are self-dual.

LEMMA 6.5 *Failure of $\mathbb{D}$-majorization in a completely r-semisimple restriction semigroup implies the existence of a Λ_k-configuration or its dual.*

Proof. Choose projections f, g, h such that $f > g$, $f > h$, $g \ \mathbb{D} \ h$, minimizing the length of any zigzag from g to h. By complete r-semisimplicity, the projections in such a zigzag form an antichain; by minimality, there are no further comparibility relations among the projections. $\square$

The family of semigroups Λ_k, introduced at the end of Section 4, concretely realize Λ_k-configurations. With the addition of a zero element, Figure 3 exhibits Λ_4 in alternative, eggbox, form. The elementary properties of these semigroups are summarized in the following result.

PROPOSITION 6.6 *For $k \geq 1$, Λ_k and its dual are completely r-semisimple restriction semigroups that do not satisfy $\mathbb{D}$-majorization. For odd values of k, Λ_k and its dual are isomorphic. The semigroup Λ_1 is isomorphic to B_0^1.*

In combination with the following, the only additional fact needed to complete the proof that they form a series of critical semigroups for B_0 (Theorem 4.4) is that $\Delta_k \in \mathbf{B}_0$. This will follow immediately from Theorem 10.3: see Proposition 10.4.

PROPOSITION 6.7 *For $k \geq 1$, the restriction semigroup Λ_k is generated, as such, by $k+1$ elements and has the property that every restriction subsemigroup generated, as such, by fewer than $k+1$ elements satisfies $\mathbb{D}$-majorization and, moreover, belongs to the variety generated by Δ_k.*

Proof. Clearly Λ_k is generated by $\{\alpha_1, \ldots, \alpha_k, \phi\}$. Let G be a subset of Λ_k of cardinality less than $k+1$ and S the restriction subsemigroup it generates. Suppose $\epsilon_1, \epsilon_{k+1} \in S$ and $e_1 \mathbb{D}\, e_{k+1}$ in S (and therefore in Λ_k). There is a unique $\mathbb{D}$-zigzag from ϵ_1 to ϵ_{k+1} in Λ_k (the same one as in Δ_k), so it is necessary that S contain $\alpha_1, \ldots, \alpha_k$. But from the definition of the products in Λ_k, it is clear that no α_i belongs to the restriction subsemigroup generated by $\Lambda_k \backslash \{\alpha_i\}$. That is, $\alpha_i \in G$. Thus if $\epsilon_1 \mathbb{D}\, \epsilon_{k+1}$ in S, then $G = \{\alpha_1, \ldots, \alpha_k\}$ and S is a restriction subsemigroup of Δ_k (and so satisfies $\mathbb{D}$-majorization).

But in the alternative case, $\mathbb{D}$-majorization of S is immediate from the definition. Further, by Proposition 6.3, in any case S is then a subdirect product of its principal r-factors, one of which is possibly the two-element semilattice, another of which is possibly the trivial semigroup, and the remainder of which must be restriction subsemigroups of Δ_k, yielding the final claim. $\square$

7 Identities satisfied in Brandt semigroups

We define a sequence of $(\cdot, ^+, ^*)$-words $W_n, n \geq 1$, in the variables $x_1, x_2, \ldots$ that, by the lemma below, encapsulate the notion of a $\mathbb{D}$-zigzag. The identities defined in Proposition 7.2 then collectively encapsulate the notion of $\mathbb{D}$-majorization. The sequence is defined inductively as follows:

$W_1(x_1) \equiv x_1^*$;

$W_{2n}(x_1, \ldots, x_{2n}) \equiv (x_{2n}\, W_{2n-1}(x_1, \ldots, x_{2n-1}))^+$ for $n \geq 1$;

$W_{2n+1}(x_1, \ldots, x_{2n+1}) \equiv (W_{2n}(x_1, \ldots, x_{2n})\, x_{2n+1})^*$, for $n \geq 1$.

The sequence begins x_1^*, $(x_2 x_1^*)^+$, $((x_2 x_1^*)^+ x_3)^* \ldots$.

For any word W in this signature, $\overline{W}$ denotes the dual word (obtained by reversing the order *and* interchanging $^+$ and *). Thus the sequence of words dual to the above is defined inductively by:

$\overline{W_1}(x_1) \equiv x_1^+$;

$\overline{W_{2n}}(x_1, \ldots, x_{2n}) \equiv (\overline{W_{2n-1}}(x_1, \ldots, x_{2n-1})\, x_{2n})^*$;

$\overline{W_{2n+1}}(x_1, \ldots, x_{2n+1}) \equiv (x_{2n+1}\, \overline{W_{2n}}(x_1, \ldots, x_{2n}))^+$, for $n \geq 1$.

This sequence begins x_1^+, $(x_1^+ x_2)^*$, $(x_3(x_1^+ x_2)^*)^+, \ldots$.

LEMMA 7.1 *Let S be a restriction semigroup and $a_1, a_2, \ldots$ nonzero elements of S.*

1. *If $a_1 \mathbb{L} a_2 \mathbb{R} \cdots \mathbb{L} a_{2n}$ holds, where $n \geq 1$, then $W_{2n}(a_1, \ldots, a_{2n}) = a_{2n}^+$. If S is a Brandt semigroup, then $a_1 \mathbb{L} a_2 \mathbb{R} \cdots \mathbb{L} a_{2n}$ holds if and only if $W_{2n}(a_1, \ldots, a_{2n}) \neq 0$;*

2. *if $a_1 \mathbb{L} a_2 \mathbb{R} \cdots \mathbb{L} a_{2n} \mathbb{R} a_{2n+1}$ holds, where $n \geq 0$, then $W_{2n+1}(a_1, \ldots, a_{2n+1}) = a_{2n+1}^*$. If S is a Brandt semigroup, then $a_1 \mathbb{L} a_2 \mathbb{R} \cdots \mathbb{L} a_{2n} \mathbb{R} a_{2n+1}$ holds if and only if $W_{2n+1}(a_1, \ldots, a_{2n+1}) \neq 0$.*

The dual statements also hold (interchanging $\mathbb{L}$ and $\mathbb{R}$ and replacing the words by their duals, as above).

Proof. Lemma 6.2 will be used freely. For convenience, abbreviate $W_i(a_1, \ldots, a_i)$ to W_i, for the moment. We prove the first statements of 1. and 2. simultaneously by induction, the base case ($n = 0$ in 2.) being obvious. Assuming that $W_{2n-1} = a_{2n-1}^*$ and $a_{2n-1} \mathbb{L} a_{2n}$, then $W_{2n} = (a_{2n} W_{2n-1})^+ = (a_{2n} a_{2n}^*)^+ = a_{2n}^+$. That $W_{2n+1} = a_{2n+1}^*$ follows similarly.

If S is a Brandt semigroup, then using the properties of products in such semigroups mentioned earlier, if $W_{2n} \neq 0$ then $W_{2n-1} \neq 0$ and $a_{2n}^* = W_{2n-1}^+ = W_{2n-1}$; invoking the induction hypothesis, $W_{2n-1} = a_{2n-1}^*$. So $a_{2n} \mathbb{L} a_{2n-1}$. The argument for W_{2n+1} is similar. □

In the following, the symbol e is a new variable representing an 'arbitrary' projection, that is, $e = x^+$ (or x^*) for some new variable x.

PROPOSITION 7.2 *Any Brandt semigroup satisfies the following $(\cdot, {}^+, {}^*)$-identities:*

(E_{2n}) $\quad W_{2n}(ex_1, x_2, \ldots, x_{2n-1}, ex_{2n}) = W_{2n}(ex_{2n}, x_{2n-1}, \ldots, x_2, ex_1)$, *for $n \geq 1$;*

($\overline{E_{2n}}$) $\quad \overline{W_{2n}}(x_1 e, x_2, \ldots, x_{2n-1}, x_{2n} e) = \overline{W_{2n}}(x_{2n} e, x_{2n-1}, \ldots, x_2, x_1 e)$, *for $n \geq 1$;*

(E_{2n+1}) $\quad W_{2n+1}(ex_1, x_2, \ldots, x_{2n}, x_{2n+1} e) = \overline{W_{2n+1}}(x_{2n+1} e, x_{2n}, \ldots, x_2, ex_1)$, *for $n \geq 0$, where E_1 is to be interpreted as $W_1(ex_1 e) = \overline{W_1}(ex_1 e)$.*

Proof. Let $x_1, x_2, \ldots$ belong to S. In the first equation, both sides evaluate to zero unless ex_1 and ex_{2n} are nonzero, in which case $(ex_1)^+ = e$ and $(ex_{2n})^+ = e$. Applying the first part of the previous lemma to each side of the equation, either both sides are zero, or both sides are nonzero, the left hand side evaluating to $(ex_{2n})^+ = e$ and the right hand side to $(ex_1)^+ = e$. Thus the equation is satisfied. The second equation is dual to the first.

For the third equation, in the case $n \geq 1$, apply the second part of the lemma to the left hand side, and apply its dual to the right hand side. Once again, either both sides are zero or both sides are nonzero; in the latter case, the left hand side evaluates to $(x_{2n+1} e)^* = e$ and the right hand side to $(ex_1)^+ = e$. In the case $n = 0$, once again, both sides are zero unless $ex_1 e$ is nonzero, in which case $(ex_1 e)^* = (ex_1 e)^+$, which is precisely the equation E_1 (evaluated in S).□

To illustrate, E_2 is the identity $(ex_1(ex_2)^*)^+ = (ex_2(ex_1)^*)^+$ and, as noted during the proof, the identity E_1 is $(ex_1 e)^+ = (ex_1 e)^*$, which from Result 2.2 is just the defining identity for strict inverse semigroups, interpreted in the signature of restriction semigroups. Referring to the definition of $\mathbf{loc}(\mathbf{V})$ early in Section 3 and to the defining identity for $\mathbf{SM}$ in Result 3.3, the following holds.

PROPOSITION 7.3 *The identity E_1 defines the variety* $\mathbf{loc}(\mathbf{SM})$ *of restriction semigroups that are locally semilattices of monoids.*

The identity $\overline{E_{2n+1}}$ dual to E_{2n+1} is

$$\overline{W_{2n+1}}(x_1e, x_2, \ldots, x_{2n}, ex_{2n+1}) = W_{2n+1}(ex_{2n+1}, x_{2n}, \ldots, x_2, x_1e),$$

which is in fact equivalent to the latter: simply reverse the order of the variables in each identity and exchange the left and right hand sides.

The next result complements Proposition 7.2 in justifying the claim that the set of identities stated there intuitively encapsulates $\mathbb{D}$-majorization, when applied in the corollary that follows and, ultimately, in Theorem 8.1.

LEMMA 7.4 *Let S be a restriction semigroup, $a_1, a_2, \ldots \in S$ and $e \in P_S$.*

(1) Suppose $a_1 \mathbb{L} a_2 \mathbb{R} \cdots \mathbb{L} a_{2n}$ and $e > a_1^+, a_{2n}^+$, where $n \geq 1$. If S satisfies the identity E_{2n}, then $a_1^+ = a_{2n}^+$.

(2) Suppose $a_1 \mathbb{R} a_2 \mathbb{L} \cdots \mathbb{R} a_{2n}$ and $e > a_1^*, a_{2n}^*$, where $n \geq 1$. If S satisfies the identity $\overline{E_{2n}}$, then $a_1^* = a_{2n}^*$.

(3) Suppose $a_1 \mathbb{L} a_2 \mathbb{R} \cdots \mathbb{R} a_{2n+1}$ and $e > a_1^+, a_{2n+1}^*$, where $n \geq 0$. If S satisfies the identity E_{2n+1}, then $a_1^+ = a_{2n+1}^*$.

Proof. (1) Here $a_1 = ea_1$ and $a_{2n} = ea_{2n}$, so

$$W_{2n}(ea_1, a_2, \ldots, a_{2n-1}, ea_{2n}) = W_{2n}(a_1, \ldots, a_{2n}) = a_{2n}^+,$$

applying Lemma 7.1. Reading, instead, from right to left,

$$W_{2n}(ea_{2n}, a_{2n-1}, \ldots, a_2, ea_1) = W_{2n}(a_{2n}, a_{2n-1}, \ldots, a_2, a_1) = a_1^+.$$

The conclusion follows.

(2) The argument is dual to that in (1).

(3) First assume $n \geq 1$. Here $a_1 = ea_1$ and $a_{2n+1} = a_{2n+1}e$, so

$$W_{2n+1}(ea_1, a_2, \ldots, a_{2n}, a_{2n+1}e) = W_{2n+1}(a_1, \ldots, a_{2n+1}) = a_{2n+1}^*.$$

Reading from right to left and dualizing,

$$\overline{W_{2n+1}}(a_{2n+1}e, a_{2n}, \ldots, a_2, ea_1) = \overline{W_{2n+1}}(a_{2n+1}, \ldots, a_1) = a_1^+,$$

applying the dual statements in Lemma 7.1. Again, the conclusion follows. In the case $n = 0$, where $e > a_1^+, a_1^*$, then $a_1 = ea_1e$ and $a_1^+ = W_1(ea_1e) = \overline{W_1}(ea_1e) = a_1^*$. $\square$

COROLLARY 7.5 *Let S be a restriction semigroup. If S satisfies the identities defined in Proposition 7.2, then it satisfies $\mathbb{D}$-majorization.*

Proof. If S does not satisfy $\mathbb{D}$-majorization, then there exist a projection e and distinct $\mathbb{D}$-related projections f and g such that $e > f, e > g$. There exist elements $a_1, a_2, \ldots$ of S such that either (1) $f \mathbin{\mathbb{R}} a_1 \mathbin{\mathbb{L}} a_2 \mathbin{\mathbb{R}} \cdots \mathbin{\mathbb{L}} a_{2n} \mathbin{\mathbb{R}} g$, (2) $f \mathbin{\mathbb{L}} a_1 \mathbin{\mathbb{R}} a_2 \mathbin{\mathbb{L}} \cdots \mathbin{\mathbb{R}} a_{2n} \mathbin{\mathbb{L}} g$, or (3) $f \mathbin{\mathbb{R}} a_1 \mathbin{\mathbb{L}} a_2 \mathbin{\mathbb{R}} \cdots \mathbin{\mathbb{R}} a_{2n+1} \mathbin{\mathbb{L}} g$ (or the dual situation to (3)). These correspond precisely to the cases analyzed in the lemma. $\square$

The dependencies among these identities are the topic of the rest of this section.

LEMMA 7.6 *The following relationships hold in any restriction semigroup, for $k \geq 2$:*

1. $\overline{W_k}(x_1^+, x_1, x_2, \ldots, x_{k-1}) = W_{k-1}(x_1, x_2, \ldots, x_{k-1})$;
2. $W_k(x_1^*, x_1, x_2, \ldots, x_{k-1}) = \overline{W_{k-1}}(x_1, x_2, \ldots, x_{k-1})$.

Proof. The second equation is the dual of the first. We prove the first by induction on k. In the base case, $k = 2$, recall that $\overline{W_2}(x_1, x_2) \equiv (x_1^+ x_2)^*$. Thus $\overline{W_2}(x_1^+, x_1) = (x_1^+ x_1)^* = x_1^* = W_1(x_1)$.

Now if $k > 2$ and $k = 2n$,

$$\begin{aligned}
\overline{W_{2n}}(x_1^+, x_1, x_2, \ldots, x_{2n-1}) &= (\overline{W_{2n-1}}(x_1^+, x_1, \ldots, x_{2n-2}) x_{2n-1})^* \\
&= (W_{2n-2}(x_1, \ldots, x_{2n-2}) x_{2n-1})^* \\
&= W_{2n-1}(x_1, \ldots, x_{2n-1}).
\end{aligned}$$

A similar argument applies when $k = 2n + 1$. $\square$

PROPOSITION 7.7 *For $n \geq 1$,*

1. *The identities E_{2n} and $\overline{E_{2n}}$ are consequences of the identity E_{2n+1};*
2. *The identity E_{2n-1} is a consequence of each of E_{2n} and $\overline{E_{2n}}$.*

Proof. Assume E_{2n+1} holds. For x_{2n} substitute ex_{2n} and for x_{2n+1} substitute $(ex_{2n})^+$. Since $(ex_{2n})^+ \leq e$, we obtain

$$W_{2n+1}(ex_1, x_2, \ldots, ex_{2n}, (ex_{2n})^+) = \overline{W_{2n+1}}((ex_{2n})^+, ex_{2n}, \ldots, x_2, ex_1).$$

From the first equation in Lemma 7.6, the right hand side equals $W_{2n}(ex_{2n}, \ldots, ex_1)$, which is the right hand side of E_{2n}. Applying the definition to the left hand side,

$$W_{2n+1}(ex_1, x_2, \ldots, ex_{2n}, (ex_{2n})^+) = (W_{2n}(ex_1, \ldots, ex_{2n})(ex_{2n})^+)^* = W_{2n}(ex_1, \ldots, ex_{2n}),$$

which is the left hand side of E_{2n}. (Here we have used the fact that, from the definition, $W_{2n}(x_1, \ldots, x_{2n}) \leq x_{2n}^+$.)

Since E_{2n+1} is equivalent to its dual, it also implies $\overline{E_{2n}}$.

Again by duality, it remains only to show that E_{2n-1} is a consequence of E_{2n}. The argument is similar to the above. Assume E_{2n} holds. For x_{2n-1} substitute $x_{2n-1}e$ and for x_{2n} substitute $(x_{2n-1}e)^*$. We obtain

$$W_{2n}(ex_1, x_2, \ldots, x_{2n-1}e, (x_{2n-1}e)^*) = W_{2n}((x_{2n-1}e)^*, x_{2n-1}e, \ldots, x_2, ex_1).$$

From the second equation in Lemma 7.6, the right hand side equals $\overline{W_{2n-1}}(x_{2n-1}e, x_{2n-2}, \ldots, x_2, ex_1)$, which is the right hand side of E_{2n-1}. Applying the definition to the left hand side,

$$\begin{aligned} W_{2n}(ex_1, x_2, \ldots, x_{2n-1}e, (x_{2n-1}e)^*) &= ((x_{2n-1}e)^* W_{2n-1}(ex_1, x_2, \ldots, x_{2n-2}, x_{2n-1}e))^+ \\ &= W_{2n-1}(ex_1, x_2, \ldots, x_{2n-2}, x_{2n-1}e), \end{aligned}$$

which is the left hand side of E_{2n-1}.

Observe that when $n = 1$, the specified reduction does yield the interpretation of E_1 stated above. □

COROLLARY 7.8 *The three families of identities exhibited in Proposition 7.2 are pairwise equivalent. Moreover, within each family, each identity implies all of the earlier ones (that is, for lower values of n).*

We use the semigroups Λ_k, introduced in Section 4 and studied briefly in Section 6, to complete the analysis of the interdependence of the identities exhibited in Proposition 7.2. Recall that for odd k, E_k is self-dual.

PROPOSITION 7.9 *For $k \geq 1$, the restriction semigroup Λ_k does not satisfy E_k. However for $k \geq 2$, it satisfies the identities E_ℓ for all $\ell < k$. For even k, Λ_k satisfies $\overline{E_k}$.*

Hence no further dependencies hold among the identities exhibited in Proposition 7.2.

Proof. It is clear from Lemma 7.4 that Λ_k does not satisfy E_k. Now when the identity E_ℓ is evaluated in Λ_k, it is actually evaluated within a restriction subsemigroup S_ℓ, say, generated by $\ell + 1 < k + 1$ elements. According to Proposition 6.7, S_ℓ satisfies $\mathbb{D}$-majorization and so satisfies E_ℓ, by Proposition 7.2.

In the case that k is even, the same argument cannot be used direct, for $\overline{E_k}$, the identity

$$\overline{W_k}(x_1e, x_2, \ldots, x_{k-1}, x_ke) = \overline{W_k}(x_ke, x_{2n-1}, \ldots, x_2, x_1e),$$

involves the same $k+1$ variables that appear in E_k. However it is straightforward to verify that for any nonzero projection ϵ and any element α of Λ_k, either $\alpha\epsilon = 0$ or $\alpha\epsilon = \alpha$. Thus when the identity is evaluated in Λ_k, either both sides evaluate to zero, or x_1e and x_1 evaluate to the same value, and likewise for x_ke. So if both sides do not evaluate to zero, then the identity is evaluated within a restriction subsemigroup generated by at most k elements and the argument above applies once more. □

8 Strict restriction semigroups and a basis for their identities

We combine the results of the two previous sections. By analogy with the use of the term for inverse semigroups, we call a restriction semigroup *strict* if it is a subdirect product of monoids and completely 0-r-simple semigroups (the latter term having being defined in Section 5). Another quite different characterization of the variety $\mathbf{B}$ will be given in Theorem 12.1.

THEOREM 8.1 *The following are equivalent for a restriction semigroup S:*

(1) S belongs to $\mathbf{B}$, the variety of restriction semigroups generated by the Brandt semigroups (equivalently, by the strict inverse semigroups);

(2) S satisfies any one of the sequences of identities listed in Proposition 7.2;

(3) S satisfies $\mathbb{D}$-majorization;

(4) S is strict.

Proof. Proposition 7.2 asserts that (1) implies that S satisfies all the identities exhibited in Proposition 7.2; Corollary 7.8 then asserts the equivalence of that statement with (2); that (2) implies (3) then follows from Corollary 7.5; that (3) implies (4) is the combination of Proposition 6.3 and Proposition 6.1; that (4) implies (1) is a consequence of Proposition 5.8. $\square$

Clearly, Proposition 7.9 provides a direct proof that $\mathbf{B}$ is not finitely based. In the next two sections, we shall also provide explicit defining identities for the varieties $\mathbf{B}_2$ and $\mathbf{B}_0$.

In the remainder of this section, we consider some further properties of the variety $\mathbf{B}$. According to Result 2.2, the strict inverse semigroups coincide with the inverse semigroups that are locally semilattices of groups. The analogue for restriction semigroups does not hold. See Section 3 for the definition of $\mathbf{loc}(\mathbf{V})$ and recall from Proposition 7.3 that $\mathbf{loc}(\mathbf{SM})$ is defined by the identity E_1.

PROPOSITION 8.2 *The variety $\mathbf{B}$ is properly contained in the variety $\mathbf{loc}(\mathbf{SM})$.*

Proof. By the theorem, $\mathbf{B}$ satisfies E_1. By Proposition 7.9, Λ_k satisfies E_1 but not E_k, for $k \geq 2$. $\square$

In Section 3 we defined, for any monoid variety $\mathbf{N}$, the class $\mathbf{mon}(\mathbf{N})$ of restriction semigroups, all of whose submonoids belong to $\mathbf{N}$, and pointed out that in general this class is not a subvariety.

PROPOSITION 8.3 *Let $\mathbf{N}$ be a variety of monoids. Then $\mathbf{B} \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B} \cap \mathbf{loc}(\mathbf{SN})$ and is therefore a subvariety of $\mathbf{B}$. It contains the subvariety $\mathbf{B}_2 \vee \mathbf{N}$.*

In particular, the class of strict restriction semigroups, all of whose submonoids are trivial, is the subvariety $\mathbf{B} \cap \mathbf{loc}(\mathbf{SL})$, defined within $\mathbf{B}$ by the additional identity $exe = (exe)^2$.

Proof. If $S \in \mathbf{B} \cap \mathbf{loc}(\mathbf{N})$, then by Proposition 8.2, $S \in \mathbf{loc}(\mathbf{SM})$, so for each $e \in P_S$, eSe is a union of monoids from $\mathbf{N}$. But by Result 3.3, eSe then belongs to $\mathbf{SN}$. Conversely, for each $e \in P_S$, the submonoid $\mathbb{H}_e$ is a submonoid of eSe and therefore belongs to $\mathbf{N}$. Since the submonoids of B_2 are trivial, $\mathbf{B}_2 \subseteq \mathbf{B} \cap \mathbf{mon}(\mathbf{T})$ and so $\mathbf{B}_2 \vee \mathbf{N} \subseteq \mathbf{B} \cap \mathbf{mon}(\mathbf{N})$. $\square$

Proposition 3.1 demonstrated that triviality of submonoids does not imply triviality of $\mathbb{H}$. See Propositions 9.6 and 10.9 for more specifics.

9 The variety generated by B_2

In this section, we obtain a basis of identities for B_2, a structural description of the members of $\mathbf{B}_2$, and further properties of this variety.

In essence, we specialize the previous section to the $\mathbb{H}$-combinatorial strict restriction semigroups. The semigroups Δ_k, which are at the heart of this paper, have this property. The following lemma allows the property of being $\mathbb{H}$-combinatorial to be defined equationally within the class of strict restriction semigroups. This is not true in general (since free restriction semigroups have this property [9]).

LEMMA 9.1 *The following are equivalent for nonzero elements a, b of a completely 0-r-simple restriction semigroup S:*

(i) $a \,\mathbb{H}\, b$;

(ii) $a(a^+b)^* = a$;

(iii) $a(a^+b)^* \neq 0$.

Proof. If $a \,\mathbb{H}\, b$, then $a(a^+b)^* = ab^* = a$. Clearly (ii) implies (iii). Assuming (iii), then $a^+b \neq 0$, so $a^+ = b^+$ and $a^+b = b$, in which case, necessarily $a^* = b^*$. $\square$

PROPOSITION 9.2 *A completely 0-r-simple restriction semigroup is $\mathbb{H}$-combinatorial if and only if it satisfies the identity $x(x^+y)^* = y(y^+x)^*$.*

Proof. Let S be such a semigroup. Suppose S is $\mathbb{H}$-combinatorial and $a, b \in S$. We must show that $a(a^+b)^* = b(b^+a)^*$. This is clearly true if either a or b is zero. Otherwise, using the fact that $\mathbb{H}$ is symmetric, either both terms are zero or both are nonzero and $a \,\mathbb{H}\, b$, in which case $a = b$ and the equation holds by virtue of (ii), again using symmetry.

Conversely, suppose the identity is satisfied and that a, b are nonzero, $\mathbb{H}$-related elements of S. Then, once again using symmetry, (ii) of the lemma implies $a = b$. $\square$

THEOREM 9.3 *The following are equivalent for a restriction semigroup S:*

(1) S belongs to $\mathbf{B}_2$, the variety generated by the combinatorial Brandt semigroup B_2 (equivalently, by the combinatorial strict inverse semigroups);

(2) S satisfies the identity $x(x^+y)^* = y(y^+x)^*$, together with any one of the sequences of identities listed in Proposition 7.2;

(3) S is a subdirect product of $\mathbb{H}$-combinatorial completely 0-r-simple restriction semigroups;

(4) S is a subdirect product of restriction subsemigroups of combinatorial Brandt semigroups;

(5) S is $\mathbb{H}$-combinatorial and strict.

Proof. Apart from (4), the equivalences are obtained immediately from Theorem 8.1 by application of Proposition 9.2. That (4) follows from (3) was shown in Proposition 5.8. □

COROLLARY 9.4 $\mathbf{B} = \mathbf{B}_2 \vee \mathbf{M}$.

Proof. On any Brandt semigroup B, Green's relation $\mathcal{H}$ is a congruence. Therefore B/μ is combinatorial and so, by the theorem, belongs to $\mathbf{B}_2$. Now follows the first statement of Result 3.2 applies. □

COROLLARY 9.5 *On any strict restriction semigroup, $\mathbb{H}$ is a congruence.*

Proof. Let S be such a semigroup. Applying Result 3.2 and the last corollary, $S/\mu \in \mathbf{B}_2$ and so S/μ is $\mathbb{H}$-combinatorial. Since congruences respect the relation $\mathbb{H}$, it follows that $\mathbb{H} = \mu$ and so $\mathbb{H}$ is a congruence. □

COROLLARY 9.6 *The variety $\mathbf{B}_2$ is strictly contained in $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$.*

Proof. The semigroup TR_2, introduced in the proof of Proposition 3.1, is embeddable in a Brandt semigroup and has trivial submonoids, so it belongs to $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$. But it is not $\mathbb{H}$-combinatorial and so does not belong to $\mathbf{B}_2$. □

10 The variety generated by B_0

In this section, we obtain a basis of identities for B_0, a structural description of the members of $\mathbf{B}_0$, and further properties of this variety. We also consider the variety $\mathbf{B}_0 \vee \mathbf{M}$.

Recall from Result 2.7 that B_0 satisfies the identity $xyx = x^2y^2$. We will also need the fact that $\mathbf{B}_2$ satisfies $x^2 = x^+x^*$, as can be derived from the results of the previous section or by simply verifying that it holds in B_2.

LEMMA 10.1 *The following are equivalent for an $\mathbb{H}$-combinatorial, completely 0-r-simple semigroup S:*

(i) S satisfies $xyx = x^2y^2$;

(ii) S contains no regular elements other than projections;

(iii) S does not contain a restriction subsemigroup isomorphic to B_2.

Proof. According to the previous section, $S \in \mathbf{B}_2$. Suppose S satisfies the identity in (i) and that a is a regular, nonzero element of S, with inverse b, say. Then $a = aba = a^2b^2$, which, as a consequence of the identity $x^2 = x^+x^*$, is a projection. So (ii) holds. Clearly (ii) implies (iii). To prove (iii) implies (i), suppose that there exist $a, b \in S$ such that $aba \neq a^2b^2$. Note that since a^2, b^2 are projections, $a^2b^2 = 0$, for otherwise $a^2 = b^2 \neq 0$, so that $a, b \in P_S$, $a = b$ and $aba = a = a^2b^2$. So $aba \neq 0$. From $ab \neq 0$, it follows that $a^* = b^+$ and $(ab)^+ = a^+$. From $ba \neq 0$ if follows that $b^* = a^+$. Put $e = a^+$, $f = a^*$. Now $(ab)^* \leq b^*$ and by primitivity, equality follows. Thus $ab \mathbb{H}\ e$ and, by assumption, $ab = e$; similarly, $ba = f$. Then $\{e, f, a, b, 0\}$ is a restriction subsemigroup isomorphic to B_2. □

COROLLARY 10.2 *The lattice $\mathcal{L}(\mathbf{B}_2)$ comprises the chain $\mathbf{B}_2 \succ \mathbf{B}_0 \succ \mathbf{SL} \succ \mathbf{T}$.*

Proof. Let $\mathbf{V} \in \mathcal{L}(\mathbf{B}_2)$. Note first that since every member of $\mathbf{B}_2$ is $\mathbb{H}$-combinatorial, $\mathbf{V} \cap \mathbf{M} = \mathbf{T}$. If $\mathbf{V} \subseteq \mathbf{SM}$, then by Result 3.4, either $\mathbf{V} = \mathbf{T}$ or $\mathbf{V} = \mathbf{SL}$. Otherwise, by Result 3.6, (and complete r-semisimplicity of the members of $\mathbf{B}_2$), $\mathbf{B}_0 \subseteq \mathbf{V}$. To complete the proof, it remains to show only that $\mathbf{B}_2 \succ \mathbf{B}_0$. By Lemma 10.1, B_2 does not satisfy the identity $xyx = x^2y^2$, which is satisfied in B_0, so the inclusion is strict. Now suppose that $S \in \mathbf{B}_2$ but $S \notin \mathbf{B}_0$. Then, applying Theorem 8.1, some subdirect factor of S does not satisfy the identity $xyx = x^2y^2$ and so contains a copy of B_2. Thus the variety generated by S contains B_2 and is therefore all of $\mathbf{B}_2$. □

THEOREM 10.3 *The following are equivalent for a restriction semigroup S:*

(1) S belongs to $\mathbf{B}_0$;

(2) S satisfies the identities $xyx = x^2y^2$ and $x(x^+y)^* = y(y^+x)^*$, together with any one of the sequences of identities listed in Proposition 7.2;

(3) S is $\mathbb{H}$-combinatorial and strict, and contains no regular elements other than projections;

(4) S satisfies $\mathbb{D}$-majorization, is $\mathbb{H}$-combinatorial, and contains no regular elements other than projections.

Proof. Clearly (1) implies (2). That (2) implies (1) is now immediate from $\mathbf{B}_2 \succ \mathbf{B}_0$ and the fact that B_2 does not satisfy $xyx = x^2y^2$. Now the equivalence with (3) follows from Theorem 9.3 and Lemma 10.1. The equivalence with (4) follows from Theorem 8.1. □

For example, the semigroups Δ_k belong to $\mathbf{B}_0$. In conjunction with Proposition 6.7, this fact completes the proof that the Λ_k's form a series of critical semigroups for B_0 (Theorem 4.4):

PROPOSITION 10.4 *Each semigroup Λ_k has the property that every restriction subsemigroup generated by fewer than $k+1$ elements belongs to $\mathbf{B}_0$.*

Recall from Section 1 that a *limit variety* is a variety that is nonfinitely based, but all of whose proper subvarieties are finitely based.

COROLLARY 10.5 *The variety* $\mathbf{B}_0$ *is a limit variety of restriction semigroups.*

Proof. See Corollary 10.2. □

Other than varieties of monoids, the only other candidates for limit varieties of restriction semigroups are $\mathbf{B}^+$ and $\mathbf{B}^-$, for according to Result 3.6, any variety that does not consist of semilattices of monoids contains either $\mathbf{B}_0$, $\mathbf{B}^+$ or $\mathbf{B}^-$. (The situation for semilattices of monoids is easily disposed of using Proposition 3.5.) Moreover, every proper subvariety either of $\mathbf{B}^+$ or of $\mathbf{B}^-$ consists of semilattices of *commutative* monoids [16, Proposition 2.13] and, again using Proposition 3.5, is therefore finitely based, since all varieties of commutative monoids have this property [25]. However it is unknown whether or not the semibicyclic semigroup B^+ (or, dually, B^-) is finitely based.

The last task of this section is to consider the subvariety $\mathbf{B}_0 \vee \mathbf{M}$ of $\mathbf{B}$. While the following theorem is phrased relative to $\mathbf{B}$, a formulation similar to Theorem 10.3 may easily be deduced.

THEOREM 10.6 *The following are equivalent for a strict restriction semigroup* S*:*

(1) $S \in \mathbf{B}_0 \vee \mathbf{M}$*;*

(2) S *satisfies the identities* $(xyx)^+ = (x^2y^2)^+$ *and* $(xyx)^* = (x^2y^2)^*$*;*

(3) S/μ *contains no regular elements other than projections;*

(4) if $e, f \in P_S$ *then* $\mathbb{R}_e \cap \mathbb{L}_f$ *and* $\mathbb{L}_e \cap \mathbb{R}_f$ *cannot both be nonempty.*

Proof. By Corollary 9.5, $\mu = \mathbb{H}$ on S, so the identities in (2) are equivalent to the property that S/μ satisfies $xyx = x^2y^2$ and thus, by Theorem 10.3, equivalent to $S/\mu \in \mathbf{B}_0$. Applying Result 3.2, (1) and (2) are equivalent.

The equivalence of (1) and (3) follows similarly, based on (4) of Theorem 10.3.

Now suppose that e and f are distinct projections of S, and that there exist $x \in \mathbb{R}_e \cap \mathbb{L}_f$ and $y \in \mathbb{L}_e \cap \mathbb{R}_f$. Then $xy \; \mathbb{H} \; e$ and $yx \; \mathbb{H} \; f$ so that, in S/μ, $x\mu$ is regular, with inverse $y\mu$. But since μ is P-separating, x cannot be a projection. So $S/\mu \notin \mathbf{B}_0$. The converse proceeds in reverse, by taking inverse images of appropriate elements of S/μ. Thus (1) and (4) are equivalent. □

When interpreted in terms of partial eggboxes, (4) means that no 'squares' exist in such diagrams. Note that in an $\mathbb{H}$-combinatorial semigroup S, (4) reduces to stating that each regular element of S is a projection. The next result refutes a plausible conjecture about $\mathbf{B}_0 \vee \mathbf{M}$.

PROPOSITION 10.7 *It is not the case that a strict restriction semigroup belongs to* $\mathbf{B}_0 \vee \mathbf{M}$ *if and only if each of its regular elements belongs to a submonoid.*

Proof. In fact, it is not enough that each of its regular elements be a projection (and therefore the $\mathbb{H}$-combinatorial property is necessary in Theorem 10.3(4)).

Let T be the Brandt semigroup $B(Z, I)$, where Z is the group of integers and $I = \{1, 2\}$. Take the restriction subsemigroup $(I \times N_0 \times I) \cup \{0\}$, where N_0 is the monoid of nonnegative integers under addition. Finally, let S be the result of further deleting the elements $(1, 0, 2)$ and $(2, 0, 1)$. It is straightforward to check that no product of the remaining elements can yield either of the two deleted ones. So S is a full subsemigroup of T and therefore a strict restriction semigroup, whose projections are $(1, 0, 1)$, $(2, 0, 2)$ and 0. It is again straightforward to check that these are the only regular elements of S. Yet S does not satisfy (4) of Theorem 10.6 (and $S/\mu \cong B_2$). □

COROLLARY 10.8 $\mathbf{B} = \mathbf{B}_2 \vee \mathbf{M} \succ \mathbf{B}_0 \vee \mathbf{M} \succ \mathbf{SL} \vee \mathbf{M} = \mathbf{SM} \succ \mathbf{M}$.

Proof. Suppose $\mathbf{B}_0 \vee \mathbf{M} \subseteq \mathbf{V} \subset \mathbf{B}_2 \vee \mathbf{M}$. Applying Result 3.3, for every $S \in \mathbf{V}$, $S/\mu \in \mathbf{V} \cap \mathbf{B}_2 = \mathbf{B}_0$, using $\mathbf{B}_2 \succ \mathbf{B}_0$. Applying the result again, $S \in \mathbf{B}_0 \vee \mathbf{M}$. The same reasoning yields the second covering. The last is from Result 3.4. □

Similarly to Proposition 9.6, there exist members of $\mathbf{B}_0 \vee \mathbf{M}$ all of whose submonoids are trivial but that are not $\mathbb{H}$-combinatorial (that is, do not belong to $\mathbf{B}_0$.) In fact, the semigroup TR_2 itself again provides an example, since (4) of the theorem is satisfied.

PROPOSITION 10.9 *The variety* $\mathbf{B}_0$ *is strictly contained in* $(\mathbf{B}_0 \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$.

11 Complete r-semisimplicity and the semigroups Ψ_k

The goal of this section is to characterize varieties consisting of completely r-semisimple semigroups by the absence of B^+, B^- and the semigroups Ψ_k that were introduced in Section 4. Together, these semigroups concretely realize the failure of the definition of complete r-semisimplicity through 'Ψ_k-configurations'.

Throughout this section, Figure 1 will be a handy visual reference. An alternative, intuitive, way to view Ψ_k is to start with an 'infinite eggbox', with projections $e_1, e_2, \ldots, e_k, \ldots$ (pictorially, extend Figure 3 indefinitely) and impose the relation $e_1 > e_{k+1}$ and its consequences, putting all undefined products equal to zero. For convenience, we briefly restate the formal definition. Let k be a positive, even integer. Partially order the set $\{e_n : n \geq 1\}$ by $e_n < e_m$ if and only if $m \equiv n \mod k$ and $m < n$. Let Y be the semilattice obtained by adjoining an element 0 that is the meet of each pair of elements from distinct chains. For each e_n, denote the identity map 1_{Ye_n} by ϵ_n.

Let $\Psi_k = \{\alpha_n : n \geq 1\} \cup E_{T_Y}$, where $\alpha_n \in T_Y$ is defined for $n = 1, 2, \ldots$ by the equations (1):

$$\alpha_n : \begin{cases} Ye_n & \longrightarrow \quad Ye_{n+1} \quad \text{if } n \text{ is odd}, \\ Ye_{n+1} & \longrightarrow \quad Ye_n \quad \text{if } n \text{ is even}. \end{cases}$$

Proposition 4.2 summarized the basic properties that were easily provable at that juncture. Recall that $\{\alpha_n : n \geq 1\} \cup \{0\}$ is a null semigroup and refer to the equations (2) given there. The proof of the following central result was outlined in Section 4 and can now be completed.

PROPOSITION 11.1 *Each semigroup Ψ_k has the property that every restriction subsemigroup generated by fewer than k elements belongs to $\mathbf{B}_0$.*

Proof. Let T be a restriction subsemigroup of Ψ_k that is generated by a set G of fewer than k elements. As stated in the outline, following Proposition 4.2, we show that T satisfies $\mathbb{D}$-majorization. Then Theorem 10.3(4) shows that $T \in \mathbf{B}_0$, since the cited proposition asserts that the additional hypotheses stated in that part of the theorem are satisfied in Ψ_k and therefore in T.

The set N of non-projections of Ψ_k is the union of the sets $N_r = \{\alpha_n : n \equiv r \mod k\}$, $r = 1, \ldots, k$; the poset of nonzero projections is the union of the ω-chains $\{\epsilon_n : n \equiv r \mod k\}$, $1 \leq r \leq k$. Using the equations (2), $N_r \cup \{0\}$ is closed under multiplication by projections, on both the left and the right, for each r. Suppose G is the union of a set of projections with a set $\{\alpha_{n_1}, \ldots, \alpha_{n_\ell}\}$ of $\ell < k$ non-projections, where $n_i \equiv r_i \mod k$, for $i = 1, \ldots, \ell$. Then $T \subseteq \bigcup\{N_{r_i} : i = 1, \ldots, \ell\} \cup P_{\Psi_k}$.

Suppose $\epsilon_m \mathbb{D} \epsilon_n$ in T, where $m < n$. There is a unique $\mathbb{D}$-zigzag from ϵ_m to ϵ_n, which if m and n are odd (the other cases being similar) is $\alpha_m \mathbb{L} \alpha_{m+1} \mathbb{R} \cdots \mathbb{L} \alpha_{n-1}$. By the previous paragraph, the length of this zigzag must be less than k, so $n - m < k$. However if ϵ_m and ϵ_n have a common upper bound, necessarily $m \equiv n \mod k$, which is a contradiction. $\square$

The rest of this section is devoted to the proof of the following theorem, cited in Section 4 as Theorem 4.3, which determines the limits of applicability of the series of critical semigroups Ψ_k for B_0. Most of the effort is devoted to a characterization (Theorem 11.8) of varieties of completely r-semisimple semigroups by the exclusion not only of the semigroups Ψ_k but also of B^+ and B^-. Once the theorem is proved, we will also show (Corollary 11.10) how to characterize in absolute terms the varieties that exclude every Ψ_k.

THEOREM 11.2 *A variety of restriction semigroups that contains B_0 consists of completely r-semisimple semigroups if and only if it contains no semigroup Ψ_k.*

The following is now an immediate result of applying this theorem to Theorem 4.1.

COROLLARY 11.3 *No variety of completely r-semisimple restriction semigroups that contains B_0 is finitely based.*

Recall Result 2.6 and the discussion that precedes it regarding completely semisimple inverse semigroups: they are characterized by exclusion of bicyclic subsemigroups. Exclusion of the semibicyclic semigroups does not characterize complete r-semisimplicity of restriction semigroups, as witnessed by the semigroups Ψ_k themselves. A further significant contrast is that although any periodic inverse semigroup is necessarily completely semisimple, the analogue for restriction semigroups is far from true, again witnessed by the Ψ_k's.

A Ψ_k*-configuration* in a restriction semigroup, where k is a positive *even* integer, consists of a standard $\mathbb{D}$-zigzag $a_1 \mathbb{L} a_2 \mathbb{R} a_3 \cdots \mathbb{L} a_k$ of distinct elements a_i, such that $e_1 > e_{k+1}$ is the only proper comparability relation in the associated sequence of projections. See the top part of Figure 1 for a visualization in mapping form. A dual Ψ_k-configuration consists of a $\mathbb{D}$-zigzag $a_1 \mathbb{R} a_2 \cdots \mathbb{R} a_k$ whose associated sequence of projections satisfies the same property.

A Ψ_k-configuration is *pure* if, in the notation above, the $\mathbb{D}$-zigzag is of minimum length among all $\mathbb{D}$-zigzags between distinct, comparable projections of S. Having the semigroup Ψ_k clearly in mind as the model, the standard zigzag $\alpha_1 \mathbb{L} \cdots \mathbb{L} \alpha_k$, together with the associated sequence of projections $\epsilon_1, \ldots, \epsilon_{k+1} < \epsilon_1$, forms such a configuration in that semigroup.

The following result is the analogue of Lemma 6.5.

LEMMA 11.4 *Failure of complete r-semisimplicity in a restriction semigroup S implies that S contains either (a) B^+, (b) B^- or (c) a pure Ψ_k-configuration.*

Proof. Let $a_1, a_2, \ldots, a_k$ be a (not necessarily standard) $\mathbb{D}$-zigzag of (as yet, not necessarily even) length k, minimum such that the associated sequence of projections $e_1, e_2, \ldots, e_{k+1}$ satisfies $e_1 > e_{k+1}$.

If $k = 1$, then in the standard situation $e_1 = a_1^+$ and $e_{k+1} = a_1^*$. By Result 2.8, a_1 generates B^+, as a restriction semigroup. In the dual situation, $e_1 = a_1^*$ and $e_{k+1} = a_1^+$ and a_1 generates the dual semigroup B^-.

If $k > 1$, assume first that the zigzag is standard. Minimality ensures that $e_1 > e_{k+1}$ is the *only* proper comparability among the projections $e_1, \ldots, e_{k+1}$. We will show that k must be even. Suppose otherwise: then the zigzag concludes with $\cdots a_{k-1} \mathbb{R} a_k$, that is, $e_{k+1} = a_k^*$. Put $a_{k+1} = e_{k+1}a_1$ and $e_{k+2} = a_{k+1}^* < e_2$. (Strict inequality is actually proven in (ii) of Lemma 11.5 below.) Since $e_{k+1} < e_1$, $a_{k+1}^+ = e_{k+1}$. In this parity, $a_k^* = a_{k+1}^+$ so $(a_k a_{k+1})^+ = a_k^+ = e_k$ and $(a_k a_{k+1})^* = a_{k+1}^* = e_{k+2}$. But this results in a $\mathbb{D}$-zigzag $a_2 \mathbb{R} \cdots \mathbb{L} a_{k-1} \mathbb{R} a_k a_{k+1}$, of length $k-1$, with associated sequence of projections $e_2, \ldots, e_k, e_{k+2}$, where $e_2 > e_{k+2}$, contradicting the minimality of k.

Thus k is necessarily even. In the same notation, $a_2 \mathbb{R} \cdots a_k \mathbb{R} a_{k+1}$ is then the dual of a standard $\mathbb{D}$-zigzag, again of length k, with associated sequence of projections $e_2, \ldots, e_{k+2} < e_2$ and, again by minimality, no further proper comparability among these projections. Dualizing the entire argument shows that if the original zigzag is the *dual* of a standard one, then the zigzag $a_2 \mathbb{L} \cdots a_k \mathbb{L} a_{k+1}$ is a standard one, again of minimum length k. So a pure Ψ_k-configuration exists once more. $\square$

Example 11.11 shows that at the level of individual semigroups this lemma does not translate into exclusion of B^+, B^- and the semigroups Ψ_k themselves.

The *extended* Ψ_k*-configuration* $E\Psi_k$ is obtained from a Ψ_k-configuration by iterating the last statement of Lemma 5.2, starting at $e_{k+1} < e_1$. (The first step has already been used in the proof of the previous lemma.) The result may be visualized in Figure 1, although in general there will be relations among the projections in addition to those shown.

Define the sequence $a_{k+1}, a_{k+2} \ldots$, and the sequence $e_{k+2}, e_{k+3} \ldots$, recursively, as follows, with $e_{k+1} = a_k^+$ the starting point. Compare these with the equations (4) and (3) that hold in

Ψ_k. For $n > k$, with $n \equiv r \mod k$, where $1 \leq r \leq k$:

$$\begin{cases} a_n = e_n a_r \quad \text{and} \quad e_{n+1} = a_n^* \quad \text{if } n \text{ (and so } r\text{) is odd,} \\ a_n = a_r e_n \quad \text{and} \quad e_{n+1} = a_n^+ \quad \text{if } n \text{ (and so } r\text{) is even.} \end{cases}$$

For $n = 1, \ldots, k$, these equations already hold, by virtue of the definition of the original zigzag. Thus $E\Psi_k$ is defined to be the sequence a_n, $n \geq 1$, together with the associated sequence of projections e_n, $n \geq 1$. Compare the following with the properties of the semigroup Ψ_k exhibited in Section 4.

LEMMA 11.5 *Given a Ψ_k-configuration, the extended Ψ_k-configuration $E\Psi_k$ has the following properties.*

(i) If $n \geq 1$ is odd, then $a_n^+ = e_n$ and $a_n^* = e_{n+1}$; if n is even, then $a_n^* = e_n$ and $a_n^+ = e_{n+1}$ (cf (3)).

(ii) If $n > m$ and $n \equiv m \mod k$, then $e_n < e_m$. Thus the poset of projections of $E\Psi_k$ is the union of the ω-chains $e_r > e_{k+r} > e_{2k+r} > \cdots$, $r = 1, \ldots, k$.

(iii) Hence if $m \equiv n \mod k$, then for m odd, $e_m a_n = a_{\max(m,n)} = a_n e_{m+1}$; and for m even, $e_{m+1} a_n = a_{\max(m,n)} = a_n e_m$ (cf (2)).

Proof. It may be helpful to refer to Figure 1 (even though relations may hold in $E\Psi_k$ that do not in Ψ_k itself). In all cases, the residue classes modulo k are taken to be $1, 2, \ldots, k$.

(i) In each case, the second equation is just a repeat of the definition. To prove the remaining equations, we show first that $e_n \leq e_r$, where $n \equiv r \mod k$. For $n = k+1$, this is given; in general, if n is odd, then $e_{n+1} = (e_n a_r)^* \leq a_r^* = e_{r+1}$; if n is even, then $e_{n+1} = (a_r e_n)^+ \leq a_r^+$ and either $a_r^+ = e_{r+1}$, if $r < k$, or $a_r^+ = e_{k+1} < e_1$ if $r = k$.

Therefore for all $n \geq 1$, if n is odd, then $a_n^+ = (e_n a_r)^+ = (e_n a_r^+)^+ = e_n$ and if n is even, then $a_n^* = e_n$, similarly.

(ii) The first statement is essentially an iteration of Lemma 5.2, starting from the given $e_{k+1} < e_1$. For the induction step, if n is odd, $m < n$ and $m \equiv n \equiv r \mod k$, then $a_n = e_n a_r = e_n e_m a_r = e_n a_m$, so $e_{n+1} = a_n^* = (e_n a_m)^* \leq a_m^* = e_{m+1}$. Suppose $e_{n+1} = e_{m+1}$. Then $a_m = a_m e_{m+1} = a_m e_{n+1} = a_m a_n^* = a_m (e_n a_m)^* = e_n a_m$, by one of the 'ample' identities, whence $e_m = a_m^+ \leq e_n$, contradicting the induction hypothesis. The even case is dual.

(iii) These are immediate from (ii), using (i) and the defining equations. □

Two features of this construction are important to bear in mind. The first is that if the initial Ψ_k-configuration is not pure, there may be identifications of projections within the layers. The second was alluded to earlier: even in the pure case, in general there will be comparability relations among the projections that are not present in that diagram (nor in Ψ_k itself), as witnessed by Example 11.11, for instance. The proof of Proposition 11.7 shows how to remove those unwanted relations. It makes use of the following modified version of the Δ-semigroups, which will also find application in the proof of Proposition 12.4.

Let $p > 1$ be an *odd* integer. Let Γ_p be obtained from Δ_p by adjoining the mapping $\alpha_{p+1} : Y_{e_1} \to Y_{e_{p+1}}$ in T_Y. The only new nonzero products within T_Y correspond to the equations $\epsilon_1 \alpha_{p+1} = \alpha_{p+1} = \alpha_{p+1} \epsilon_{p+1}$, so the result is again a restriction subsemigroup of T_Y. Figure 6 and 7 illustrate Γ_3 in eggbox form and mapping forms.

In the case $p = 1$, α_1 already maps Y_{e_1} to Y_{e_2} in Δ_1. In that case, it is convenient to set $\alpha_2 = \alpha_1$ in order to define Γ_1.

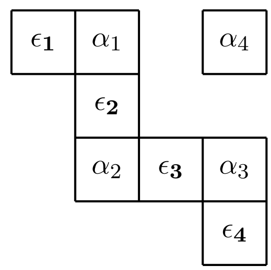

Figure 6: The nonzero elements of the semigroup Γ_3, in eggbox form

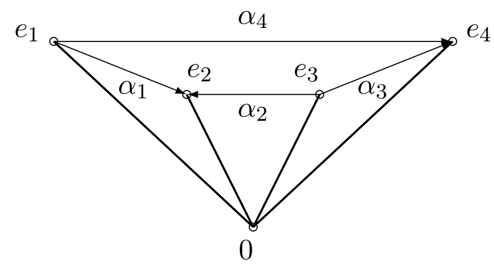

Figure 7: Γ_3 as a semigroup of mappings.

PROPOSITION 11.6 *For $p \geq 1$, $\Gamma_p \in \mathbf{B}_0$.*

Proof. The semigroups are strict, $\mathbb{H}$-combinatorial and contain no mutually inverse elements other than projections, so Theorem 10.3 applies. $\square$

Now for any $p \geq 1$, the sequence $\alpha_1 \mathbb{L} \alpha_2 \cdots \mathbb{R} \alpha_p \mathbb{L} \alpha_{p+1}$ in Γ_p, together with the sequence of projections $\epsilon_1, \ldots, \epsilon_{p+1}, \epsilon_{p+2} = \epsilon_1$, forms a 'degenerate' Ψ_{p+1}-configuration, in the sense that it satisfies the definition with the exception that, rather than $\epsilon_{p+2} < \epsilon_1$, these projections are equal. It will be convenient to extend the formal definition of the extended Ψ_{p+1}-configuration to include this case. The recursion formulas for α_n and ϵ_n, $n \geq 1$, proceed in a formal manner, although now they merely cycle through the original elements: $\alpha_n = \alpha_r$ and $\epsilon_n = \epsilon_r$, where $n \equiv r \mod p+1$, $r \in \{1, \ldots, p+1\}$.

PROPOSITION 11.7 *If a restriction semigroup S contains a Ψ_k-configuration, then $\mathbf{V}(S)$ contains the semigroup Ψ_k itself.*

Proof. Observe that since $a_1^+ \parallel a_1^*$, then by Result 2.8, $B_0 \in \mathbf{V}(S)$.

The original Ψ_k-configuration comprises the $\mathbb{D}$-zigzag $a_1 \mathbb{L} a_2 \cdots a_{k-1} \mathbb{L} a_k$, with associated sequence of projections $e_1, \ldots, e_k, e_{k+1} < e_1$. In Γ_{k-1}, form the 'degenerate' Ψ_k-configuration considered above: the sequence $\alpha_1 \mathbb{L} \alpha_2 \cdots \alpha_{k-1} \mathbb{L} \alpha_k$, with associated sequence of projections $\epsilon_1, \ldots, \epsilon_k, \epsilon_{k+1} = \epsilon_1$. (In the case $k = 2$, the sequence is $\alpha_1 \mathbb{L} \alpha_2 = \alpha_1$, with projections $\epsilon_1, \epsilon_2, \epsilon_3 = \epsilon_1$.)

In $\Gamma_{k-1} \times S$, put $c_i = (\alpha_i, a_i)$, for $1 \leq i \leq k$; and put $g_i = (\epsilon_i, e_i)$, for $1 \leq i \leq k+1$. Then $c_1 \mathbb{L} c_2 \mathbb{R} \cdots c_{k-1} \mathbb{L} c_k$ is a $\mathbb{D}$-zigzag of distinct elements, with associated sequence $g_1, \ldots, g_{k+1} < g_1$ of distinct projections. Together they constitute a Ψ_k-configuration in $\Gamma_{k-1} \times S$, for since $\{\epsilon_1, \ldots, \epsilon_k\}$ is an antichain, the same is true of $\{g_1, \ldots, g_k\}$ and, further, $g_{k+1} \not< g_i$ for $2 \leq i \leq k$.

Construct in parallel the extended Ψ_k-configurations in S and in $\Gamma_{k-1} \times S$ and the formal extension in Γ_{k-1} considered above. Clearly $c_n = (\alpha_n, a_n)$ and $g_n = (\epsilon_n, e_n)$ for all $n \geq 1$. Now if $n \equiv r \mod k$, then $c_n = (\alpha_r, a_n)$ and $g_n = (\epsilon_r, e_n)$.

From Lemma 11.5(ii), the poset of projections of $E\Psi_k$, in $\Gamma_{k-1} \times S$, is the union of the ω-chains $g_r > g_{k+r} > g_{2k+r} > \cdots$, $r = 1, \ldots, k$. Now, however, the first components of projections from distinct ω-chains are incomparable. Hence the poset of projections is the cardinal sum of those chains (and therefore isomorphic to the poset of nonzero projections of Ψ_k itself). In particular, the members of the sequence $g_1, g_2, \ldots$ are distinct. From (i) of the lemma, in turn the members of the sequence $c_1, c_2, \ldots$ are also distinct.

Let $T = \{c_n : n \geq 1\} \cup \{g_n : n \geq 1\} \cup (\{0\} \times S)$. From (iii) of the lemma, the following products hold in T, cf equations (2) in Ψ_k itself.

If $m \equiv n \mod k$, then for m odd, $g_m c_n = c_{\max(m,n)} = c_n g_{m+1}$; and for m even, $g_{m+1} c_n = c_{\max(m,n)} = c_n e_m$.

Now in Γ_{k-1} the *only* nonzero products, other than those of projections, are the following: for r odd, $\epsilon_r \alpha_r = \alpha_r = \alpha_r \epsilon_{r+1}$; and for r even, $\epsilon_{r+1} \alpha_r = \alpha_r = \alpha_r \epsilon_r$. Thus the products above are the *only* products in T, other than those of projections, that do not fall into $\{0\} \times S$ (again cf the corresponding statement in Ψ_k itself).

It follows that T is a subsemigroup of $\Gamma_{k-1} \times S$. Once more from Lemma 11.5, the following hold: if $n \geq 1$ is odd, then $c_n^+ = g_n$ and $c_n^* = g_{n+1}$; if n is even, then $c_n^* = g_n$ and $c_n^+ = g_{n+1}$, cf equations (3) in Ψ_k itself. So T is a restriction subsemigroup of $\Gamma_{k-1} \times S$.

Finally, define $\Theta : T \longrightarrow \Psi_k$ as follows. For $n \geq 1$, $c_n \Theta = \alpha_n$ and $g_n \Theta = \epsilon_n$; and $(0, s)\Theta = 0$ for all $s \in S$. Comparing the properties above with the properties of Ψ_k in Section 4, it is clear that Θ is a surjective homomorphism. So Ψ_k divides $\Gamma_{k-1} \times S$ and therefore belongs to $\mathbf{V}(S)$. $\square$

In view of Proposition 11.4, this proposition completes the proof of sufficiency in the following theorem. Necessity is clear.

THEOREM 11.8 *A variety of restriction semigroups consists of completely r-semisimple semigroups if and only if it contains neither B^+ nor B^-, nor any of the semigroups Ψ_k.*

Theorem 11.2 now follows immediately from the next lemma.

LEMMA 11.9 *The restriction semigroups $B^+ \times B_0$ and $B^- \times B_0$ each contain a Ψ_2-configuration. Therefore $\mathbf{B}^+ \vee \mathbf{B}_0$ and $\mathbf{B}^- \vee \mathbf{B}_0$ each contain Ψ_2.*

Proof. The final statement is a consequence of Proposition 11.7. Represent B_0 as $\{e, a, f, 0\}$, where $e = a^+$ and $f = a^*$, and use the notation for B^+ from Section 2.

Let $a_1 = (c, a)$ and $a_2 = (c^*, a)$. Then the standard $\mathbb{D}$-zigzag $a_1 \; \mathbb{L} \; a_2$, with associated sequence of projections $e_1 = (1, e)$, $e_2 = (c^*, f)$, $e_3 = (c^*, e)$, satisfies $e_3 < e_1$ and no other proper comparabilities, that is, it is a Ψ_2-configuration.

Construct a_3 and e_4 from the extended Ψ_2-configuration. That is, $a_3 = e_3 a_1 = (c^* c, a)$ and $e_4 = a_3^* = ((c^2)^*, f)$. Then the dual zigzag $a_2 \; \mathbb{R} \; a_3$ has associated sequence of projections $e_2, e_3, e_4 < e_2$ and, again, no other proper comparabilities, that is, it is a dual Ψ_2-configuration. Dualizing the entire process to this point therefore yields a Ψ_2-configuration in $B^- \times B_0$. $\square$

COROLLARY 11.10 *Let $\mathbf{V}$ be any variety of restriction semigroups. Then $\mathbf{V}$ contains none of the semigroups Ψ_k if and only if either it consists of completely r-semisimple semigroups, or it is contained in the variety defined by $x^+ \geq x^*$ or the variety defined by $x^+ \leq x^*$.*

Proof. Sufficiency is clear from the theorem, since no Ψ_k satisfies either $x^+ \geq x^*$ or its dual. Conversely, if $\mathbf{V}$ contains no Ψ_k and is not completely r-semisimple, then by the theorem it contains either B^+ or B^-. In either case, it cannot also contain B_0, for by Lemma 11.9 and Lemma 11.7 it would then contain Ψ_2. Applying Result 3.6(3), $\mathbf{V}$ satisfies either $x^+ \geq x^*$ or its dual. $\square$

We conclude with the example, promised above, that justifies the need for a construction such as that used in Proposition 11.7.

EXAMPLE 11.11 *There exists an $\mathbb{H}$-combinatorial, $\mathbb{D}$-simple restriction semigroup that is not completely r-semisimple, contains a pure Ψ_2-configuration, but contains neither B^+, B^- nor any semigroup Ψ_k.*

Let Y be the semilattice $(C_\omega \times C_\omega) \backslash \{(0, 0)\}$, where C_ω denotes the ω-chain of nonnegative integers, under the reverse of the usual order. Consider the Munn semigroup T_Y. For each $(i, j) \in Y$, denote by ϵ_{ij} the projection $1_{Y(i,j)}$. Given $(i, j), (k, \ell) \in Y$, there is a unique translation in T_Y that maps $Y(i, j)$ to $Y(k, \ell)$. Let A be the set of all translations $\alpha \in T_Y$ with the property that, for ordered pairs in its domain, α is strictly decreasing on the first component and strictly increasing on the second component. Clearly, A is closed under composition; it is also closed under restriction (to principal ideals) and therefore under multiplication on either side by idempotents of T_Y. Therefore the set $S = E_{T_Y} \cup A$ is a full subsemigroup – and therefore a restriction subsemigroup – of T_Y.

Every non-projection of S has infinite order, so S cannot contain any semigroup Ψ_k. Suppose $(i, j) \geq (k, \ell) \in Y$, that is, $i \leq k$ and $j \leq \ell$. If these elements are distinct, there can be no member of A that maps one to the other, that is, there can be no $\mathbb{D}$-zigzag of length one between the corresponding projections ϵ_{ij} and $\epsilon_{k\ell}$ of S. In other words, S contains neither B^+ nor B^-.

To show S is $\mathbb{D}$-simple, we show that there is a $\mathbb{D}$-zigzag of length at most three from ϵ_{10} to ϵ_{ij}, for any $(i,j) \in Y$, $(i,j) \neq (1,0)$. Let α_1 denote the member of A that maps $(1,0)$ to $(0,1)$. This gives a zigzag of length 1 from the former to the latter. Now assume $i+j > 1$. If $j = 0$, let α_2 be the member of A that maps $(i,0)$ to $(0,1)$. Thus $\alpha_1 \mathbb{L} \alpha_2$ is a zigzag of length two from $(1,0)$ to $(i,0)$, yielding, incidentally, a Ψ_2-configuration in S, which is necessarily pure (by the previous paragraph). Finally, if $j > 0$, let α_3 be the member of A that maps $(i+j,0)$ to (i,j). Then $\alpha_1 \mathbb{L} \alpha_2 \mathbb{R} \alpha_3$ is a $\mathbb{D}$-zigzag from $(1,0)$ to (i,j), where now α_2 maps $(i+j,0)$ to $(0,1)$. □

12 Characterizing B by the absence of Λ_k's

In Proposition 10.4, we completed the proof that the semigroups Λ_k comprise a second series of critical semigroups for B_0, this time finite. We now prove direct analogues of Theorem 11.2 and Corollary 11.10.

THEOREM 12.1 *Let* $\mathbf{V}$ *be a variety of restriction semigroups that contains* B_0*. Then* $\mathbf{V} \subseteq \mathbf{B}$ *if and only if it contains no semigroup* Λ_k*.*

COROLLARY 12.2 *Let* $\mathbf{V}$ *be any variety of restriction semigroups. Then* $\mathbf{V}$ *contains none of the semigroups* Λ_k *if and only if either it is contained in* $\mathbf{B}$*, or it is contained in the variety defined by* $x^+ \geq x^*$ *or the variety defined by* $x^+ \leq x^*$*.*

We refer the reader to the material on Λ_k-configurations in Section 6 and that on Ψ_k-configurations and their extensions $E\Psi_k$ in Section 11.

According to Lemma 6.5, failure of $\mathbb{D}$-majorization in a completely r-semisimple semigroup S implies the existence of a Λ_k-configuration or its dual. Observe that, by Result 2.8, if S contains such a configuration, then $B_0 \in \mathbf{V}(S)$.

LEMMA 12.3 *For any* $k \geq 1$*, if the restriction semigroup* S *contains a* Λ_k*-configuration or a dual* Λ_k*-configuration, then* $B_0 \times S$ *contains a* Λ_{k+1}*-configuration.*

Proof. Represent B_0 as $\{g, b, h, 0\}$, where $g = b^+$, $h = b^*$. First consider the case of the Λ_k-configuration. If k is even, $a_1 \mathbb{L} a_2 \mathbb{R} \cdots \mathbb{R} a_{k-1} \mathbb{L} a_k$ is a $\mathbb{D}$-zigzag in S. Then $(h,a_1) \mathbb{L} (h,a_2) \mathbb{R} \cdots \mathbb{R} (h,a_{k-1}) \mathbb{L} (b,a_k) \mathbb{R} (b,e_{k+1})$ is a zigzag of length $k+1$ in $B_0 \times S$, with associated sequence of projections $(h,e_1), \ldots, (h,e_k), (g,e_{k+1}), (h,e_{k+1})$, which forms an antichain. Here $(h,f) > (h,e_1)$ and $(h,f) > (h,e_{k+1})$, but (h,f) is incomparable with the other members of the sequence, so this constitutes a Λ_{k+1}-configuration.

If k is odd, then, similarly, $(g,a_1) \mathbb{L} (g,a_2) \mathbb{R} \cdots \mathbb{L} (g,a_{k-1}) \mathbb{R} (b,a_k) \mathbb{L} (b,e_{k+1})$ is a zigzag of length $k+1$ in $B_0 \times S$, with associated sequence of projections $(g,e_1), \ldots, (g,e_k), (h,e_{k+1}), (g,e_{k+1})$, where $(g,f) > (g,e_1)$ and $(g,f) > (g,e_{k+1})$. Again this constitutes a Λ_{k+1}-configuration.

Since for odd k, Λ_k-configurations are self-dual, the remaining case is that of a dual Λ_k-configuration for k even. But in that case, dualizing the argument in the first paragraph leads to a (dual) Λ_{k+1}-configuration in $B_0 \times S$. □

The proof of the next result uses the same approach as the proof of its analogue, Proposition 11.7. Refer to the definition of the semigroups Γ_p that precedes Proposition 11.6 and to the paragraph that follows that proposition.

PROPOSITION 12.4 *If either a Λ_k-configuration or a dual Λ_k-configuration exists in a restriction semigroup S, then $\Lambda_\ell \in \mathbf{V}(S)$, for some positive integer ℓ.*

Proof. By applying Lemma 12.3 – twice if necessary – it may be assumed, without loss of generality, that a Λ_k-configuration exists for an even value of k. It consists of the standard $\mathbb{D}$-zigzag $a_1 \mathbb{L} \cdots \mathbb{L} a_k$ and its associated sequence of projections $e_1, \ldots, e_{k+1}$, together with the projection $f > e_1, e_{k+1}$.

Again we use the sequence $\alpha_1 \mathbb{L} \cdots \alpha_{k-1} \mathbb{L} \alpha_k$ in Γ_{k-1}, with associated sequence of projections $\epsilon_1, \ldots, \epsilon_k, \epsilon_{k+1} = \epsilon_1$.

As in the proof of Proposition 11.7, in $\Gamma_{k-1} \times S$ once more put $c_i = (\alpha_i, a_i)$, for $1 \le i \le k$; and put $g_i = (\epsilon_i, e_i)$, for $1 \le i \le k+1$. Then $c_1 \mathbb{L} c_2 \mathbb{R} \cdots c_{k-1} \mathbb{L} c_k$ is a $\mathbb{D}$-zigzag of distinct elements, with associated sequence $g_1, \ldots, g_{k+1}$ of distinct projections. Here $(\epsilon_1, f) > (\epsilon_1, e_1) = g_1$ and $(\epsilon_1, f) > (\epsilon_1, e_{k+1}) = (\epsilon_{k+1}, e_{k+1}) = g_{k+1}$ but there are no other proper comparability relations among these projections. We obtain another Λ_k-configuration, therefore. Let $T = \{c_1, \ldots, c_k\} \cup \{(\epsilon_1, f), g_1, \ldots, g_{k+1}\} \cup (\{0\} \times S)$. Apart from the obvious products involving projections, all products in Γ_{k-1} yield zero, so T is a restriction subsemigroup of $\Gamma_{k-1} \times S$. Map T to Λ_k by: $c_n \mapsto \alpha_n$, for $1 \le n \le k$; $g_n \mapsto \epsilon_n$, for $1 \le n \le k+1$; $(\epsilon_1, f) \mapsto \phi$; and $(0, s) \mapsto 0$ for all $s \in S$. This is clearly a surjective homomorphism. □

This completes the discussion of the outcome of Lemma 6.5. We now examine the possible outcomes provided by Proposition 11.4.

LEMMA 12.5 *If a restriction semigroup S contains a pure Ψ_k-configuration, then $S \times S$ contains a Λ_{2k-1}-configuration. Therefore $\Lambda_\ell \in \mathbf{V}(S)$ for some ℓ. In particular, this applies to Ψ_k itself.*

Proof. The second assertion is a consequence of Proposition 12.4. The third follows from the fact that Ψ_k itself possesses such a Ψ_k-configuration, as mentioned following the definition of purity.

The proof only makes use of the first two 'layers' of the extended configuration $E\Psi_k$, specifically the $\mathbb{D}$-zigzags $a_1 \mathbb{L} a_2 \cdots \mathbb{R} a_{k-1} \mathbb{L} a_k$, with associated sequence of projections $e_1, \ldots, e_{k+1} < e_1$, and $a_{2k-1} \mathbb{R} a_{2k-2} \cdots a_{k+1} \mathbb{R} a_k$, with associated sequence of projections $e_{2k}, e_{2k-1}, \ldots, e_{k+1}, e_k > e_{2k}$. By the minimality of k (purity of the configuration), there are no other comparabilities among the projections in the second sequence. Again, it may be helpful to refer to Figure 1.

Since $(a_{k-1}, e_{2k}) \mathbb{L} (a_k, a_{2k-1}) \mathbb{R} (e_{k+1}, a_{2k-2})$, the concatenation of the three zigzags

(1) $(a_1, e_{2k}) \mathbb{L} (a_2, e_{2k}) \mathbb{R} \cdots \mathbb{R} (a_{k-1}, e_{2k})$,
(2) (a_k, a_{2k-1}) and
(3) $(e_{k+1}, a_{2k-2}) \mathbb{L} (e_{k+1}, a_{2k-3}) \mathbb{R} \cdots \mathbb{L} (e_{k+1}, a_{k+1}) \mathbb{R} (e_{k+1}, a_k)$

yields a zigzag in $S \times S$ of length $2k-1$, with associated sequence of projections

$$(e_1, e_{2k}), (e_2, e_{2k}), \ldots, (e_k, e_{2k}),\ (e_{k+1}, e_{2k-1}), (e_{k+1}, e_{2k-2}), \ldots, (e_{k+1}, e_{k+1}), (e_{k+1}, e_k).$$

The first k projections form an antichain because the same is true of their first components. The last k form an antichain because the same is true of their second components, using the remark above that was based on minimality. Comparing members of the two subsequences, the only possible comparability in the first components occurs via $e_1 > e_{k+1}$; and the only one in the second components occurs via $e_{2k} < e_k$. Thus the entire sequence forms an antichain.

Now $(e_1, e_k) > (e_1, e_{2k})$ and $(e_1, e_k) > (e_{k+1}, e_k)$ but (e_1, e_k) is not above any other member of the first k projections in the sequence, since $e_1 \not\geq e_i$ for $1 < i \leq k$; and it is not above any other member of the last k in the sequence, since $e_k \not\geq e_{k+i}$ for $1 \leq i < k$.

Therefore a Λ_{2k-1}-configuration is obtained. □

Proof of Theorem 12.1. Necessity of the first statement has already been proven in Section 7. Conversely, if on the one hand $\mathbf{V}$ fails to consist of completely r-semisimple semigroups, then by Theorem 11.2, it contains some Ψ_k and so, by Lemma 12.5, some Λ_ℓ. If, on the other hand, $\mathbf{V}$ contains a completely r-semisimple semigroup that is not strict, then the combination of Lemma 6.5 and Proposition 12.4 again shows that it contains some Λ_k. □

This leads easily to an absolute characterization of $\mathbf{B}$.

COROLLARY 12.6 *A variety of restriction semigroups is contained in* $\mathbf{B}$ *if and only if it contains no semigroup* Λ_k *and, in addition, it contains neither* B^+ *nor* B^-.

Proof. Necessity is again clear. Conversely, by Result 3.6(1), if $\mathbf{V}$ contains none of B_0, B^+ and B^-, then it consists of semilattices of monoids and so is contained in $\mathbf{B}$. □

Proof of Corollary 12.2. Sufficiency follows from the proof of the theorem and the earlier remarks on the varieties defined by $x^+ \geq x^*$ and its dual. Conversely, suppose that $\mathbf{V}$ contains no Λ_k. If $B_0 \in \mathbf{V}$, then the theorem applies. If not, then Result 3.6(3) applies. □

13 Appendix

This appendix is basically the same as that in the precursor [16], though here we restrict attention to the two-sided situation, by and large. The material in this paper is self-contained, in that only the defining identities are needed. Gould [11] was the first to make explicit the identification of the varietal definitions of restriction semigroups with the 'traditional' definitions of weakly E-ample semigroups, and it was her paper that motivated the author to investigate the lattices of varieties. The later paper by Hollings [13] surveyed 'the historical development of the study of left restriction semigroups, from the 'weakly left E-ample' perspective', taking as the definition of left restriction semigroups, however, the semigroups of partial mappings of a given set that are closed under taking the identity maps on their domains.

Together, those two papers demonstrate the equivalence of these approaches to the topic. They also provide a broad overview of the development of the various historical strands of development of the topic, including some not touched upon here, to which we refer the reader.

Here we briefly summarize these equivalences, so as to place our paper in context. Naturally, the reader is referred to [11] and [13] for a fuller exposition.

Let S be a semigroup whose set E_S of idempotents is a semilattice. Let E be a nonempty 'distinguished' subsemilattice of E_S. Define the relation $\widetilde{\mathcal{R}}_E$ on S by $a\widetilde{\mathcal{R}}_E b$ if, for all $e \in E$, $ea = a$ if and only if $eb = b$. Each $\widetilde{\mathcal{R}}_E$-class of S contains at most one member of E. Call S *weakly left E-ample* if

(1) every element a of S is $\widetilde{\mathcal{R}}_E$-related to a (necessarily unique) member of E, which may be denoted a^+;

(2) $\widetilde{\mathcal{R}}_E$ is a left congruence;

(3) for all $a \in S$, $e \in E$, $ae = (ae)^+a$.

Treating S now as a unary semigroup $(S, \cdot, ^+)$, and referring to the identities stated at the beginning of Section 1, notice that $E = \{x \in S : x^+ = x\}$, so $(x^+)^+ = x^+$ holds, the identities $x^+x = x$ and $x^+y^+ = y^+x^+$ are obvious, the identity $(xy)^+ = (xy^+)^+$ (see Lemma 1.1) follows from (2), and the left ample identity $xy^+ = (xy)^+x$ is an immediate consequence. Also as a result of that additional identity, $(x^+y)^+ = (x^+y^+)^+ = x^+y^+$.

Using the same set E of distinguished idempotents, define $\widetilde{\mathcal{L}}_E$ on S dually to $\widetilde{\mathcal{R}}_E$ and call S *weakly right E-ample* if it satisfies the duals of (1) – (3). Call S *weakly E-ample* if it is both weakly left E-ample and weakly right E-ample.

Every weakly E-ample semigroup, regarded as a biunary semigroup, therefore satisfies the defining identities for restriction semigroups, and E is its set of projections. Conversely, given any restriction semigroup $(S, \cdot, ^+, ^*)$ and putting $E = P_S$, then $a\widetilde{\mathcal{R}}_E b$ if and only if $a^+ = b^+$, and $a\widetilde{\mathcal{L}}_E b$ if and only if $a^* = b^*$, that is, $a \ \mathbb{R} \ b$ and $a \ \mathbb{L} \ b$, respectively, in our notation, from which it readily follows that S is weakly E-ample.

When regarded from the varietal point of view, the semilattice of 'distinguished idempotents' is now no longer 'distinguished': it is simply the semilattice of projections, subsidiary to the unary operation. Thus the subscript notation on the generalized Green relations plays only the historical role of distinguishing these semigroups from the earlier classes considered in the next paragraph. That is why we chose in [16] to start afresh with the notation $\mathbb{R}$, etc.

The term *weakly ample* is reserved for the special case that $E = E_S$. The term *ample* refers to case that $\widetilde{\mathcal{R}}_E = \mathcal{R}^*$, the 'potential' Green relation given by $a \ \mathcal{R}^* \ b$ if $xa = ya$ if and only if $xb = yb$ for all $x, y \in S^1$, and $\widetilde{\mathcal{L}}_E = \mathcal{L}^*$, dually. (Necessarily, $E = E_S$ [11].) The inverse semigroups, and their full inverse subsemigroups, provide a ready source of ample semigroups. The ample semigroups were originally termed the *type-A adequate* semigroups.

From the universal algebraic point of view, the great advantage of working with restriction semigroups is that they form a variety. The weakly ample and the ample semigroups form only quasi-varieties. At least in the author's view, they also exhibit the most natural generality, in that the reduced restriction semigroups comprise all monoids, whereas in the case of weakly ample and ample semigroups, they yield instead the unipotent and the cancellative monoids, respectively, (see, for example, [9, Proposition 2.5]).

On the other hand, many of the most natural examples of restriction semigroups are in fact ample. For example, the free restriction semigroups are ample [9]; and the full subsemigroups of inverse semigroups are easily seen to be ample. Thus the semigroups Δ_k, Λ_k and Ψ_k that have played such a major role in the current paper are all ample.

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Department of Mathematics, Statistics and Computer Science
Marquette University
Milwaukee, WI 53201, USA
peter.jones@mu.edu