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A common framework for restriction semigroups and regular $*$-semigroups

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ABSTRACT

Left restriction semigroups have appeared at the convergence of several flows of research, including the theories of abstract semigroups, of partial mappings, of closure operations and even in logic. For instance, they model unary semigroups of partial mappings on a set, where the unary operation takes a map to the identity map on its domain. This perspective leads naturally to dual and two-sided versions of the restriction property. From a varietal perspective, these classes of semigroups – more generally, the corresponding classes of Ehresmann semigroups – derive from reducts of inverse semigroups, now taking a to $a^+ = aa^{-1}$ (or, dually, to $a^* = a^{-1}a$, or in the two-sided version, to both).

In this paper the notion of restriction semigroup is generalized to P-restriction semigroup, derived instead from reducts of regular $*$-semigroups (semigroups with a regular involution). Similarly, [left, right] Ehresmann semigroups are generalized to [left, right] P-Ehresmann semigroups. The first main theorem is an abstract characterization of the posets P of projections of each type of such semigroup as 'projection algebras'.

The second main theorem, at least in the two-sided case, is that for every P-restriction semigroup S there is a P-separating representation into a regular $*$-semigroup, namely the 'Munn' semigroup on its projection algebra, consisting of the isomorphisms between the algebra's principal ideals under a modified composition. This theorem specializes to known results for restriction semigroups and for regular $*$-semigroups. A consequence of this representation is that projection algebras also characterize the posets of projections of regular $*$-semigroups. By further characterizing the sets of projections 'internally', we connect our universal algebraic approach with the classical approach of the so-called 'York school'.

The representation theorem will be used in a sequel to show how the structure of the free members in some natural varieties of (P-)restriction semigroups may easily be deduced from the known structure of associated free inverse semigroups.

The various strands in the historical development of the class of restriction semigroups are comprehensively reviewed in [7,8,5] (see later in this introduction) but the inspiration for the current work comes in particular from [7]. As noted above, the left restriction semigroups model unary semigroups of partial mappings on a set, with α^+ the identity map on the domain of α. The set of 'distinguished' idempotents that results need not comprise all idempotents of the semigroup. In [7], Gould formalized the connection with the so-called 'York school': the left restriction semigroups are the weakly left E-ample semigroups S, defined in terms of 'generalized Green's relations' with respect to, once more, a distinguished set (in fact a semilattice) E of idempotents of S. (See Section 6.) In the cited paper, Gould showed how to define these semigroups, and their dual and two-sided versions, in varietal terms.

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In this work, we instead regard the distinguished idempotents as the sets of projections induced from a regular involution; that is, instead of abstractly taking reducts of inverse semigroups as the starting point, we work from regular $*$-semigroups, namely semigroups with an involution $x \mapsto x^{-1}$, such that x^{-1} is an inverse of x.

Our purpose is to initiate the study of a common framework for two of these strands by using varietal language to define classes of bi-unary semigroups that include both regular $*$-semigroups and (two-sided) restriction semigroups, together with their one-sided analogues. (In fact, the one-sided versions extend the class of one-sided Ehresmann semigroups, which include the one-sided restriction semigroups.) The main thrust is found in one-sided and two-sided versions of the classical Munn representation of inverse semigroups.

In a sequel [13], we will pursue the universal algebraic aspects of the study and in so doing investigate the free objects in some of the varieties, and their subvarieties, that are defined herein. In particular, the description of the free (two-sided) restriction semigroups found in [5], and of the free one-sided restriction semigroups found, in essence, in [3], are shown to follow in an elementary fashion from the general methods developed in the two papers. A key role is played therein by Theorem 5.2, our generalization of the classical Munn representation to P-restriction semigroups.

For practical reasons associated with their representations, we prefer to work with the right-handed versions of these entities. A *right P-Ehresmann semigroup* is a semigroup endowed with a unary operation * that satisfies the following identities:

$$xx^* = x, \qquad (xy)^* = (x^*y)^*, \qquad (x^*y^*)^* = y^*x^*y^*, \qquad x^*x^* = x^*.$$

The set $P_S = \{a^* : a \in S\}$ is the set of *projections* of S. A *left P-Ehresmann semigroup* is a semigroup $(S, \cdot, ^+)$ that satisfies the dual identities, substituting $^+$ for *, in which case the set of projections is $P_S = \{a^+ : a \in S\}$. A *P-Ehresmann semigroup* is then a semigroup $(S, \cdot, ^+, ^*)$ that is a left P-Ehresmann semigroup under $^+$, a right P-Ehresmann semigroup under *, and in addition satisfies the following (as a result of which, the sets of projections coincide):

$$(x^+)^* = x^+, \qquad (x^*)^+ = x^*.$$

A *P-restriction semigroup* is a P-Ehresmann semigroup that, in addition, satisfies the 'generalized ample' identities:

$$(xy)^+x = xy^+x^*, \qquad x(yx)^* = x^+y^*x.$$

The models for these definitions are the unary and bi-unary semigroups induced from regular $*$-semigroups by setting $x^+ = xx^{-1}$ and $x^* = x^{-1}x$. As shown in Section 6, they generalize respectively the classes of right E-Ehresmann, left E-Ehresmann, E-Ehresmann and restriction semigroups. In that section we also see how they relate to certain generalizations of the latter classes studied in [7]. The structure of the paper is as follows.

In Section 1 we study the elementary properties of the semigroups defined above.

In Section 2, given any right P-Ehresmann semigroup S, we induce an operation on the poset P_S by the rule $e \star f = fef$, thereby defining 'right projection algebras'. We go on to axiomatize the algebras that arise in this way. With any right projection algebra P we associate a right P-Ehresmann semigroup that is a 'large' subsemigroup $\mathrm{Ord}\,_1P$ of the semigroup of order-preserving transformations of P. In the case of left P-Ehresmann semigroups, the operation $\times$ is defined dually.

In Section 3, we then represent any right P-Ehresmann semigroup S in the semigroup $\mathrm{Ord}\,_1P_S$, in such a way that an algebra-isomorphism is induced between the respective right projection algebras. The representation is not, in general, a representation by algebra endomorphisms of P_S.

Clearly, in the two-sided case, for the 'projection algebra' $(P_S, \times, \star)$, the operations are just the reverses of each other. In Section 4, we perform the two-sided analogue of the abstract analysis in Section 2. This entails the construction from any projection algebra P of a 'Munn-type' semigroup T_P, consisting of the algebra isomorphisms between principal ideals of P, under a 'sandwich' modification of the usual composition of partial maps. As the name implies, this construction generalizes the Munn semigroup of a semilattice. The resulting semigroup is in fact a regular $*$-semigroup. Thus not only do the projection algebras P characterize abstractly the projection algebras of P-Ehresmann semigroups, they do likewise for P-restriction semigroups and regular $*$-semigroups.

In Section 5 we represent any P-restriction semigroup S as a full subsemigroup of T_{P_S}, in such a way that the projection algebra of T_{P_S} is algebra-isomorphic to P_S. Specializations to restriction semigroups (cf [4,6]) and to regular $*$-semigroups (cf [10,11,19,14]) are discussed in Sections 6 and 7 respectively. This theorem will be applied concretely in [13] (see below).

Section 6 connects the varietal approach of this paper to the historical approach of the York school, via generalizations of Green's relations. As part of this connection, we find an internal characterization of the sets of projections of right P-Ehresmann semigroups. In terms of the generalized Green's relation $\widetilde{\mathcal{L}}_P$, defined in the usual way, the terminology we have used in this paper is shown to be consistent with the historical terminology used by members of that school (see e.g. [8]). A further consequence is to place in context the 'generalized left restriction' semigroups introduced by Gould in [7]. The material in this section is a self-contained extract of the broader approach taken in [12].

Section 7 consolidates specializations of various aspects of our work to regular $*$-semigroups and discusses the relationships between, for instance, the definition of projection algebras in this paper and the abstract characterization of the sets of projections in a regular $*$-semigroup found by Imaoka [11].

The literature of historical relevance to this paper is far too large to include in the bibliography. For instance, the excellent survey by Hollings [8] cites 79 articles on the historical development of the 'York school' approach. We recommend it for background on that aspect of this paper. In that literature, the term 'weakly left E-ample' has been used. (See also Section 6.)

Gould's notes [7] cite other manifestations – and alternative names – of left restriction semigroups, going back to work on 'function systems' in the 1960's. The term 'restriction semigroup' was motivated by the use of the term 'restriction category' by Cockett and Lack [2]. Gould's approach provided great preparation and motivation for this study, and further motivation came from considering the beautiful descriptions of the free one- and two-sided restriction semigroups in [5]. (We should note that each of these papers chooses 'left', rather than 'right', as we have generally done, in the one-sided case.)

There is a somewhat smaller literature on regular $*$-semigroups. A foundational work was that of Nordahl and Scheiblich [15]. Adair [1] studied regular $*$-bands, for which [16] provides more recent results and citations relevant to these semigroups and to the wider class of completely regular $*$-semigroups. Unfortunately, the terminology of the field has not been consistent. Alternative terms for such semigroups have been $*$-*regular semigroups* [17] and *special* $*$-*semigroups* [15]. As the choice of name implies, [15] views regular $*$-semigroups within the context of the somewhat more broadly defined $*$-*regular* semigroups. That paper contains an extensive bibliography on involutory semigroups, to 1981. More recent work includes the papers of Imaoka, Yamada and Polák cited herein.

A sequel [13] will study varieties of P-restriction semigroups and their free objects, through their relationship with varieties of regular $*$-semigroups and *their* free objects. Of particular interest are the varieties of what may be called *orthodox* P-restriction semigroups.

1. *P*-Ehresmann and *P*-restriction semigroups

Let S be a regular $*$-semigroup, that is, a semigroup with involution $a \mapsto a^{-1}$ for which a^{-1} is an inverse of a. The set of idempotents of (any semigroup) S is denoted E_S. Let P_S denote the set of *projections* of S, that is, $P_S = \{e \in E_S : e = e^{-1}\}$. The following is well known (e.g. see [10]). We include proofs both for the sake of completeness and to delineate the role of the left and right units, respectively aa^{-1} and $a^{-1}a$: note that while the first two parts use only one or the other of these, the third requires both.

Result 1.1. *Let S be a regular $*$-semigroup. Then*

(a) $P_S = \{aa^{-1} : a \in S\} = \{a^{-1}a : a \in S\}$;
(b) *if* $e, f \in P_S$, *then* $ef \in E_S$ *and* $efe = (ef)(ef)^{-1} = (fe)^{-1}(fe) \in P_S$.
(c) *if* $e \in E_S$, *then* $e = (ee^{-1})(e^{-1}e)$, *so that* $P_S^2 = E_S$.

Proof. (a) Clearly, if $e \in P_S$, then $e = ee = ee^{-1} = e^{-1}e$. Conversely, if $a \in S$, then $aa^{-1}, a^{-1}a \in P_S$.
(b) If $e, f \in P_S$, then $efe = ef^2e = ef(ef)^{-1} \in P_S$, and dually, and $(ef)^2 = (efe)(ef) = (ef)(ef)^{-1}(ef) = ef$.
(c) We have already shown that $P_S^2 \subseteq E_S$. But if $e \in E_S$, then $e^{-1} \in E_S$ and so $e = (ee^{-1})(e^{-1}e) \in P_S^2$. □

If S is any regular $*$-semigroup, consider the induced unary semigroups $(S, \cdot, {}^+)$, $(S, \cdot, {}^*)$ and bi-unary semigroup $(S, \cdot, {}^+, {}^*)$, where $a^+ = aa^{-1}$ and $a^* = a^{-1}a$ (and otherwise omit reference to isolated "$()^{-1}$" symbols). By Result 1.1, $P_S = \{a^+ : a \in S\} = \{a^* : a \in S\}$.

Lemma 1.2. *Let S be a regular $*$-semigroup. The unary semigroup $(S, \cdot, {}^*)$ satisfies:*

(1) $xx^* = x$;
(2) $(xy)^* = (x^*y)^*$;
(3) $(x^*y^*)^* = y^*x^*y^*$;
(4) $x^*x^* = x^*$.

The unary semigroup $(S, \cdot, {}^+)$ satisfies the dual identities (with ${}^+$ substituted for *):

(1^r) $x^+x = x$;
(2^r) $(xy)^+ = (xy^+)^+$;
(3^r) $(x^+y^+)^+ = x^+y^+x^+$;
(4^r) $x^+x^+ = x^+$.

Proof. (1) This is equivalent to the statement $x(x^{-1}x) = x$.
(2) $(xy)^* = (xy)^{-1}(xy) = y^{-1}x^{-1}xy = y^{-1}x^{-1}xx^{-1}xy = (x^{-1}xy)^{-1}(x^{-1}xy) = (x^*y)^*$.
(3) $(x^*y^*)^* = ((x^{-1}x)(y^{-1}y))^{-1}(x^{-1}x)(y^{-1}y) = (y^{-1}y)(x^{-1}x)(y^{-1}y) = y^*x^*y^*$.
(4) This is immediate from $xx^{-1}x = x$.

The dual statements are clear. □

Lemma 1.3. *Let S be a regular $*$-semigroup. Then the bi-unary semigroup $(S, \cdot, {}^+, {}^*)$ satisfies the identities* (1) *through* (4), *(1^r) through (4^r) and, in addition:*

(5) $(x^+)^* = x^+$ *and* $(x^*)^+ = x^*$;
(6) $(xy)^+x = xy^+x^*$ *and* $x(yx)^* = x^+y^*x$.

Proof. These are similar to the proofs in the previous lemma. □

We will term any semigroup $(S,\cdot,^*)$ that satisfies the identities (1)–(4) a *right P-Ehresmann semigroup*. The set $P_S = \{a^* : a \in S\}$ is the set of *projections* of S. Since, by (4), P_S consists of idempotents, it may be partially ordered in the usual way, by $f \leq e$ if $f = fe = ef$. A *left P-Ehresmann semigroup* is a semigroup $(S,\cdot,^+)$ that satisfies the identities (1^r)–(4^r), in which case the set of projections is $P_S = \{a^+ : a \in S\}$.

A *P-Ehresmann semigroup* is then a semigroup $(S,\cdot,^+,^*)$ that is a left P-Ehresmann semigroup under $^+$, a right P-Ehresmann semigroup under *, and in addition satisfies (5). As a result, the sets of projections coincide. A *P-restriction semigroup* is a P-Ehresmann semigroup $(S,\cdot,^+,^*)$ that satisfies (6). The term *weakly* P-*ample* is an alternative term that is consistent with historical terminology in this field (see Section 6).

It is clear from the two results above that any regular $*$-semigroup induces the P-restriction semigroup $(S,\cdot,^+,^*)$ by setting $a^+ = aa^{-1}$ and $a^* = a^{-1}a$, $a \in S$. The appropriate converse will be provided by Proposition 7.1.

Lemma 1.4. *Let $(S,\cdot,^*)$ be a right P-Ehresmann semigroup. Then S satisfies:*

(7) $(xy^*)^* = y^*x^*y^*$;
(8) $(x_1^*\cdots x_n^*)^* = x_n^*\cdots x_2^*x_1^*x_2^*\cdots x_n^*$, *for* $n \geq 2$;
(9) $(x^*)^* = x^*$, *so that* $P_S = \{a^* : a \in S\}$;
(10) $(xy)^*y^* = (xy)^*$;
(11) $(ef)^2 = ef$, *for all* $e, f \in P_S$;
(12) *if* $e, f \in P_S$, *then* $f \leq e$ *if and only if* $fe = f$; *in particular,* $(xy)^* \leq y^*$;
(13) *if* $e, f \in P_S$ *and* $ef \in P_S$, *then* $ef = fef$.

In combination with (1)–(3), (10) *is equivalent to* (4).

Proof. (7) Replace x by x^* in (2) and then apply (3).

(8) For $n = 2$, this is (3). For $n > 2$, write $(x_1^*\cdots x_n^*)^* = ((x_1^*\cdots x_{n-1}^*)x_n^*)^* = ((x_1^*\cdots x_{n-1}^*)^*x_n^*)^*$, by (2). Then the proof proceeds by induction on n.

(9) First observe that by (1) and (7), $x^* = (xx^*)^* = x^*x^*x^*$. By (8), therefore, $(x^*)^* = (x^*x^*x^*)^* = (x^*)^5 = x^*$.

(10) Applying (1) and (7) in order yields $(xy)^* = ((xy)y^*)^* = y^*(xy)^*y^*$. Then (4) yields the desired conclusion.

(11) If $e, f \in P_S$, then by (1), (3) and (4), $ef = (ef)(ef)^* = (ef)(fef) = (ef)^2$.

(12) If $e, f \in P_S$ and $fe = f$, then $ef = e(fe) = (fe)^* = f^* = f$.

(13) This is clear from (3), in conjunction with (9).

To show that (1)–(3) and (10) imply (4), recall that $x^* = (xx^*)^*$ and that the proof of (9) only uses (1)–(3). Now by (10) and (9), $(xx^*)^* = (xx^*)^*(x^*)^* = (xx^*)^*x^* = x^*x^*$. $\square$

The third property of regular $*$-semigroups proved in Result 1.1 translates into the implication $e = e^2 \Rightarrow e = e^+e^*$ in the induced P-restriction semigroup $(S,\cdot,^+,^*)$. Since any monoid may be regarded as a P-restriction semigroup, setting $a^+ = a^* = 1$ for all a, this implication is not a consequence of the defining identities.

Lemma 1.5. *Let S be a P-restriction semigroup. The implication* $e = e^2 \Rightarrow e = e^+e^*$ *is equivalent to* $E_S = P_S^2$.

Proof. Necessity is clear. Conversely, suppose $E_S = P_S^2$ and let $e \in E_S$, so that $e = fg$ for some $f, g \in P_S$. Then $e = eg$, so $e^* = (eg)^* = (e^*g)^* = ge^*g$; dually, $e^+ = fe^+f$. Thus $e^+e^* = fe^+fge^*g = fe^+ee^*g = feg = e$. $\square$

The terms *homomorphism* and *congruence* will be used appropriate to the context, with clarification where necessary. When considering topics such as fundamentality, this must be kept in mind. Let $(S,\cdot,^*)$ be a right P-Ehresmann semigroup. Denote by μ_L the greatest projection-separating (or 'P-separating') congruence on S (that respects *). Call S *left P-fundamental* if μ_L is the identical relation. It is routinely verified that S/μ_L has that property. For a description of μ_L, see Corollary 3.3. Define μ_R, and right fundamentality, dually on a left P-Ehresmann semigroup. Finally, if $(S,\cdot,^+,^*)$ is a P-Ehresmann semigroup, let μ be the largest P-separating congruence on S (that respects $^+$ and *). Call S *P-fundamental* if μ is the identical relation. Again, S/μ is P-fundamental. For a description of μ on P-restriction semigroups, see Corollary 5.3.

On a regular semigroup S, μ traditionally denotes the greatest idempotent-separating congruence and, again, S is *fundamental* if μ is the identical relation. Again, it is routinely verified that S/μ has that property. According to [10, Theorem 4], on any regular $*$-semigroup $(S,\cdot,^{-1})$, μ respects inversion. Consider the induced P-restriction semigroup $(S,\cdot,^+,^*)$. Note that any congruence on S that respects both of the induced unary operations also respects inversion (since if ρ is such a congruence and $a\rho b \in S$, then $a^{-1}\rho$ and $b^{-1}\rho$ are $\mathcal{H}$-related inverses of $a\rho = b\rho$ in S/ρ and therefore are equal.) If such a congruence is P-separating, it is also idempotent-separating, in light of Result 1.1(c). Hence S is fundamental as a regular $*$-semigroup if and only if the induced P-restriction semigroup is P-fundamental as defined above.

2. Right projection algebras

In this section we abstractly characterize the posets of projections of right P-Ehresmann semigroups. It will turn out that essentially the same characterization applies in the dual case, the two-sided case and, in fact, in the case of regular $*$-semigroups: see the discussion at the end of Section 7, where this characterization is compared with that given independently, but much earlier, by Imaoka [11] of the sets of projections of regular $*$-semigroups.

A *right projection algebra* consists of a set P and a binary operation $\star$ satisfying the following axioms:

(P1) $e \star e = e$;
(P2) $(f \star e) \star e = e \star (f \star e) = f \star e$;
(P3) $g \star (f \star e) = ((g \star e) \star f) \star e$;
(P4) $(g \star f) \star e = ((g \star f) \star e) \star (f \star e)$.

The algebra P is *monoidal* if it has an element 1 that satisfies (P5): $1 \star e = e \star 1 = e$. If P is not monoidal, denote by P^1 the algebra obtained by adjoining a new element 1 and extending the operation $\star$ in the obvious way. Apart from the proof of (P3) in the case $g = 1$, which requires the next lemma, it is straightforward to verify that $(P^1, \star)$ is a monoidal right projection algebra. If P is already monoidal, let $P^1 = P$.

Lemma 2.1. *Any right projection algebra satisfies the identities:*

(P6) $(e \star f) \star e = f \star e$ *and* $f \star (f \star e) = f \star e$.

Proof. From (P2), $f \star e = e \star (f \star e)$. Substituting e for g in (P3) yields $e \star (f \star e) = ((e \star e) \star f) \star e = (e \star f) \star e$. Next, substituting f for g in (P3) yields $f \star (f \star e) = ((f \star e) \star f) \star e = (e \star f) \star e = f \star e$. □

Any *right regular band* $(B, \cdot)$ – one that satisfies the identity $efe = fe$ – is a right projection algebra, as (P1)–(P4) are easily verified. As we shall see in Example 2.5 below, the operation $\star$ will not in general be associative. (Also see Corollary 2.9.)

Example 2.2. Any right projection algebra on two generators is a right regular band. Thus the free right projection algebra on $\{e, f\}$ is the free right regular band on $\{e, f\}$, given by the operation table:

$\star$	e	f	g	h
e	e	g	g	h
f	h	f	g	h
g	h	g	g	h
h	h	g	g	h

Proof. In view of the defining relations, it is easily verified that for any right projection algebra P that is generated by a pair $\{e, f\}$ of its members, the operation $\star$ is associative, so that $(P, \star)$ is a right regular band. It is straightforward to verify that the table is as shown. □

Lemma 2.3. *Let P be a right projection algebra. Define a relation $\leq$ on P by $f \leq e$ iff $f = f \star e$. Then:*

(i) *the relation $\leq$ is a partial order on P and, if the algebra is monoidal,* 1 *is maximum under this order;*
(ii) *if $g \leq f$ then $g \star e \leq f \star e$, for all $e, f, g \in P$.*

Proof. (i) Clearly $e \leq e$. Suppose $e \leq f$ and $f \leq e$, so that $e = e \star f$ and $f = f \star e$. Thus, using (P6), $f = f \star e = (e \star f) \star e = e \star e = e$. Next suppose $g \leq f$ and $f \leq e$, so that $g = g \star f$ and $f = f \star e$. Then $g = g \star (f \star e) = ((g \star e) \star f) \star e$ so that, by (P2), $g \star e = g$, that is, $g \leq e$. Hence $\leq$ is a partial order. If P is monoidal, then by (P5), $e \leq 1$ for all $e \in P$.
(ii) Suppose $g \leq f$. Then $g = g \star f$ so that, by (P4), $g \star e = (g \star e) \star (f \star e)$, as required. □

Proposition 2.4. *Let $(S, \cdot, ^*)$ be a right P-Ehresmann semigroup. Define a binary operation on P_S by $f \star e = (fe)^*$ ($= efe$, by (2)). Then $(P_S, \star)$ is a right projection algebra, which is monoidal if S is a monoid. The partial order induced on $(P_S, \star)$ coincides with the original partial order on P_S.*

Proof. That the operation $\star$ is well defined follows from (2). Since $f \leq e$ in P_S if and only if $f = efe$, the partial orders coincide. Now (P1) is obvious; if $e, f \in P_S$, then $e(efe)e = efe$ and $(efe)e(efe) = efe$, proving (P2); $(efe)g(efe) = e(f(ege)f)e$, proving (P3); and $e(fgf)e = (efe)(e(fgf)e)(efe)$, proving (P4). If S is a monoid, then (P5) is obvious. □

Free bands yield useful examples of right projection algebras. See [9] for the basic properties, including the standard solution to the word problem, of the free band B_X on the set X (which we may denote by B_n if $|X| = n$). Szendrei [18] pointed out that B_X has the natural structure of a regular $*$-band. Alternatively, it is straightforward from the solution to the word problem that the operation $w \mapsto w^{-1}$ that reverses the order of the letters in a word w defines a regular involution. The projections in $(B_X, \cdot, ^*)$ correspond to the palindromic words.

Example 2.5. (i) The right projection algebra P_{B_2} is the free right projection algebra on $\{e, f\}$ that was constructed in Example 2.2. Although $(P_{B_2}, \star)$ is a band, P_{B_2} is not a subband of B_2 under the original operation.
(ii) The operation $\star$ on the right projection algebra P_{B_3} is nonassociative.

Proof. (i) Suppose B_2 is generated by $\{e, f\}$. It is well known (and easily seen) that $B_2 = \{e, f, ef, efe, fe, fef\}$. As remarked above, $P_{B_2} = \{e, f, fef, efe\}$. Setting $g = fef$ and $h = efe$, it can be checked that $(P_{B_2}, \star)$ is isomorphic to the cited example. The final statement follows from the fact that $(fef)(efe) = fe$.

(ii) Suppose B_3 is generated by $\{e, f, g\}$. Once again, e, f, g are projections. The equation $(e \star f) \star g = e \star (f \star g)$ in $(P_{B_3}, \star)$ translates into the equation $g(fef)g = (gfg)e(gfg)$ in $(B_3, \cdot)$ (cf Corollary 2.9 below). Recall that for a semigroup word w in an alphabet X, $0(w)$ refers to the longest initial segment of w that does not involve all the letters that appear in w; and that if two words w_1 and w_2 are equal as members of the free band on X, then the same is true of $0(w_1)$ and $0(w_2)$. Now $0(gfefg) = gf$ and $0(gfgegfg) = gfg$, so the equation $gfefg = gfgegfg$ fails to hold in B_3. □

For any poset P, denote by $\mathrm{Ord}\,P$ the submonoid of the full transformation monoid on P consisting of its order-preserving members. Similarly to the definition of P^1 in the case of projection algebras, if P does not have a maximum element then P^1 denotes the poset obtained by adjoining one. If P already has a maximum element, put $P^1 = P$. By Lemma 2.3, in the case of a right projection algebra the definitions coincide. Put $\mathrm{Ord}\,_1P = \{\alpha \in \mathrm{Ord}\,P^1 : P^1\alpha \subseteq P\}$ (cf [4, p.698]). Clearly, if P already has a maximum element, $\mathrm{Ord}\,_1P = \mathrm{Ord}\,P$.

Now let P be a right projection algebra. For each $f \in P$, the map π_f, given by $e\pi_f = e \star f$, is an order-preserving retraction onto $f\downarrow$, as a result of Lemma 2.3(ii). Extend π_f to P^1 by setting $1\pi_f = f$.

Proposition 2.6. *Let P be a right projection algebra. For $\alpha \in \mathrm{Ord}\,_1P$, put $\alpha^* = \pi_{1\alpha}$. Then*

(i) $(\mathrm{Ord}\,_1P, \cdot, ^*)$ *is a right P-Ehresmann semigroup. In particular, if P is monoidal,* $\mathrm{Ord}\,P$ *itself is a right P-Ehresmann monoid;*
(ii) *the right projection algebra $P' = P_{\mathrm{Ord}\,_1P}$ is $\star$-isomorphic to P itself;*
(iii) $\mathrm{Ord}\,_1P$ *is left P-fundamental.*

Proof. We verify the identities (1)–(4). Throughout, $\alpha, \beta \in \mathrm{Ord}\,_1P$ and $e \in P^1$. Note that $1\alpha^* = 1\alpha$.

(1) We have $e(\alpha\alpha^*) = (e\alpha)\alpha^* = e\alpha \star 1\alpha$. Since $e \le 1$, $e\alpha \le 1\alpha$ and so $e\alpha \star 1\alpha = e\alpha$.
(2) Clearly $1(\beta\alpha) = 1(\beta^*\alpha)$.
(3) On the one hand $e(\beta^*\alpha^*)^* = e \star 1(\beta^*\alpha^*) = e \star (1\beta \star 1\alpha)$. On the other, $e(\alpha^*\beta^*\alpha^*) = ((e \star 1\alpha) \star 1\beta) \star 1\alpha$. Equality results from an application of (P3).
(4) We have $e\alpha^*\alpha^* = (e \star 1\alpha) * 1\alpha = e \star 1\alpha = e\alpha^*$.

To prove (ii), observe that for any $f \in P$, $e\pi_f{}^* = e \star 1\pi_f = e \star f = e\pi_f$. Thus $\{\pi_f : f \in P\} \subseteq P'$. The opposite inclusion also holds, since for any $\alpha \in \mathrm{Ord}\,_1P$, $1\alpha \in P$. Since $1\pi_f = f$, the map $f \mapsto \pi_f$ is a bijection of P upon P'. Now in P', $1(\pi_g \star \pi_f) = 1(\pi_g\pi_f)^* = (1\pi_g)\pi_f = g \star f$. Since $\pi_g \star \pi_f \in P'$, it must be $\pi_{g\star f}$. Hence the map $f \mapsto \pi_f$ is a $\star$-isomorphism.

To prove (iii), let $\alpha, \beta \in \mathrm{Ord}\,_1P$, $\alpha\mu_L\beta$. Then for every $f \in P^1$, $(\pi_f\alpha)^*\mu_L(\pi_f\beta)^*$, so since μ_L is P-separating, $(\pi_f\alpha)^* = (\pi_f\beta)^*$. Now $(\pi_f\alpha)^* = \pi_{1(\pi_f\alpha)} = \pi_{f\alpha}$; similarly, $(\pi_f\beta)^* = \pi_{f\beta}$. Hence $f\alpha = 1\pi_{f\alpha} = 1\pi_{f\beta} = f\beta$ and $\alpha = \beta$, as required. □

Corollary 2.7. *A poset $(P, \le)$ is the poset of projections of a right P-Ehresmann semigroup if and only if it can be endowed with the structure of a right projection algebra.*

The question arises as to when $\mathrm{End}\,_1P$, the subsemigroup of $\mathrm{Ord}\,_1P$ comprising those mappings that are endomorphisms of P^1, is a right P-Ehresmann semigroup. Note first that, at least in the case where P is monoidal, a suitable unary operation can always be defined in a rather trivial way by setting α^* to be the identity mapping on P. More relevant, then, is the question of when $\mathrm{End}\,_1P$ is a unary subsemigroup of $(\mathrm{Ord}\,_1P, \cdot, ^*)$. Clearly, this is equivalent to the property that π_e be an endomorphism, for each $e \in P$.

Lemma 2.8. *Let $(P, \star)$ be a right projection algebra and $e \in P$. Then π_e is an endomorphism if and only if $(g \star f) \star e = g \star (f \star e)$ for all $g, f \in P$.*

Proof. Let $g, f \in P^1$ and $e \in P$. Then $(g \star f)\pi_e = (g \star f) \star e$ and $g\pi_e \star f\pi_e = (g \star e) \star (f \star e)$. By (P2), these terms are equal if either $g = 1$ or $f = 1$, so suppose otherwise. By replacing g by $g \star e$ in (P3), then using (P2), and finally applying (P3) as it is stated, we obtain $(g \star e) \star (f \star e) = (((g \star e) \star e) \star f) \star e = ((g \star e) \star f) \star e = g \star (f \star e)$. □

Corollary 2.9. *Let $(P, \star)$ be a right projection algebra. The following are equivalent:*

(i) $\mathrm{End}\,_1P$ *is a unary subsemigroup of the right P-Ehresmann semigroup* $(\mathrm{Ord}\,_1P, \cdot, ^*)$*;*
(ii) *each map π_e, $e \in P$, is an endomorphism of P;*
(iii) *the operation $\star$ is associative;*
(iv) *every [some] right P-Ehresmann semigroup $(S, \cdot, ^*)$ for which $P_S \cong P$ satisfies the identity that may be represented as $g(fef)g = (gfg)e(gfg)$, $e, f, g \in P_S$ (which may be converted into a formal identity by setting $e = x^*, f = y^*, g = z^*$, for instance.)*

In that case, $(P, \star)$ satisfies $e \star f \star e = f \star e$, that is, $(P, \star)$ is a right regular band.

Proof. The equivalence of (i)–(iii) is clear from Lemma 2.8. The identity in (iv) is simply a restatement of associativity in terms of the definition of $e \star f$. To prove the final statement, assume associativity holds. Then by (P1) $(P, \star)$ is a band and, by (P2) satisfies the stated identity. □

As noted prior to Example 2.2, every right regular band is a right projection algebra. Example 2.2(ii) demonstrates that the equivalent conditions of this corollary are not satisfied in general. A stronger condition than those of the corollary, as witnessed by part (i) of that example, is that $(P_S, \cdot)$ actually be a subband of S, in which case (cf Proposition 6.2) the operation $\star$ coincides with the original operation on S.

This is an appropriate point at which to consider the circumstances under which $(P, \star)$ is a semilattice, that is, a commutative band.

Lemma 2.10. *If $(P, \star)$ is a semilattice, then the poset $(P, \leq)$ is a semilattice. The converse does not hold.*

Proof. If $(P, \star)$ is a commutative band, then it is immediate that for all $e, f \in P$, $e \star f$ is their meet in $(P, \leq)$. To show that the converse does not hold, let $(P, \star)$ be the three-element right regular band obtained by adjoining a zero to the right zero semigroup $\{e, f\}$. Then P is a right projection algebra. As a partially ordered set, P is the three-element, non-chain, semilattice. □

Proposition 2.11. *Let $(P, \star)$ be a right projection algebra. The following are equivalent:*

(i) *$(P, \star)$ is a semilattice;*
(ii) *$(P, \star)$ is commutative;*
(iii) *for every [some] right P-Ehresmann semigroup $(S, \cdot, ^*)$ for which $P_S \cong P$, $(P_S, \cdot)$ is a subsemilattice of $(S, \cdot)$, that is, S satisfies the identity that may be represented as $ef = fe$, $e, f \in P_S$;*
(iv) *in the language of Section 6, $(\text{Ord}\,_1P, \cdot, ^*)$ is a right E-Ehresmann semigroup.*

In that event, again in the language of Section 6, $(\text{End}\,_1P, \cdot, ^*)$ is a right restriction semigroup.

Proof. That (i) implies (ii) is clear. If S is a right P-Ehresmann semigroup, then the equation $e\star f = f\star e$ in $(P_S, \star)$ is equivalent to the equation $fef = efe$ in $(P_S, \cdot)$. Since $ef, fe \in E_S$, the latter is equivalent to $ef = fe$. In that case, $ef = efe \in P_S$. Thus (ii) is equivalent to (iii). Moreover, commutativity of P_S clearly implies that $g(fef)g = (gfg)e(gfg)$, for all $e, f, g \in P_S$, so by Corollary 2.9, (iii) implies (i).

Looking ahead to Section 6, a right E-Ehresmann semigroup is simply a right P-Ehresmann semigroup S for which P_S is a subsemilattice. Thus the equivalence of (iii) and (iv) follows from Proposition 2.6.

Assuming (iii), say, it follows from (i) and Corollary 2.9 that $\text{End}\,_1P$ is again a right P-Ehresmann semigroup. According to Proposition 6.2(iv), we need to show that $\text{End}\,_1P$ satisfies the identity $\alpha(\beta\alpha)^* = \beta^*\alpha$, that is, for all $e \in P$, $e\alpha \star (1\beta\alpha) = (e \star 1\beta)\alpha$. This follows from the fact that α is an endomorphism. □

The following additional properties of right projection algebras will be useful in the sequel.

Lemma 2.12. *Any right projection algebra also satisfies the following identities:*

(P7) $(g \star f) \star e = (g \star f) \star (f \star e)$;
(P8) $((g \star (f \star e)) \star f) \star e = g \star (f \star e)$.

Proof. (P7) Applying (P4) we have $(g \star f) \star e = ((g \star f) \star e) \star (f \star e)$. Then applying (P3), (P2) and (P3) in that order:

$$((g \star f) \star e) \star (f \star e) = ((((g \star f) \star e) \star e) \star f) \star e = (((g \star f) \star e) \star f) \star e = (g \star f) \star (f \star e).$$

(P8) First we observe that, applying (P3) twice and then (P7):

$$(g \star (f \star e)) \star f = (((g \star e) \star f) \star e) \star f = (g \star e) \star (e \star f) = (g \star e) \star f.$$

Thus $((g \star (f \star e)) \star f) \star e = ((g \star e) \star f) \star e = g \star (f \star e)$, applying (P3). □

A *left projection algebra* consists of a set P and a binary operation $\times$ that satisfies the duals of (P1)–(P4) (and the dual of (P5) if monoidal), with $\times$ replacing $\star$. In the dual of Lemma 2.3, the partial order induced on P is defined by $f \leq e$ iff $f = e \times f$. Clearly, given any right projection algebra $(P, \star)$, the reverse operation $e \times f = f \star e$ induces a left projection algebra and vice versa. In the dual of Proposition 2.4, the left P-Ehresmann semigroup $(S, \cdot, ^+)$ induces the left projection algebra $(P_S, \times)$, where $e \times f = (ef)^+ = efe$. The dual of Proposition 2.6 is the following, where $\text{Ord}\,^r_1P$ is the reverse semigroup of $\text{Ord}\,_1P$, functions being written on the left of their arguments and composition being denoted by $\circ$. For $f \in P$, σ_f denotes the member of $\text{Ord}\,^rP$ that is dual to π_f. That is, $\sigma_f(e) = f \times e$, $e \in P$.

Proposition 2.13. *Let P be a left projection algebra. Then $\text{Ord}\,^r_1P$ is a left P-Ehresmann semigroup, where for $\alpha \in \text{Ord}\,^rP$, $\alpha^+ = \sigma_{\alpha(1)}$. The left projection algebra $P' = P_{\text{Ord}\,^r_1P}$ is $\times$-isomorphic to P itself. $\text{Ord}\,^r_1P$ is right P-fundamental.*

3. A representation for right *P*-Ehresmann semigroups

In this section, we represent any right P-Ehresmann semigroup in Ord $_1P_S$ by means of a one-sided generalization of the classical Munn representation of any inverse semigroup. This representation and its dual have antecedents in the literature of the 'York school', in particular in the papers [4,6] cited in Section 5. Likewise, they have antecedents in the literature on regular $*$-semigroups in the work of T. Imaoka (see Section 7). Yet we feel that their usefulness, especially in the 'one-sided' situation, has yet to be fully realized.

A right P-Ehresmann semigroup $(S, \cdot, ^*)$ that is a monoid necessarily satisfies $1^* = 11^* = 1$. In that case, put $S^1 = S$. Otherwise, S^1 denotes the monoid obtained by adjoining an identity in the usual way and defining $1^* = 1$. In either case, P_{S^1} is clearly monoidal. Analogous constructions apply in the dual and two-sided cases.

Lemma 3.1. *Let $(S, \cdot, ^*)$ be a right P-Ehresmann semigroup. For any $a \in S$, define $\theta_a : f \mapsto (fa)^*, f \in P_S^1$. Then $\theta_a \in$ Ord $_1P_S$. In particular, for any $e \in P_S$, $\theta_e : f \mapsto (fe)^* = efe$ defines an order-preserving retraction of P_S^1 onto $e\downarrow$.*

Proof. Let $a \in S$ and $f, g \in P_S^1, f \le g$. Then $(fa)^* = (f(ga))^* \le (ga)^*$, by (12). Clearly the image of θ_a is contained in P_S itself. For $e \in P_S$, θ_e restricts to the identity map on $e\downarrow$, by the definition of the order on P_S^1. □

Theorem 3.2. *Let $(S, \cdot, ^*)$ be a right P-Ehresmann semigroup. Then the map $\theta : a \mapsto \theta_a$ is a P-separating *-homomorphism of S into the right P-Ehresmann semigroup* Ord $_1P_S$ *that induces a $\star$-isomorphism of P_S onto the right projection algebra P' of* Ord $_1P_S$.

Proof. Let $a, b \in S, e \in P_S^1$. Then $(e\theta_a)\theta_b = ((ea)^*b)^* = (eab)^* = e\theta_{ab}$, using (2). In particular, $e\theta_{a^*} = a^*ea^* = e \star a^*$, according to Proposition 2.4. That is, $\theta_{a^*} = \pi_{a^*}$. But $(\theta_a)^* = \pi_{1\theta_a}$, where $1\theta_a = (1a)^* = a^*$. Hence $\theta_{a^*} = (\theta_a)^*$ for all $a \in S$.

Let $e, f \in P_S^1$ and suppose $\theta_e = \theta_f$. By the last statement of the lemma above, $e = f$. So θ separates P_S. Finally, as noted in the proof of Proposition 2.6, $P' = \{\pi_f : f \in P\}$, so the restriction of θ to P_S maps it onto P'. Since the respective operations $\star$ are defined analogously, in terms of the respective products, the restriction is a $\star$-isomorphism. □

Corollary 3.3 (*Cf Remarks in Section 1*)**.** *On any right P-Ehresmann semigroup $(S, \cdot, ^*)$, $\mu_L = \{(a, b) : (ea)^* = (eb)^* \quad \forall e \in P_S^1\}$, which is the congruence induced by the homomorphism θ. Thus the image of S in* Ord $_1P_S$ *is left P-fundamental.*

Proof. The indicated relation is clearly the congruence induced by θ and therefore separates projections. But if $a\mu_L b$, then $(ea)^*\mu_L(eb)^*$ for all $e \in P_S^1$ and so $(ea)^* = (eb)^*$. The last statement follows from the remarks at the end of Section 1. □

In general, the representation above will not be by algebra homomorphisms. Since each map π_f is in the image of S under θ, then by Corollary 2.9 a necessary condition for this to occur is that $(P_S, \star)$ be a (necessarily right regular) band, equivalently, S itself satisfies the identity that may be represented as $g(fef)g = (gfg)e(gfg)$.

Proposition 3.4 (*Cf [4, Lemma 2.7], [6, Lemma 4.1]*)**.** *Let $(S, \cdot, ^*)$ be a right P-Ehresmann semigroup. The representation θ is by endomorphisms of P_S^1 if and only if S satisfies the identity that may be represented as $(efa)^* = (fa)^*(ea)^*(fa)^*$.*

Proof. This identity is merely a restatement of the condition $((e \star f)a)^* = (ea)^* \star (fa)^*$, using $((e \star f)a)^* = ((ef)^*a)^* = ((ef)a)^*$, where the last equality follows from (2). □

By analogy with previous work [4,6] on what is for us the special case where P_S is a semilattice, a right P-Ehresmann semigroup satisfying the identity stated in the proposition may be termed *right P-hedged*. (The additional modifier 'weakly' used there appears to be redundant.) In the final section of [6] it was shown that not every (right) P-Ehresmann semigroup is right P-hedged, even in case P is a semilattice.

Clearly, all of the above dualizes for left P-Ehresmann semigroups. We will need some of the details in the sequel.

Lemma 3.5. *Let $(S, \cdot, ^+)$ be a left P-Ehresmann semigroup. For any $a \in S$, define $\psi_a : f \mapsto (af)^+, f \in P_S^1$. Then $\psi_a \in$ Ord $_1P_S$. In particular, for any $e \in P_S$, $\psi_e : f \mapsto (ef)^+ = efe$ defines an order-preserving retraction of P_S^1 onto $e\downarrow$.*

Proposition 3.6. *Let $(S, \cdot, ^+)$ be a left P-Ehresmann semigroup. Then the map $\psi : a \mapsto \psi_a$ is a P-separating $^+$-homomorphism of S into the left P-Ehresmann semigroup* Ord $^r_1P_S$ *that induces a $\times$-isomorphism of P_S onto the left projection algebra P' of* Ord $^r_1P_S$.

4. Projection algebras

A *projection algebra* is an algebra $(P, \times, \star)$ that is a left projection algebra under $\times$, a right projection algebra under $\star$ and, further, $\times$ and $\star$ are the reverses of each other, that is, $e \star f = f \times e$. In that case, the two operations induce the same partial order on P.

The analog of Proposition 2.4 and its dual is the following, which is evident from the last sentence of the cited proposition.

Proposition 4.1. *Let $(S, \cdot, ^+, ^*)$ be a P-Ehresmann semigroup. Define binary operations on P_S by $e \times f = (ef)^+ = efe$ and $e \star f = (ef)^* = fef$. Then $(P_S, \times, \star)$ is a projection algebra, which is monoidal if S is a monoid. The partial order induced on $(P_S, \times, \star)$ coincides with the original partial order on P_S.*

Corollary 4.2. *The poset of projections of any regular $*$-semigroup is a projection algebra.*

Examples 2.2 and 2.5 illustrate this corollary.

We now generalize the concept of the Munn semigroup of a semilattice to projection algebras. Refer to Section 7 for the relationship between the construction below and earlier generalizations to regular $*$-semigroups.

If P is any projection algebra, let T_P denote the 'generalized Munn semigroup' whose underlying set consists of all $\star$-preserving (and thus $\times$-preserving) order isomorphisms between principal ideals of P and whose product will be defined after the next, preparatory, lemma. Note that each principal ideal is a subalgebra, since if $e, f \le g$, then $e \star f \le f \le g$. In order to define the new product, we first need to introduce a class of members of T_P.

Recall that for each $g \in P$, π_g is defined by $e\pi_g = e \star g$, $e \in P$. For any $f, g \in P$, let $\pi_{g,f}$ be the restriction of π_g to $(g \star f)\downarrow$. Clearly $\pi_{g,g}$ is the identity map on $g\downarrow$. The partial maps $\pi_{g,f}$ will turn out to be the idempotents of T_P, and the maps $\pi_{g,g}$ will be its projections.

Lemma 4.3. *For any $f, g \in P$, $\pi_{g,f} \in T_P$ and $\pi_{g,f}^{-1} = \pi_{f,g}$. Further, $\pi_{g,f} = \pi_{g,g\star f} = \pi_{f\star g,f} = \pi_{f\star g,g\star f}$.*

Proof. Each mapping $\pi_{g,f}$ is clearly order-preserving. Next we show that $\pi_{g,f}$ and $\pi_{f,g}$ are mutually inverse. It suffices to show that if $e \le g \star f$, then $(e \star g) \star f = e$. Now $e \le g \star f$ is equivalent to $e = e \star (g \star f)$. But the equation $((e \star (g \star f)) \star g) \star f = e \star (g \star f)$ is precisely (P8).

To show $\pi_{g,f}$ is $\star$-preserving, suppose $x, y \in (g \star f)\downarrow$. First observe that

$$(x \star g) \star (y \star g) = (((x \star g) \star g) \star y) \star g = ((x \star g) \star y) \star g,$$

by (P3) and (P2). Next, since $x \in (g \star f)\downarrow$, $x = (x \star g) \star f$, as shown in the previous paragraph. Further, $x \star y \le y \le g \star f \le f$, so $x \star y = (x \star y) \star f$. Thus, applying (P3),

$$x \star y = ((x \star g) \star f) \star y = (((x \star g) \star f) \star y) \star f = (x \star g) \star (y \star f) = (x \star g) \star y.$$

In combination, we obtain $(x \star g) \star (y \star g) = (x \star y) \star g$.

To prove the final statement, we first observe that $\pi_{g,f}$ and $\pi_{g,g\star f}$ are defined by the same rule and, by (P6), have the same domains and ranges. As a result, $\pi_{f,g} = \pi_{f,f\star g}$, and so $\pi_{g,f} = \pi_{f,g}^{-1} = \pi_{f,f\star g}^{-1} = \pi_{f\star g,f}$. In combination, these then yield $\pi_{g,f} = \pi_{g,g\star f} = \pi_{(g\star f)\star g,g\star f} = \pi_{f\star g,g\star f}$, again by (P6). $\square$

The product on T_P is defined as follows. Let $\alpha, \beta \in T_P$, where the range of α is $f\downarrow$ and the domain of β is $g\downarrow$. Put $\alpha \star \beta = \alpha\pi_{g,f}\beta$, where the composition is that in the symmetric inverse semigroup $\mathcal{I}_P$. For $\alpha \in T_P$, let α^{-1} be its inverse in $\mathcal{I}_P$. For any subset X of P, the identity map on X will be denoted 1_X. The following lemma may help accustom the reader to the basic properties of this product.

Lemma 4.4. *Let $\alpha, \beta \in T_P$, as just described. Then $\alpha \star \beta : (g \star f)\alpha^{-1}\downarrow \longrightarrow (f \star g)\beta\downarrow$ and $\alpha \star \beta \in T_P$. It follows that for all $g, f \in P$, $1_{f\downarrow} \star 1_{g\downarrow} = \pi_{g,f}$, otherwise written as $\pi_{f,f} \star \pi_{g,g} = \pi_{g,f}$.*

Proof. Since $g \star f$ belongs to the range of α and $f \star g$ belongs to the domain of β, the first statement is immediate. By Lemma 4.3, the product and its inverse are $\star$-preserving and so are order isomorphisms between principal ideals of P. Now $1_{f\downarrow} \star 1_{g\downarrow} = 1_{f\downarrow}\pi_{g,f}1_{g\downarrow} = \pi_{g,f}$ (since the domain and range of the two last terms coincide). $\square$

Theorem 4.5. *Under the binary and unary operations defined above, $(T_P, \star, ^{-1})$ is a regular $*$-semigroup whose projection algebra is isomorphic to P. Further, the induced bi-unary semigroup $(T_P, \star, ^+, ^*)$ is a P-restriction semigroup whose projection algebra is that of $(T_P, \star, ^{-1})$ and is thus again isomorphic to P.*

For $\alpha \in T_P$, α^+ and α^* are the identity maps on its domain and range, respectively. The idempotents of T_P are precisely the partial maps $\pi_{e,f}$, $e, f \in P$, and its projections are the identity maps on the principal ideals of P, that is, the maps $\pi_{e,e}$, $e \in P$.

The regular $*$-semigroup $(T_P, \star, ^{-1})$ is fundamental as a regular semigroup and as a regular $*$-semigroup, and P-fundamental as a P-restriction semigroup.

Proof. To prove associativity, let $\alpha, \beta, \gamma \in T_P$, where $\alpha : e\downarrow \longrightarrow f\downarrow$, $\beta : g\downarrow \longrightarrow h\downarrow$ and $\gamma : k\downarrow \longrightarrow l\downarrow$. Put $a = (g \star f)\alpha^{-1}$ and $b = (f \star g)\beta$, so that $\alpha \star \beta : a\downarrow \longrightarrow b\downarrow$. Thus the domain of $(\alpha\star\beta)\star\gamma$ is generated by $(k\star b)(\alpha\star\beta)^{-1} = (k\star b)\beta^{-1}\pi_{g,f}\alpha^{-1}$. Now $b \le h$, so $k \star b = k \star (b \star h) = (((k \star h) \star b) \star h) = (k \star h) \star b$, where we have applied (P3). Here both $k \star h$ and b lie in the range of β, so $((k \star h) \star b)\beta^{-1} = (k \star h)\beta^{-1} \star b\beta^{-1}$.

Put $c = (k\star h)\beta^{-1} \le g$. We have shown that $(k\star b)\beta^{-1} = c \star (f \star g)$. Now by (P3), $c \star (f \star g) = ((c \star g) \star f) \star g = (c \star f) \star g$. Further, $((c \star f) \star g)\pi_{g,f}^{-1} = ((c \star f) \star g) \star f = (c \star g) \star f = c \star f$, similarly.

Hence the domain of $(\alpha \star \beta) \star \gamma$ is generated by $(c \star f)\alpha^{-1}$. Now by the definition of c, it generates the domain of $\beta \star \gamma$. Thus the domain of $\alpha \star (\beta \star \gamma)$ is also generated by $(c \star f)\alpha^{-1}$. That their ranges are also equal follows by symmetry.

When defined, $x((\alpha \star \beta) \star \gamma) = ((x\alpha \star g)\beta \star k)\gamma$. Similarly, $x(\alpha \star (\beta \star \gamma)) = (x\alpha \star c)(\beta \star \gamma) = ((x\alpha \star c)\beta \star k)\gamma$. It suffices to prove, then, that $(x\alpha \star c)\beta \star k = (x\alpha \star g)\beta \star k$. Now since $c \le g$, then as above $x\alpha \star c = (x\alpha \star g) \star c$ and so $(x\alpha\star c)\beta = (x\alpha\star g)\beta\star c\beta = (x\alpha\star g)\beta\star(k\star h)$. It follows that $(x\alpha\star c)\beta\star k = ((x\alpha\star g)\beta\star(k\star h))\star k$. Put $y = (x\alpha\star g)\beta \le g\beta = h$. Now, applying (P3) twice and the fact that $y \le h$, $((y \star (k \star h)) \star k = (((y \star h) \star k) \star h) \star k = ((y \star k) \star h) \star k = y \star (h \star k)$. Further, again since $y \le h$, $y \star (h \star k) = (y \star h) \star (h \star k) = (y \star h) \star k = y \star k$, where this time (P7) was applied. Thus $((y \star (k \star h)) \star k = y \star k$, as required.

Let $\alpha \in T_P$, $\alpha : e\downarrow \longrightarrow f\downarrow$, say. Then $\alpha^+ = \alpha \star \alpha^{-1} = \alpha\pi_{f,f}\alpha^{-1} = \alpha 1_{f\downarrow}\alpha^{-1} = 1_{e\downarrow}$. Dually, $\alpha^* = \alpha^{-1} \star \alpha = 1_{f\downarrow}$. Thus α^{-1} is an inverse of α and T_P is a regular semigroup. Clearly $(\alpha^{-1})^{-1} = \alpha$. If $\beta \in T_P$, where $\beta : g\downarrow \longrightarrow h\downarrow$, then

$$(\alpha \star \beta)^{-1} = (\alpha\pi_{g,f}\beta)^{-1} = \beta^{-1}\pi_{f,g}\alpha^{-1} = \beta^{-1} \star \alpha^{-1}.$$

Hence S is a regular $*$-semigroup. From the discussion in the first paragraph of the proof, it is clear that the set P' of projections of T_P consists of the identity maps on the principal ideals $e\downarrow$, (that is, the maps $\pi_{e,e}$), $e \in P$. To avoid confusion, we denote the operation on the right projection algebra induced on P' by $\odot$. If $e, f \in P$, then

$$1_{e\downarrow} \odot 1_{f\downarrow} = 1_{f\downarrow} \star 1_{e\downarrow} \star 1_{f\downarrow} = \pi_{e,f} \star 1_{f\downarrow} = \pi_{e,f}\pi_{f,f\star e}1_{f\downarrow} = \pi_{e,f}\pi_{f,e}1_{f\downarrow} = 1_{(e\star f)\downarrow}1_{f\downarrow} = 1_{(e\star f)\downarrow},$$

where we have used the formulas proven in Lemmas 4.3 and 4.4 and the fact that $e \star f \le f$. Thus the map $e \mapsto 1_{e\downarrow}$ is an isomorphism of $(P, \star)$ with $(P', \odot)$.

Now by Result 1.1(c), $E_{T_P} = P' \odot P' = \{\pi_{e,f} : e, f \in P\}$, using Lemma 4.4.

That the final statements of the proposition are equivalent was demonstrated at the end of Section 1. Let $\alpha, \beta \in T_P$ and suppose $\alpha\mu\beta$. By the description of μ in Corollary 5.3 below (or by [10, Theorem 4]), $\alpha^+ = \beta^+$ and $(\gamma \star \alpha)^* = (\gamma \star \beta)^*$ for all $\gamma \in P'$ such that $\gamma \le \alpha^+$. Thus $\alpha, \beta : e\downarrow \longrightarrow f\downarrow$, for some $e, f \in P$, and for all $g \le e$, $1_{g\downarrow} \star \alpha$ and $1_{g\downarrow} \star \beta$ have the same range. Now by Lemma 4.4, $1_{g\downarrow} \star \alpha$ maps $g\downarrow = (e \star g)\downarrow$ to $(g \star e)\alpha\downarrow = g\alpha\downarrow$; and similarly for β. It follows that $g\alpha = g\beta$, so that $\alpha = \beta$. □

Corollary 4.6. *In the special case that the projection algebra is a semilattice, T_P is the usual Munn semigroup on P.*

Proof. By the proof of Lemma 2.10, the operation $\star$ provides the meet in the semilattice $(P, \le)$. Thus T_P consists of the usual isomorphisms between principal ideals of that semilattice. Consider the definition of the product $\alpha \star \beta$, as stated prior to Lemma 4.4. Now by Lemma 4.3, $\pi_{g,f} = \pi_{f\star g, g\star f} = 1_{(f\star g)\downarrow}$ and so $\alpha \star \beta = \alpha\beta$, the usual product. □

We observe that the construction of T_P may be repeated with $\times$ in place of $\star$, with appropriate dualization. The resulting semigroup will then be the reverse semigroup T_P^r. The next result should be viewed as an extension of Corollary 2.7.

Corollary 4.7. *Let $(P, \le)$ be a poset. Then the following are equivalent:*

1. *P can be endowed with the structure of a projection algebra;*
2. *P is the poset of projections of a regular $*$-semigroup;*
3. *P is the poset of projections of a P-Ehresmann semigroup;*
4. *P is the poset of projections of a P-restriction semigroup.*

5. Representations for *P*-restriction semigroups

Let $(S, \cdot, ^+, ^*)$ be a P-Ehresmann semigroup. Then $(P_S, \times, \star)$ is a projection algebra. Thus we may combine the two representations $\theta : S \longrightarrow \mathrm{Ord}\,{}_1P_S$ and $\psi : S \longrightarrow \mathrm{Ord}\,{}_1^rP_S$ defined in Section 3 into a representation $S \longrightarrow \mathrm{Ord}\,{}_1P \times \mathrm{Ord}\,{}_1^rP_S$. This was the approach of Imaoka [10,11] for regular $*$-semigroups and for Gomes and Gould [6] for Ehresmann semigroups, extending work in [4]. See the further discussion following Corollary 5.3.

We prefer, however, to make use of the additional structure provided by the identities (6) to provide a representation of P-restriction semigroups S in the 'Munn'-type semigroup T_{P_S}. As noted below, this representation specializes to 'classical' representations both of regular $*$-semigroups and of restriction semigroups (and at root, of inverse semigroups, which are common to both classes of semigroups). We first consider how the identities (6) are reflected in properties of the two homomorphisms θ and ψ. Note that although ψ was regarded as a homomorphism into $\mathrm{Ord}\,{}_1^rP_S$, we may equally well regard it as an antihomomorphism into $\mathrm{Ord}\,{}_1P_S$ itself. In the next proposition, we consider all the maps θ_a and ψ_a as members of $\mathrm{Ord}\,{}_1P_S$. Note that for any $f \in P$, the dual map σ_f now coincides with π_f, and we use the latter notation solely. Recall that in this context $e\pi_f = e \star f = fef$, $e, f \in P_S^1$.

Let $a \in S$. Recall that $\theta_a : P_S^1 \longrightarrow P_S$ is defined by $e\theta_a = (ea)^*$, $e \in P_S^1$. Now denote by θ'_a the restriction of θ_a to the principal ideal $a^+\downarrow$ of P_S. Clearly, θ'_a is again order-preserving on its domain, and its range includes $a^+\theta = (a^+a)^* = a^*$. Dually, denote by ψ'_a the restriction of ψ_a to $a^*\downarrow$.

Lemma 5.1. *Let $(S, \cdot, ^+, ^*)$ be a P-restriction semigroup. Let $a \in S$. Then*

(i) *for all $e \in P_S^1$ and $a \in S$, $e\theta_a = (a^+ea^+)\theta_a$, that is, $\theta_a = \pi_{a^+}\theta_a = \pi_{a^+}\theta'_a$;*
(ii) *$\theta_a\psi_a = \pi_{a^+}$ and $\psi_a\theta_a = \pi_{a^*}$;*
(iii) *thus $\theta_a\psi_a\theta_a = \theta_a$ and $\psi_a\theta_a\psi_a = \psi_a$;*
(iv) *the partial maps θ'_a and ψ'_a are mutually inverse order-isomorphisms between the principal ideals $a^+\downarrow$ and $a^*\downarrow$ of P_S; further, θ'_a is $\star$-preserving and ψ'_a is $\times$-preserving.*

Proof. In all the relevant cases, dualization yields the second statement from the first. To prove (i), observe that $((a^+ea^+)a)^* = (a^+ea)^* = (a(ea)^*)^* = (ea)^*a^*(ea)^* = (ea)^*$, applying (1), (6), (7), and (10) in turn.

(ii) Let $e \in P_S^1$. We must show that $e\theta_a\psi_a = a^+ea^+$. By (6), $(a(ea)^*)^+ = (a^+ea)^+ = (a^+ea^+)^+ = a^+ea^+$.

(iii) This is immediate from (i) and (ii).

(iv) The first statement is clear from (i) and the fact that for any $f \in P_S$, π_f is a retraction onto $f\downarrow$. To prove that θ'_a is $\star$-preserving, let $e, f \in a^+\downarrow$. Observe that $e\theta'_a \star f\theta'_a \in a^*\downarrow$, so it suffices to show that the image of this element under ψ_a is $e \star f$. First we compute $a(e\theta_a \star f\theta_a) = a((ea)^* \star (fa)^*) = a((fa)^*(ea)^*(fa)^*)$. By (6), $a(fa)^* = a^+fa = fa$. Repeating this process twice, we obtain $a(fa)^*(ea)^*(fa)^* = fefa$ and so $(e\theta'_a \star f\theta'_a)\psi_a = (fefa)^+ = (fefa^+)^+ = fef = e \star f$. □

According to this lemma, the image of the map $\theta' : a \mapsto \theta'_a$ is contained in T_{P_S}. The image of $\psi' : a \mapsto \psi'_a$ is again a subset of T_{P_S}, but will turn out to be a subsemigroup of the reverse semigroup $T^r_{P_S}$. Denote by γ the anti-isomorphism $T_{P_S} \longrightarrow T^r_{P_S}$ that is induced by inversion. Recall that a subsemigroup of a semigroup S is *full* if it contains all the idempotents of S.

Theorem 5.2. *Let $(S, \cdot, ^+, ^*)$ be a P-restriction semigroup, with projection algebra $(P_S, \times, \star)$. Then θ' is a $^+$- and *-preserving homomorphism of S onto a full subsemigroup of the regular $*$-semigroup T_{P_S}, which induces an isomorphism between their respective projection algebras. Dually, S is represented via ψ in $T^r_{P_S}$, with the same properties. The representations are related by $\psi' = \theta'\gamma$.*

If, moreover, S is the P-restriction semigroup that is induced from some regular $*$-semigroup $(S, \cdot, ^{-1})$, then θ' also preserves the inverse operation from the latter semigroup.

Proof. In view of the results of this section, for the statements in the first paragraph it only remains to prove that $\theta'_a \star \theta'_b = \theta'_{ab}$ for all $a, b \in S$, and that the image is full. According to Lemma 4.4, the domain of $\theta'_a \star \theta'_b$ is generated by $(b^+ \star a^*)\theta'^{-1}_a = (b^+ \star a^*)\psi'_a = (a^*b^+a^*)\psi'_a = (a(a^*b^+a^*))^+$. Applying (6) and the duals of (7), (10), and (4) in turn, we then obtain $(ab^+a^*)^+ = ((ab)^+a)^+ = (ab)^+a^+(ab)^+ = (ab)^+$. But $(ab)^+$ generates the domain of θ'_{ab}. Similarly, the ranges are identical.

If $e \le (ab)^+$, then, similarly to the proof of Theorem 3.2, $e\theta'_{ab} = (e(ab))^* = ((ea)b)^* = ((ea)^*b)^* = (e\theta'_a)\theta'_b$. Now $e(\theta'_a \star \theta'_b) = e\theta'_a\pi_{b^+,a^*}\theta'_b = ((e\theta'_a) \star a^*)\theta'_b$. But since $e \le a^+$, $e\theta'_a \le a^*$ and thus $e\theta'_a = (e\theta'_a) \star a^*$. So $e(\theta'_a \star \theta'_b) = e\theta'_{ab}$.

To prove that the image is full, recall from Theorem 4.5 and Lemma 4.4 that the idempotents of T_{P_S} are precisely the maps $\pi_{e,f}$, $e, f \in P_S$, and that $\pi_{e,f} = \pi_{f,f} \star \pi_{e,e} = \theta'_f \star \theta'_e = \theta'_{fe} \in S\theta'$.

To prove the final statement, we need to verify that for any $a \in S$, $\theta'_{a^{-1}} = \psi'_a$. Now since $(a^{-1})^+ = a^*$ and $(a^{-1})^* = a^+$, the domains and ranges of these two partial maps agree; and if $f \le a^*$, then $f\theta' = (fa^{-1})^* = (fa^{-1})^{-1}(fa^{-1}) = afa^{-1} = (af)(af)^{-1} = (af)^+ = f\psi'$. (This also follows from the last paragraph of Section 1.) □

Corollary 5.3 (*Cf Remarks in Section* 1). *Let S be a P-restriction semigroup and $a, b \in S$. Then $a\mu b$ if and only if $a^+ = b^+$ and $(ea)^* = (eb)^*$ for all $e \in P$, $e \le a^+$, and if and only if $a^* = b^*$ and $(af)^+ = (bf)^+$ for all $f \in P$, $f \le a^*$. Thus the image of S in T_{P_S} under θ' is P-fundamental.*

Proof. The indicated relation is clearly the congruence induced by θ' and therefore separates projections. Clearly, if $a\mu b$, then $a^+\mu b^+$, so $a^+ = b^+$; and if $e \in P$, then $ea\mu eb$, so that $(ea)^* = (eb)^*$. The dual statement follows similarly. The last statement follows from remarks in Section 1. □

In Section 7, we relate the specialization of Theorem 5.2 to regular $*$-semigroups with the literature on that topic. In Section 6, we do the same for restriction semigroups.

Analysis of the proof of Theorem 5.2 makes clear that the identities (6) play an integral role. As discussed in [6], using the combination of representations alluded to in the introduction to this section should be key to extending our results to a more general setting. We will not pursue that approach here. However, it is again interesting (cf Section 3 and [6, Lemma 4.1]) to determine when the (total) maps θ_a are not merely order-preserving but are algebra homomorphisms. In the case of a general P-Ehresmann semigroup, this is simply a combination of Proposition 3.4 and its dual, that is, the combination of the right P-hedged property, introduced following that proposition, and its dual.

In the case of P-restriction semigroups, stronger statements may be made. As we show below, the one-sided P-hedged properties reduce to a common identity that can be stated in terms of projections only. It would be of interest to study further the semigroups that satisfy this identity and to investigate how it might be extended beyond the context of P-restriction semigroups, where we already have the representation in T_P. (Note that in the context of [6], the 'ample' identities satisfied by restriction semigroups – see Section 6 – imply the 'hedged' properties, whereas the identities (6) do not imply the identity in (iii) of the next corollary, as was observed in Example 2.5.)

Corollary 5.4. *Let $(S, \cdot, ^+, ^*)$ be a P-restriction semigroup. The (total) map θ_a is $\star$-preserving if and only if π_{a^+} is $\star$-preserving and if and only if $a^+fa^+ea^+fa^+ = a^+fefa^+$ for all $e, f \in P_S$. In the terminology of Section* 3, *the following are therefore equivalent:*

(i) *θ is a representation by $\star$-endomorphisms of the right projection algebra $(P^1_S, \star)$;*
(ii) *ψ is a representation by $\times$-endomorphisms of the left projection algebra $(P^1_S, \times)$;*
(iii) *S satisfies the identity that may be expressed as $gfgegfg = gfefg$, where $e, f, g \in P_S$;*
(iv) *the operation $\star$ is associative, equivalently the operation $\times$ is associative.*

In that case, $(P_S, \star)$ is a right regular band (and $(P_S, \times)$ is a left regular band).

Proof. To prove the first set of statements, let $a \in S$, $e, f \in P_S$. By Lemma 5.1(i) and (iv), $e\theta_a \star f\theta_a = e\pi_{a^+}\theta'_a \star f\pi_{a^+}\theta'_a = (e\pi_{a^+} \star f\pi_{a^+})\theta'_a$. Again by (i), $(e\star f)\theta_a = (e\star f)\pi_{a^+}\theta'_a$. The first statement is then clear. The second one is simply a restatement in terms of the operations on S.

It is clear from the above that every map θ_a is $\star$-preserving if and only if every map π_g, $g \in P_S$, is $\star$-preserving. According to Corollary 2.9, this is equivalent to associativity of $\star$ and thus to (iv). Now $(e\star f)\pi_g = g(fef)g$ and $e\pi_g \star f\pi_g = (gfg)(geg)(gfg)$. Thus (i) is equivalent to (iii). Now (iii) is self-dual, and so is also equivalent to (ii).

The final statement follows from Corollary 2.9 and its dual. □

6. Projection sets and the 'York school' approach

In this section we show how the classes of semigroups that we have defined generalize various classes previously considered by others, focusing on the approach of Fountain et al: the so-called 'York school'. We rely considerably on a historical survey by Hollings [8] and the unpublished, but widely cited, notes of Gould [7]. The former includes a welcome tabulation of the terminology used by the York school. The varietal approach used in the latter formed the model for the author's approach to this paper. We refer the reader to these two papers for further information. We should note that where we have concentrated on 'right' properties in this paper, the authors of those two references have chosen the dual perspective.

The present section is essentially an extraction of the relevant facts from the author's analysis [12] of the topics contained herein in their broadest context: that in which the sets P are entirely arbitrary, at least *a priori*. However, we have made it largely self-contained.

As usual, $\mathcal{L}$ denotes the relation on a semigroup S defined by $\{(a, b) : S^1a = S^1b\}$. For any nonempty subset P of the set E_S of idempotents of S, define

$$\widetilde{\mathcal{L}}_P = \{(a, b) : ae = a \Leftrightarrow be = b, \quad \forall e \in P\}.$$

It is easily verified that $\mathcal{L} \subseteq \widetilde{L}_P$ and that, when restricted to P, $\mathcal{L}$ and $\widetilde{\mathcal{L}}_P$ coincide. Recall that a *right unit* for an element a of S is an idempotent e of S such that $ae = a$. The set of right units of a that belong to P is denoted a_P. Thus $a \widetilde{\mathcal{L}}_P b$ if and only if $a_P = b_P$.

Following [8], we call S *weakly right P-abundant* if every $\widetilde{\mathcal{L}}_P$-class of S contains a member of P. There appears to be no standard nomenclature for the general property that every $\widetilde{\mathcal{L}}_P$-class contain a *unique* member of P. With the understanding that the prefix 'P-' will clarify any ambiguity, we propose the term *weak right P-adequacy* to describe this situation in general terms. Traditionally, the term 'adequacy' and its variants have been used exclusively in case P is a subsemilattice of E_S. (We should note that, rather than 'weakly right P-abundant' and 'weakly right P-adequate', the terms 'right P-semiabundant' and 'right P-semiadequate' have also been used – at least with E, rather than P, as the prefix – for example by Hollings [8].)

If S is weakly right P-adequate, as defined above, then for each $a \in S$, denote by a^* the unique element of $a\widetilde{\mathcal{L}}_P \cap P$. Under this assignment, $(S, \cdot, ^*)$ becomes a unary semigroup and $a\widetilde{\mathcal{L}}_P b$ if and only if $a^* = b^*$, for all $a, b \in S$. As elsewhere in this paper, let $P_S = \{a^* : a \in S\}$. Clearly, $P_S = P$ for the unary operation just defined.

The sets of projections of right P-Ehresmann semigroups have considerable structure, though not the structure of a band, or even a semilattice, that has traditionally been assumed when studying abundancy and, especially, adequacy.

Let S be any semigroup. A *right projection-set* is a nonempty subset P of E_S that satisfies the following properties, the first two of which are intrinsic, the third extrinsic.

(Pr1) $efe \in P$ for all $e, f \in P$;
(Pr2) $P^2 \subseteq E_S$;
(Pr3) for each $a \in S$, a_P contains a least member under the usual partial order on E_S.

In addition, we consider the following property (which is sometimes instead denoted (CR) in the literature):

(cr) $\widetilde{\mathcal{L}}_P$ is a right congruence on S.

That the set P_S of projections of a right P-Ehresmann semigroup $(S, \cdot, ^*)$ satisfies (Pr1) is immediate from (3); (Pr2) is just (11); if $a \in S$, then $a^* \in a_P$ by (1) and, for $e \in a_P$, $a^* = (ae)^* \leq e$, by (12), so that a^* is the least element of a_P and (Pr3) is satisfied.

The term *right E-Ehresmann semigroup* has been used for weakly right P-adequate semigroups such that $P = E$ is a semilattice and satisfies (cr). (Actually, that is the terminology of Gould in [7]; Hollings simply uses the term *right Ehresmann*, which in [7] specifically assumes that $P = E_S$, in addition.) The main result of this section demonstrates that the terminology *right P-Ehresmann semigroup* used throughout our paper is consistent with the historical usage. As mentioned earlier, this result appears in [12] as a consequence of much more general considerations. To keep this paper self-contained, we include a direct proof.

Theorem 6.1 (*[12, Proposition 9]*). *The following are equivalent for a semigroup S:*

(i) *S is weakly right P-adequate with respect to a subset P of E_S that satisfies (Pr1), (Pr2) and (cr);*
(ii) *S contains a right projection-set P for which $\widetilde{\mathcal{L}}_P$ is a right congruence;*
(iii) *S can be endowed with a unary operation * such that $(S, \cdot, ^*)$ is a right P-Ehresmann semigroup.*

In that case, the subsets P in (i) and (ii) coincide with the set P_S of projections in (iii).

Proof. (i) ⇒ (ii). Only (Pr3) need be verified. Let $a \in S$ and let a^* be the unique member of $a\widetilde{\mathcal{L}}_P \cap P$, according to the definition of weak right P-adequacy. Since $a\widetilde{\mathcal{L}}_P a^*$ and $a^* \in E_S$, $a^* \in a_P$. For any $e \in a_P$, $e \in a^*_P$, that is, $a^*e = a^*$. Now by (Pr1), $ea^* = ea^*e \in P$ and, since $ea^* \,\mathcal{L}\, a^*$, $ea^*\widetilde{\mathcal{L}}_P a^*$, so that $ea^* = a^*$, by assumption. Thus $a^* \leq e$, as required.

(ii) ⇒ (i). Let $a \in S$ and let g be the least element of a_P prescribed by (Pr3). If $ae = a$, then $ge = g$ and conversely (since $a = ag$), so $g\widetilde{\mathcal{L}}_P a$. Suppose $e, f \in P$ and $e\widetilde{\mathcal{L}}_P f$, that is, $e \,\mathcal{L}\, f$. Then (Pr3) implies that $e \leq f$ and $f \leq e$, so that $e = f$. Hence S is weakly right P-adequate.

Observe that, as a result of the proof so far, for each $a \in S$, the element a^* of a_P defined by (i) coincides with that defined in (ii) by (Pr3). Since for $a, b \in S$, $a\widetilde{\mathcal{L}}_P b$ if and only if $a^* = b^*$, it is now apparent that (cr) is equivalent to satisfaction of the identity (2). (Cf [12, Lemma 8], which generalizes [8, Lemma 4.8]).

(ii) ⇒ (iii). For $a \in S$, define a^* to be the member of P determined by (Pr3). Now (1) follows from the fact that $a^* \in a_P$; (4) from $P \subseteq E_S$; and (2) is immediate from (cr). To prove (3), let $x, y \in S$ and put $e = x^*, f = y^*$. Then, by (Pr2), $(ef)(fef) = ef$ and, by (Pr1), $fef \in P$. Thus $fef \in (ef)_P$ and, by (Pr3), $(ef)^* = (fef)(ef)^* = fef$.

(iii) ⇒ (ii). It was shown earlier that P_S is a right projection-set. Now (cr) follows from (2), as noted above. □

We conclude this vein of study by citing a further result from [12] that clarifies the role of right projection-sets themselves and thus the distinct role of (cr) in this paper. It is shown there (Corollary 7) that a semigroup S contains a projection-set if and only if S can be endowed with a unary operation * such that $(S, \cdot, ^*)$ satisfies (1), (3), (4), (9) and (10), and if and only if it is weakly right P-adequate with respect to a nonempty subset P of E_S that satisfies (Pr1) and (Pr2). In terms of the current paper – and the literature on this general topic – the property (cr) has been essential in order to obtain "Munn-type" representations of the kind found herein. Of course, this property is also one naturally held by regular *-semigroups, one of the classes of semigroups that motivated this paper.

Generalized right restriction semigroups were defined by Gould [7] (actually, she defined the dual of this notion) as the semigroups that, in our language, are weakly right P-adequate with P a band. The result cited in the last paragraph was further specialized in [12] to provide identities for such semigroups (cf [7, Corollary 3.6]) and for those in which, even stronger, P is a semilattice, in other words a commutative subsemigroup of S (cf [7, Corollary 3.10]).

It is appropriate here to characterize several natural specializations of the right P-Ehresmann property. Observe from (ii) of the next result that if P_S is a subband of a right P-Ehresmann semigroup, then the right projection algebra $(P_S, \star)$ is isomorphic to $(P_S, \cdot)$, so that $\star$ is associative and Corollary 2.9 applies. Following that corollary, an example was given to show that the latter property is a strictly weaker one. In light of (iii) and (iv) below, Proposition 2.11 is also of particular relevance. The generalized right restriction semigroups were defined above. A *right restriction semigroup* is a right E-Ehresmann semigroup that, in addition, satisfies the 'right ample' condition (ar), which in terms of the operation * may be expressed as the identity $x(yx)^* = y^*x$. The older term is *weakly right E-ample semigroup*.

Before stating the proposition, we note from [12, Lemma 10] that if a weakly P-adequate semigroup satisfies (ar), then necessarily P is a subband. (This is true without any additional hypotheses on P at all.)

Proposition 6.2. *Let $(S, \cdot, ^*)$ be a right P-Ehresmann semigroup.*

(i) *If $P_S = E_S$, then P_S is a subband of S;*
(ii) *P_S is a subband of S if and only if S satisfies $(x^*y^*)^* = x^*y^*$, in which case $(P_S, \cdot)$ is a right regular band, the operations $\cdot$ and $\star$ on P_S coincide, so that the right projection algebra $(P_S, \star)$ is isomorphic to $(P_S, \cdot)$, and S is a generalized right restriction semigroup that, in addition, satisfies (cr);*
(iii) *P_S is a semilattice if and only if S is a right E-Ehresmann semigroup;*
(iv) *S is a right restriction semigroup if and only if P_S is a semilattice and S satisfies (ar).*

Proof. (i) This is immediate from (11).
(ii) The first equivalence is clear. Under this hypothesis, $efe = (fe)^* = fe$, that is, the band P_S is right regular. Thus $f \star e = (fe)^* = fe$. That the resulting semigroups are generalized right restriction follows from the discussion above.
(iii) and (iv) follow from the definitions and the earlier discussion. □

Corollary 6.3. *If S is a generalized right restriction semigroup, then the representation θ in Theorem 3.2 is by order-preserving maps of the (right regular) subband $(P_S^1, \cdot)$. In particular, if S is a right P-Ehresmann semigroup that satisfies (ar), the representation is by endomorphisms of the band P_S^1.*

Proof. The first statement follows from (ii) of the proposition. For the second we first recall from the remark preceding the proposition that P_S is necessarily a (right regular) band. We then apply Proposition 3.4 by showing that S is right P-hedged, that is, that $(efa)^* = (fa)^*(ea)^*(fa)^*$ for all $e, f \in P_S, a \in S$. Here the right hand side is just $(ea)^*(fa)^*$. Now two applications of (ar) yield $a(ea)^*(fa)^* = efa$ and then (2) gives $(efa)^* = (a(ea)^*(fa)^*)^* = (a^*(ea)^*(fa)^*)^* = (ea)^*(fa)^*$. □

Recall from Section 1 that on any right P-Ehresmann semigroup S, μ_L denotes the greatest P-separating congruence on S. A description of μ_L was given in Corollary 3.3. The following result follows immediately from the fact that for $a, b \in S$, $a\widetilde{\mathcal{L}}_P b$ if and only if $a^* = b^*$, so that a congruence on S (that respects *) separates P_S if and only if it is contained in $\widetilde{\mathcal{L}}_P$.

Proposition 6.4. *On any right P-Ehresmann semigroup $(S, \cdot, ^*)$, μ_L is the greatest congruence on S that is contained in $\widetilde{\mathcal{L}}_P$.*

The traditional approach to this general topic was based on the relation $\mathcal{L}^*$, rather than $\widetilde{\mathcal{L}}_P$. See [12] for discussion of how the results of this section specialize to that situation. Furthermore, historically an intermediate stage involved the generalization from reference to $\mathcal{L}^*$ to reference to the case $P = E_S$. In our situation, this requirement is of no interest as, by Proposition 6.2, it forces P to be a subband. As noted below, a plausible substitute is to posit that $P^2 = E_S$.

We leave it to the reader to formulate the dual versions of the above definitions and results, other than noting that (cl) and (al) denote the duals of (cr) and (ar), respectively.

Turning now to the two-sided case, we follow historical precedent by dropping the adjective 'right' or 'left' from the terminology above to define P-semiabundant, P-semiadequate and P-Ehresmann semigroups as those that have, in the respective cases, both the right and left properties and the same sets of projections on both sides. A *projection-set* is then a nonempty subset P of E_S that satisfies the intrinsic properties (Pr1) and (Pr2) and the extrinsic properties (Pr3) and its dual.

The two-sided version of Theorem 6.1 then states the following, demonstrating the consistency of our terminology in Section 1 with historical usage.

Theorem 6.5. *The following are equivalent for a semigroup S:*

(i) *S is weakly P-adequate with respect to a subset P of E_S that satisfies (Pr1), (Pr2), (cr) and (cl);*
(ii) *S contains a projection-set P for which $\widetilde{\mathcal{L}}_P$ is a right congruence and the dual relation $\widetilde{\mathcal{R}}_P$ is a left congruence;*
(iii) *S can be endowed with unary operations $^+$ and * such that $(S, \cdot, ^+, ^*)$ is a P-Ehresmann semigroup.*

In the two-sided case, we have also defined in Section 1 the P-restriction semigroups: the P-Ehresmann semigroups that, in addition, satisfy the identities (6). In the case that P_S is a semilattice, those identities reduce respectively to (al) and (ar), and so S is a restriction semigroup. We term the two identities in (6) the generalized left and right ample conditions (gal) and (gar), respectively. From Proposition 6.2 and its dual, if P_S is a band in a P-Ehresmann semigroup, then it is both left and right regular, whence a semilattice. According to the comments that precede that proposition, this is necessarily the case if P satisfies (al) and (ar).

Finally, we deduce from Theorem 5.2 the known representation of restriction semigroups in the 'classical' Munn semigroup.

Corollary 6.6. *Let $(S, \cdot, ^+, ^*)$ be a P-restriction semigroup that satisfies $x^*y^* = y^*x^*$, that is, S is a restriction semigroup. Then T_{P_S} is the Munn semigroup of the semilattice P_S, which is an inverse semigroup, and the image of S under θ' is a full subsemigroup of T_{P_S} whose idempotents coincide with its projections, that is, it is an ample semigroup, in the traditional terminology.*

The relation $\widetilde{\mathcal{H}}_P$ is defined to be the intersection of $\widetilde{\mathcal{L}}_P$ and $\widetilde{\mathcal{R}}_P$. Then (cf Proposition 6.4) on any P-Ehresmann semigroup, the congruence μ is the greatest congruence contained in $\widetilde{\mathcal{H}}_P$.

7. Specialization to regular ∗-semigroups

The appropriate converse to the fact that regular ∗-semigroups induce P-restriction semigroups is provided by the next proposition. The proof can be expedited by using Yamada's characterization [20] of the sets of projections in regular ∗-semigroups as 'P-systems', within the class of regular semigroups. A *P-system* in a regular semigroup S is a subset P of E_S such that (a) for any $a \in S$, there exists a unique inverse a' of a (in the general sense) for which $aa', a'a \in P$, (b) for any $a \in S$, $a'Pa \subseteq P$, where a' is defined as in (a), and (c) $P^2 \subseteq E_S$.

Proposition 7.1. *If a P-restriction semigroup S satisfying $E_S = P_S^2$ is a regular semigroup, then it can be endowed with the structure of a regular ∗-semigroup.*

Proof. Let S be such a semigroup. We show that $P = P_S$ is a P-system in S. Clearly, (c) is satisfied. See [9] for the basic properties of regular $\mathcal{D}$-classes that we use in the following. Let $a \in S$. By regularity, the $\mathcal{L}$-class L_a contains an idempotent e, say. From $e = ee^*$ it follows that $e^*e \mathrel{\mathcal{L}} e$; and from $e = e^+e^*$ (see Lemma 1.5) that $e^*e = e^*e^+e^* \in P$. Thus $e^* = e^*e \mathrel{\mathcal{L}} e$. From $a \mathrel{\mathcal{L}} e$ it follows that $a^* = e^*$ (either by application of (1) and (3) or by using the relation $\widetilde{\mathcal{L}}_P$ defined in Section 6). Thus $a^* \mathrel{\mathcal{L}} a$. Dually, $a^+ \mathrel{\mathcal{R}} a$. Let a' be the inverse of a that belongs to $R_{a^*} \cap L_{a^+}$. If a'' is any inverse of a such that $aa'', a''a \in P$, then since $aa'' \mathrel{\mathcal{R}} a^+$ and $a''a \mathrel{\mathcal{L}} a^*$, $aa'' = a^+$ and $a''a = a^*$, whence $a'' = a'$, since no $\mathcal{H}$-class contains more than one inverse of a. Thus property (a) is satisfied in the definition of P-system.

Again let $a \in S$, with a' as above, and now let $e \in P$. Then $a'ea = a'(aa')ea = a'(a^+ea) = a'(a(ea)^*) = a^*(ea)^* = (ea)^* \in P$, applying identity (6) and property (12). Thus (b) is satisfied. □

P-systems are the analogues in regular ∗-semigroups of the projection-sets in Section 6, the *internal* characterization of the respective sets of projections. We now return to the *external* characterization. As noted at the start of Section 2, the external characterization of the sets of projections of right P-Ehresmann semigroups was obtained independently of Imaoka's characterization in [11] of the sets of projections of regular ∗-semigroups, and the form of the latter is superficially quite different. By Corollary 4.7, they are equivalent, so it behooves us to make the connection explicit.

Imaoka defined a *P-groupoid (with respect to θ)* in the following way. Let P be a set and $\theta : e \mapsto \theta_e$ a mapping of P to the full transformation semigroup on P. Suppose the pair (P, θ) satisfies the following axioms:

(P′1) $e\theta_e = e$;
(P′2) $\theta_e\theta_e = \theta_e$;
(P′3) $e\theta_f\theta_e = f\theta_e$;
(P′4) $\theta_e\theta_f\theta_e = \theta_{f\theta_e}$;
(P′5) $\theta_f\theta_e\theta_f\theta_e = \theta_f\theta_e$.

Then P becomes a partial groupoid under the partial operation $ef = e\theta_f$, defined if and only if $e\theta_f = f\theta_e$.

The connection is straightforward. On the one hand, if $(P, \star)$ is a right projection algebra, define $f\theta_e = f \star e$. That is, θ_e is the mapping we have denoted by π_e. On the other hand, if (P, θ) is a P-groupoid, define the (complete) operation $\star$ on P by $f \star e = f\theta_e$. We will show how each axiom system is a consequence of the other.

Axiom (P′1) is just our (P1); (P′2) is one part of our (P2); (P′3) is the first part of our (P6); (P′4) is our (P3); and (P′5) follows from (P7) by an application of (P3). Conversely, the second part of our (P2) follows from setting $g = e$ in (P3) and applying the first equation in (P6), in other words, from (P′4) and (P′3); the displayed equation in the proof of (P7) in Lemma 2.12 shows that (P2) and (P3), in combination with (P′5) then imply (P4). This establishes the equivalence direct.

Clearly Imaoka's partial operation is essentially a restriction of our operation $\star$. As in Section 4, the verification of the abstract characterization of projection sets relied on the construction of a semigroup of the appropriate type having the initial set as its set of projections. Imaoka [11] constructed a regular $*$-semigroup based on his earlier representation theorem in [10], which essentially entailed the pairing of the two one-sided representations considered in our Section 5. Implicitly, the representing semigroup is the Munn semigroup of the P-groupoid and is thus equivalent to our semigroup T_P.

Another external characterization of the set of projections of a regular $*$-semigroup was given by Yamada [19]. Nambooripad and Pastijn [14], more generally, characterized the set of projections of a '$*$-regular semigroup' and constructed a 'Munn-type' semigroup based on its biordered set of idempotents. They specialized both the characterization and the representation to the case of regular $*$-semigroups (which are there termed 'special $*$-semigroups) in [14, Theorem 3.8]. They went on to explicitly relate their characterization with Imaoka's, so we refer the reader to that paper for details.

Finally, Theorem 5.2 then specializes to a P-separating representation of any regular $*$-semigroup S in T_{P_S} which, when interpreted in the language of the papers cited in the preceding two paragraphs, is equivalent to the representations found therein.

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