Varieties of P-restriction semigroups

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Abstract

The restriction semigroups, in both their one-sided and two-sided versions, have arisen in various fashions, meriting study for their own sake. From one historical perspective, as 'weakly E-ample' semigroups, the definition revolves around a 'designated set' of commuting idempotents, better thought of as projections. This class includes the inverse semigroups in a natural fashion. In a recent paper, the author introduced P-restriction semigroups in order to broaden the notion of 'projection' (thereby encompassing the regular $*$-semigroups). That study is continued here from the varietal perspective introduced for restriction semigroups by V. Gould. The relationship between varieties of regular $*$-semigroups and varieties of P-restriction semigroups is studied. In particular, a tight relationship exists between varieties of orthodox $*$-semigroups and varieties of 'orthodox' P-restriction semigroups, leading to concrete descriptions of the free orthodox P-restriction semigroups and related structures. Specializing further, new, elementary paths are found for descriptions of the free restriction semigroups, in both the two-sided and one-sided cases.

In [13], the author introduced P-restriction semigroups as a common generalization of the restriction semigroups (or 'weakly E-ample' semigroups) and the regular $*$-semigroups, defining them as bi-unary semigroups – semigroups with two additional unary operations, $^+$ and * – satisfying a set of simple identities. The projections in such a semigroup are the elements of the form x^+ (or, equivalently, x^*). If $(S, \cdot, ^{-1})$ is a regular $*$-semigroup and $^+$ and * are defined respectively by $x^+ = xx^{-1}$ and $x^* = x^{-1}x$, then its bi-unary reduct is a P-restriction semigroup. In view of the tight connections exhibited between the two classes in [13] and in Proposition 2.1 herein, it is natural to conjecture that the (bi-unary) variety generated by (the reducts of) regular $*$-semigroups is precisely the variety of P-restriction semigroups. By analogy, the variety of restriction semigroups is indeed generated by the inverse semigroups in this fashion. We place this conjecture and the cited example in the following more general context.

For any variety $\mathbf{V}$ of regular $*$-semigroups, let $C\mathbf{V}$ [resp. $\mathbf{P}C\mathbf{V}$] denote the variety of regular $*$-semigroups [P-restriction semigroups] whose projections generate a member of $\mathbf{V}$. For instance, if $\mathbf{SL}$ is the variety of ($*$-) semilattices, $C\mathbf{SL}$ and $\mathbf{P}C\mathbf{SL}$ are the varieties of inverse semigroups and of restriction semigroups, respectively.

The general question is then: *for which varieties* $\mathbf{V}$ *of regular* $*$*-semigroups is it true that the variety of* P*-restriction semigroups generated by* $C\mathbf{V}$ *is precisely* $\mathbf{P}C\mathbf{V}$*?* It is shown that for a given variety $\mathbf{V}$, a positive answer to this question is equivalent to the following: for each nonempty set X, the free P-restriction semigroup in $\mathbf{P}C\mathbf{V}$, on X, is isomorphic to the bi-unary subsemigroup of the free regular $*$-semigroup in $C\mathbf{V}$, on X, that is generated by X. The latter is in fact the subsemigroup, in the usual sense, generated by the union of X and the set of projections.

As noted above, this question has a positive answer for $\mathbf{V} = \mathbf{SL}$, as a consequence of the description in [8] of the free restriction semigroups. We provide an alternative proof that is elementary, within the context of our general results. We can then deduce that the question has a positive answer for *any* variety $\mathbf{V}$ of $*$-bands. The associated varieties of regular $*$-semigroups consist of orthodox $*$-semigroups, where orthodoxy in the context of P-restriction semigroups means that the projections generate a subband. As a consequence, for any variety $\mathbf{V}$ of $*$-bands the free semigroups in $\mathbf{P}C\mathbf{V}$ are embedded naturally in the corresponding free semigroups in $\mathbf{V}$. In the case of most interest, where $\mathbf{V}$ contains $\mathbf{SL}$, the structure of the latter was determined in general in [28, 16] and the embedding may be described quite explicitly in terms of that structure.

The results in the orthodox case rely on connections established herein between orthodox P-restriction semigroups and restriction semigroups. These connections suggest that the former class is worthy of continued study.

In a future work, we show that the question above also has a positive answer in the case where $\mathbf{V}$ is the variety of completely simple $*$-semigroups (in which case $C\mathbf{V} = \mathbf{V}$). There the question will be considered in a somewhat broader context. Whether the original conjecture holds remains open. See Proposition 4.9 for a summary and discussion.

In the penultimate section of the paper, we briefly consider the one-sided analogues of some of our results, including a similarly elementary (but not entirely analogous) alternative path to a description of free left restriction semigroups.

Varieties of restriction semigroups *per se* do not play a direct role in this paper. Study of these varieties has recently been initiated by Cornock [6] and by the author [14, 15].

For background on restriction semigroups, we recommend the excellent survey by Hollings [10]. There, and until recently elsewhere in the literature of the 'York school', the term 'weakly left E-ample' was used. Both Gould's notes [9] and [8] cite other manifestations – and alternative names – of [left] restriction semigroups, going back to work on 'function systems' in the 1960's. The term 'restriction semigroup' was motivated by the use of the term 'restriction category' by Cockett and Lack [4]. Gould's approach provided motivation for this study and its prequel [13], and further inspiration came from reflection on the beautiful descriptions of the free one- and two-sided restriction semigroups in [8].

For general semigroup theory, we refer the reader to [11]. For regular $*$-semigroups (which are frequently termed $*$-regular semigroups and in [18] are termed special $*$-semigroups), see the various papers cited in the sequel, in particular the foundational work by Nordahl and Scheiblich [19].

1 Preliminaries

A *P-restriction semigroup* [13] is a bi-unary semigroup $(S, \cdot, ^+, ^*)$ that satisfies the following identities and their 'duals', which are obtained by writing each statement from right to left, replacing $^+$ by *, and vice versa. (For instance, the dual of (1) is $xx^* = x$ and that of (6) is $x(yx)^* = x^+y^*x$.)

(1) $x^+x = x$; (2) $(xy)^+ = (xy^+)^+$; (3) $(x^+y^+)^+ = x^+y^+x^+$; (4) $x^+x^+ = x^+$;

(5) $(x^+)^* = x^+$; (6) $(xy)^+x = xy^+x^*$.

The set $P_S = \{a^+ : a \in S\}$ is the set of *projections* of S; E_S denotes the set of idempotents of S; E_S is partially ordered in the usual way by $e \leq f$ if $e = ef = fe$. By (4), $P_S \subseteq E_S$, by (5), $P_S = \{a^* : a \in S\}$, and by (9) below, $P_S = \{a \in S : a = a^+\} = \{a \in S : a = a^*\}$.

For the purposes of this paper, the relevant generalized Green's relations may be defined as $\widetilde{\mathcal{R}}_P = \{(a, b) : a^+ = b^+\}$, $\widetilde{\mathcal{L}}_P = \{(a, b) : a^* = b^*\}$ and $\widetilde{\mathcal{H}}_P = \widetilde{\mathcal{L}}_P \cap \widetilde{\mathcal{R}}_P$. A P-restriction semigroup is *P-combinatorial* if $\widetilde{\mathcal{H}}_P = \iota$, the identical relation. Section 6 of [13] connects these definitions with the historical 'York-school' development of ample semigroups and their generalizations.

A *regular ∗-semigroup* is a unary semigroup $(S, \cdot, ^{-1})$ for which the unary operation is a 'regular' involution, that is, $(ab)^{-1} = b^{-1}a^{-1}$, $(a^{-1})^{-1} = a$ and $aa^{-1}a = a$. Setting $a^+ = aa^{-1}$ and $a^* = a^{-1}a$, the induced semigroup $(S, \cdot, ^+, ^*)$ becomes a P-restriction semigroup. The regular ∗-semigroups with commuting projections are just the inverse semigroups.

RESULT 1.1 *[13, Lemma 1.3, Result 1.1, Lemma 1.5] Let S be a regular ∗-semigroup. Then the bi-unary semigroup $(S, \cdot, ^+, ^*)$ is a P-restriction semigroup that, in addition, satisfies $E_S = P_S^2$. The generalized Green's relations then coincide with the usual ones.*

For P-restriction semigroups in general, the property $E_S = P_S^2$ is equivalent to the implication $e = e^2 \Rightarrow e = e^+e^*$.

The term *P-full* will be used for P-restriction semigroups that satisfy $E_S = P_S^2$. Following standard usage, the adjective 'full' will be reserved for subsemigroups that contain all the idempotents of the larger one. By (11) below, if $P_S = E_S$ it follows that P_S must be a subsemigroup of S. But it was shown in [13, Section 6] that that property is equivalent to P_S being a subsemilattice.

For the purposes of this paper, a *restriction* semigroup may be *defined* to be a P-restriction semigroup S for which P_S is a subsemilattice of S. The identities (6) then reduce to what have been termed the 'ample' conditions. That this definition coincides with the definition of 'weakly E-ample' semigroup was shown in [13, Section 6]. Its equivalence with other formulations of the same concept may be found in [9] and [10]. The regular ∗-semigroups whose bi-unary reducts are restriction semigroups are simply the inverse semigroups.

We shall at times consider the larger classes of left, right, and two-sided P-Ehresmann semigroups, which were studied in some depth in [13]. A *left P-Ehresmann semigroup* is a semigroup $(S, \cdot, ^+)$ that satisfies (1) through (4); a *right P-Ehresmann semigroup* is a semigroup

$(S,\cdot,^*)$ that satisfies their duals; and a *P-Ehresmann semigroup* is then a semigroup $(S,\cdot,^+,^*)$ that is a left P-Ehresmann semigroup under $^+$, a right P-Ehresmann semigroup under *, and in addition satisfies (5). (The sections of [13] devoted to one-sided representations were more naturally treated in terms of the operation *. Focusing on $^+$, as we do here, follows recent treatments such as [9].) Thus a P-restriction semigroup is a P-Ehresmann semigroup that, in addition, satisfies the 'generalized ample identity' (6) and its dual.

We shall refer to the conclusions of the following result by reference to its itemizations (8) – (12), following the numbering in [13]. Implicitly, such reference will include the statements of their duals.

RESULT 1.2 *[13, Lemma 1.4] A left P-Ehresmann semigroup $(S,\cdot,^+)$ satisfies:*

(8) $(e_1\cdots e_n)^+ = e_1\cdots e_n\cdots e_1$, *for all* $e_1,\ldots,e_n \in P_S$, $n \geq 2$;

(9) $(x^+)^+ = x^+$;

(10) $x^+(xy)^+ = (xy)^+$;

(11) $(ef)^2 = ef$, *for all* $e, f \in P_S$;

(12) if $e, f \in P_S$, *then* $f \leq e$ *if and only if* $fe = f$; *in particular,* $(xy)^+ \leq x^+$.

In general, the terms 'homomorphism' and 'congruence' will be used appropriate to context. So for regular $*$-semigroups, they should be 'unary', that is, respect the inversion operation, and for P-restriction semigroups they should be 'bi-unary', that is, respect both unary operations, and so on. However, we will frequently make the preservation properties explicit, for clarity's sake. Note that any congruence on a regular $*$-semigroup that respects both of the induced unary operations also respects inversion, since if ρ is such a congruence and $a\rho b$, then $a^{-1}\rho$ and $b^{-1}\rho$ are $\mathcal{H}$-related inverses of $a\rho = b\rho$ in S/ρ and therefore [11, Theorem 2.3.4] are equal.

Let $(S,\cdot,^+,^*)$ be a P-restriction semigroup. The congruence μ is the greatest (bi-unary) congruence on S that is contained in $\widetilde{\mathcal{H}}_P$; S is called *P-fundamental* if $\mu = \iota$. As noted in [13], S/μ is P-fundamental. According to [13, Corollary 5.3], if $a, b \in S$, then $a\ \mu\ b$ if and only if $a^+ = b^+$ and $(ea)^* = (eb)^*$ for all $e \in P_S, e \leq a^+$. From this characterization it follows that if S is P-fundamental, the same is true for its full, bi-unary subsemigroups. It was also observed there that, when applied to (the reducts of) regular $*$-semigroups, P-fundamentality coincides with the usual definition for regular semigroups.

The *projection algebra* of a P-Ehresmann semigroup $(S,\cdot,^+,^*)$ is $(P_S,\times,\star)$, where for $e, f \in P_S$, $e \times f = efe$ and $e \star f = fef$. The projection algebra of a regular $*$-semigroup is that of its bi-unary reduct. In [13, Section 4], the projection algebras of P-Ehresmann semigroups (equivalently of P-restriction semigroups, and of regular $*$-semigroups, by [13, Corollary 4.7]) were characterized axiomatically as *projection algebras*. With any projection algebra $(P,\times,\star)$ is associated a 'generalized Munn semigroup' T_P. We refer the reader to [13] for the details. Only the properties of T_P and of the representation of S in T_{P_S} are needed in the sequel.

RESULT 1.3 *[13, Theorems 4.5 and 5.2, Corollary 5.3] With any projection algebra* $(P, \times, \star)$ *there is associated a fundamental regular* $*$*-semigroup* $(T_P, \star, ^{-1})$*, whose projection algebra is algebra-isomorphic to* P*. Further, the induced bi-unary semigroup* $(T_P, \star, ^+, ^*)$ *is a* P*-fundamental,* P*-restriction semigroup whose projection algebra is that of* $(T_P, \star, ^{-1})$ *and is thus again isomorphic to* P*.*

Given any P*-restriction semigroup* $(S, \cdot, ^+, ^*)$*, with projection algebra* $(P_S, \times, \star)$*, there is a bi-unary homomorphism* θ' *of* S *onto a full subsemigroup of the regular* $*$*-semigroup* T_{P_S}*, which separates the members of* P_S *and induces an isomorphism between the respective projection algebras. The congruence induced by* θ' *is* μ*. If, moreover,* S *is induced from some regular* $*$*-semigroup* $(S, \cdot, ^{-1})$*, then* θ' *also preserves the inverse operation from that semigroup.*

Observe that since, by Result 1.1, any regular $*$-semigroup is P-full, any full P-restriction subsemigroup is again P-full. According to Result 1.3, for any P-restriction semigroup S, S/μ is P-full. In particular, any fundamental P-restriction semigroup has this property.

The P-restriction semigroups S with $|P_S| = 1$ are necessarily restriction semigroups. Following the standard terminology, we call such semigroups *reduced*. In essence, they are just the monoids, regarded as P-restriction semigroups by setting $a^+ = a^* = 1$ for all a. The least congruence on a P-restriction semigroup S whose quotient is reduced is denoted σ and is just the least semigroup congruence that identifies all of its projections (since, as noted in [9], for any $a, b \in S$, a^+, b^+, a^*, b^* belong to P_S and so are all σ-related.)

If S is a regular $*$-semigroup, then σ is the least group congruence on S. The regular $*$-semigroups whose bi-unary reducts are reduced are just the groups.

The following will be useful in the sequel.

LEMMA 1.4 *Let* $(S, \cdot, ^+, ^*)$ *be a* P*-restriction semigroup that is generated, as a bi-unary semigroup, by a set* X*. Then* S *is generated, as a semigroup, by* $X \cup P_S$*.*

Proof. Let F denote the free bi-unary semigroup generated by X. Then F is the subsemigroup of the free semigroup on the set consisting of X itself and the four symbols $($, $)^+$, $($ and $)^*$ that is generated recursively as follows: $X \subset F$; if $u \in F$, then $(u)^+, (u)^* \in F$; if $u, v \in F$, then $uv \in F$. When interpreted in S, each of the first two operations results in a member of P_S. The result then follows easily by induction. □

2 The P-core

The *P-core* C_S of a [right, left] P-Ehresmann semigroup S is the subsemigroup $\langle P_S \rangle$ generated by P_S.

PROPOSITION 2.1 *The* P*-core of a left* P*-Ehresmann semigroup* $(S, \cdot, ^+)$ *is a regular, unary subsemigroup of* S*. The* P*-core of a* P*-Ehresmann semigroup* $(S, \cdot, ^+, ^*)$ *is a regular* $*$*-semigroup for which the induced unary operations coincide with the restrictions of those on* S*. It is therefore a bi-unary subsemigroup of* S*. The* P*-core of a regular* $*$*-semigroup coincides with its (idempotent-generated) core.*

Proof. First suppose S is left P-Ehresmann. By definition, $C_S = \{e_1 \cdots e_n : e_1, \ldots, e_n \in P_S, n \geq 1\}$. If $x \in C_S$, $x = e_1 \cdots e_n$, say, then by (8), $x^+ = x(e_n \cdots e_1) \in C_S$, so that C_S is a unary subsemigroup and $x = x^+x = x(e_n \cdots e_1)x$. Thus C_S is regular. Now suppose S is P-Ehresmann. For x as above, put $x^{-1} = e_n \cdots e_1$. Then $(x^{-1})^{-1} = x$. Now $x^* = x^{-1}x$ and $x^+ = xx^{-1}$. Since $x^{-1} = x^{-1}xx^{-1} = x^*x^{-1} = x^{-1}x^+$, x^{-1} is an inverse of x in the $\mathcal{H}$-class $R_{x^*} \cap L_{x^+}$ of C_S. Such an $\mathcal{H}$-class can contain at most one inverse of x [11, Theorem 2.3.4]. It follows that the definition of x^{-1} is independent of the expression for x as a product of projections. The now well-defined map $x \mapsto x^{-1}$ is clearly an involution. Finally, in the case that S is a regular $*$-semigroup, $E_S = P_S^2$ and the last statement is evident. □

In the one-sided case, C_S may not be a regular $*$-semigroup. For example, any nontrivial left zero semigroup S becomes a left P-Ehresmann semigroup under the operation $e^+ = e$. Now $C_S = S$, which clearly is not a regular $*$-semigroup.

A P-Ehresmann semigroup and its P-core share the same projection algebra. This provides a much simpler path to [13, Corollary 4.7], which used the generalized Munn semigroup to show that the projection algebras of P-Ehresmann semigroups form the same class as the projection algebras of regular $*$-semigroups.

3 Varieties of regular $*$-semigroups

The regular $*$-semigroups form a variety **RS**, under the operations $\{\cdot, ^{-1}\}$. In conjunction with the associativity identity, it is defined by $(xy)^{-1} = y^{-1}x^{-1}$, $(x^{-1})^{-1} = x$ and $xx^{-1}x = x$. These semigroups were introduced by Nordahl and Scheiblich [19]. The following list identifies the subvarieties of **RS** that will frequently be encountered in the sequel.

T: trivial $*$-semigroups.
G: groups.
SL: $*$-semilattices (where the involution is just the identity automorphism).
RB: rectangular $*$-bands.
B: $*$-bands. (See [2], where the lattice of varieties of $*$-bands was determined.)
CR: completely regular $*$-semigroups. (See [20].)
I: inverse semigroups.
O: regular $*$-semigroups that are orthodox as regular semigroups, which we term orthodox $*$-semigroups. (The lattice of varieties was first studied by Adair [1].)
ES: regular $*$-semigroups S that are E-solid as regular semigroups, that is, the core of S is completely regular.

If **V** is any variety of regular $*$-semigroups, let C**V** consist of those *regular $*$-semigroups* S for which C_S belongs to **V**. It is routinely verified that C**V** is a subvariety of **RS** (also see Proposition 4.1 below) and that the operation **V** $\mapsto C$**V** is a closure operation on the lattice of varieties of regular $*$-semigroups. The following identifications are clear: **G** $= C$**T**, **I** $= C$**SL**, **O** $= C$**B** and **ES** $= C$**CR**.

3.1 Free regular $*$-semigroups

Let $\mathbf{V}$ be a variety of regular $*$-semigroups and X a nonempty set. Formally, the free (unary) semigroup in $\mathbf{V}$ on X consists of a pair $(\iota_X, F\mathbf{V}_X)$, where $F\mathbf{V}_X \in \mathbf{V}$, $\iota_X : X \to F\mathbf{V}_X$, and for any $S \in \mathbf{V}$ and $\theta : X \to S$, there is a unique (unary) homomorphism $\bar{\theta} : F\mathbf{V}_X \to S$ such that $\iota_X\bar{\theta} = \theta$. We shall generally omit explicit reference to the map ι_X.

Recall that the free involutory semigroup FI_X on X is the free semigroup on the set $X \cup X^{-1}$, where X^{-1} is a set of formal inverses for the elements of X in the usual way, the inverse of $x_1 \cdots x_n$ being $x_n^{-1} \cdots x_1^{-1}$. Clearly $F\mathbf{RS}_X$ is the quotient of FI_X by the fully invariant congruence that identifies w with $ww^{-1}w$ for all $w \in FI_X$. More generally, $F\mathbf{V}_X$ is the quotient of FI_X by a fully invariant congruence, denoted $\rho_{\mathbf{V}}$. Solutions to the word problem for, or structural descriptions of, free semigroups $F\mathbf{V}_X$ are usually given, either implicitly or explicitly, with reference to $\rho_{\mathbf{V}}$. For example, words u and v in FI_X are related under $\rho_{\mathbf{G}}$ if and only if $\bar{u} = \bar{v}$, where the 'bar' operation produces the reduced word associated with a given involutorial word.

Polák [22] showed that the word problem is solvable in free regular $*$-semigroups. The solution does not *per se* provide structural properties of these semigroups. It is reasonable to conjecture that they are combinatorial, that is, Green's relation $\mathcal{H}$ is the identical relation. This is known to be true in the monogenic case ($|X| = 1$), as a result of the proof by Yamada and Imaoka [30] that in this case the resulting semigroups are orthodox (and thus combinatorial by the result of Szendrei [27]). Also see §7 below. Circumstantial support in the general case comes from another result of Polák [21]: the free 'regular unary' semigroups are combinatorial.

Until this conjecture is resolved, we shall settle for proving fundamentality in the case that X is countably infinite, using a modification of the following construction, which was shown to the author by T.E. Hall many years ago. Let S be a semigroup and let $I_L = \{\ell_s : s \in S\}$ be a set in one-one correspondence with S. Let $\overline{S}_L = S \cup I_L$, retaining the product in S, making every element ℓ_s a left zero for $\overline{S}_L$, and defining $s\ell_t = \ell_{st}$ for $s, t \in S$. Then it is easily verified that $\overline{S}_L$ is a semigroup that is an ideal extension of the left zero semigroup I_L by the semigroup S^0. Clearly it contains S as a subsemigroup. Recall that a semigroup is *right reductive* if whenever $as = bs$ holds for all $s \in S$, then $a = b$. Clearly, every monoid has this property.

PROPOSITION 3.1 *If S is a right reductive, regular semigroup, then $\overline{S}_L$ is a fundamental regular semigroup in which S embeds. In general, S may be embedded in $\overline{S^1}_L$ in this fashion. If S is orthodox [resp. E-solid], then so is $\overline{S^1}_L$.*

Proof. Suppose $a, b \in \overline{S}_L$ and $a \;\mu\; b$. Then $a\ell_s \;\mu\; b\ell_s$, for all $s \in S$. But $\mu \subseteq \mathcal{H}$, and $\mathcal{H}$ restricts to the identical relation on I_L, so $a\ell_s = b\ell_s$, that is, $\ell_{as} = \ell_{bs}$, and so $as = bs$, for all $s \in S$. Since S is right reductive, this implies that $a = b$. Since the core of $\overline{S}_L$ is the union of the core of S with the left zero semigroup I_L, the last statement is clear. $\square$

We modify this construction in order to preserve the property of being a regular $*$-semigroup. To begin with, however, let S be any monoid. Denote by $\overline{S}_R = S \cup I_R$ the semigroup dual to $\overline{S}_L$, where $I_R = \{r_t : t \in S\}$ is a right zero semigroup. Now let $I = I_L \times I_R$, a rectangular band. Let $\overline{S} = S \cup I$, retaining the products in S and I and, for $a \in S$ and $(\ell_s, r_t) \in I$, putting $a(\ell_s, r_t) = (\ell_{as}, r_t)$ and $(\ell_s, r_t)a = (\ell_s, r_{ta})$.

PROPOSITION 3.2 *Given any semigroup S, $\overline{S}$ is a semigroup that is isomorphic to the subsemigroup of $\overline{S}_L \times \overline{S}_R$ that comprises $\{(s, s) : s \in S\} \cup I$. If S is regular, then so is $\overline{S}$.*

If S is a regular semigroup that is both left and right reductive, then $\overline{S}$ is fundamental. In general, S may be embedded in $\overline{S^1}$ in this fashion.

Proof. Mapping $a \in S$ to (a, a) and mapping I identically clearly takes $\overline{S}$ onto the indicated subset of $\overline{S}_L \times \overline{S}_R$. In the latter semigroup, $(a, a)(\ell_s, r_t) = (a\ell_s, ar_t) = (\ell_{as}, r_t)$ and dually, so the map is an isomorphism and the subset is a subsemigroup, which is clearly then a subdirect product of its factors.

Since $\overline{S}$ is the union of S with a rectangular band, it is regular if S is regular. On any regular semigroup, the congruence μ has the property that if $\phi : S \to T$ is a surjective homomorphism, then $a\ \mu_S\ b$ implies $a\phi\ \mu_T\ b\phi$. Hence a regular subdirect product of fundamental regular semigroups is again fundamental. By Proposition 3.1, both $\overline{S}_L$ and $\overline{S}_R$ are fundamental, so $\overline{S}$ is likewise. □

PROPOSITION 3.3 *Let $(S, \cdot, ^{-1})$ be a regular $*$-semigroup and define $\overline{S}$ as above. The assignment $(\ell_s, r_t)^{-1} = (\ell_{t^{-1}}, r_{s^{-1}})$ is a regular involution on I. Under the involution that extends that on S and that on I, $(\overline{S}, \cdot, ^{-1})$ becomes a regular $*$-semigroup.*

Thus any regular $*$-semigroup may be embedded in a fundamental regular $*$-semigroup.

If $\mathbf{V}$ is a variety of regular $*$-semigroups that is closed under taking ideal extensions of rectangular $*$-bands by members of $\mathbf{V}$, then any member of $C\mathbf{V}$ is embeddable in a fundamental member of $C\mathbf{V}$. In particular, the embedding preserves the properties of being orthodox and of being E-solid.

Proof. The proof of the statements in the first two paragraphs is routine. The next statement follows from the observation that $C_{\overline{S}} = C_S \cup I$ is an ideal extension of I by C_S^0. The final statements then hold because $\mathbf{O} = C\mathbf{B}$ and $\mathbf{ES} = C\mathbf{CR}$. □

COROLLARY 3.4 *The free regular $*$-semigroup of countably infinite rank is fundamental. The same is true with respect to any subvariety of the form $C\mathbf{V}$, where $\mathbf{V}$ is as in the previous proposition. In particular, it is true for the variety of E-solid regular $*$-semigroups (and for the variety of orthodox $*$-semigroups).*

Proof. The result follows from universal algebraic principles (cf [3, Theorem 1.12]) and the fact that fundamentality is inherited by regular subdirect products. For the sake of completeness, we provide a direct proof. In the usual way, it suffices to prove the statement for the (relatively) free regular $*$-*monoid* F on the countably infinite set X. We regard its elements as members of the free involutorial semigroup FI_X on X and so as words in $X \cup X^{-1}$. Suppose $u, v \in FI_X$ and $u\ \mu_F\ v$ in F. Let Z be the set of $x \in X$ such that either x or x^{-1} appears in u or v. Let $x \in X - Z$.

Let $G = \overline{F}$, the countable, fundamental, regular $*$-semigroup (in this case, monoid) constructed above. It is clear from the construction that G is generated, as a monoid, by X together with a single element y, say, of I. Define $\phi : X \longrightarrow X \cup \{y\}$ as follows. Let ϕ be the identity map on Z; let $x\phi = y$; and on $X - (Z \cup \{x\})$ let ϕ be any bijection upon $X - Z$. By freedom, ϕ extends to a homomorphism of F upon G that restricts to the identity map on the (free $*$-) subsemigroup generated by Z. Now since $u \; \mu_F \; v$, $u\phi \; \mu_G \; v\phi$, as noted in the proof of Proposition 3.2, and so $u\phi = v\phi$. But then $u = v$. Hence F itself is fundamental.

The final statement – the case $\mathbf{V} = C\mathbf{B}$ – is parenthesized because it is already known that every free orthodox $*$-semigroup is combinatorial [27]. □

We turn now to a brief description of certain free orthodox $*$-semigroups, to which reference will be made in §6. Kad̆ourek and Szendrei [16] provided a concrete description of $F\mathbf{CW}_X$, for every variety $\mathbf{W}$ of $*$-bands that contains $\mathbf{SL}$. Such varieties $\mathbf{W}$ are in one-one correspondence with the self-dual varieties of bands, considered simply as semigroups. (Self-duality in this context means that if a band satisfies a (semigroup) identity $u = w$, then it satisfies the identity obtained by writing u and w in reverse order. Note that free bands themselves have a natural interpretation as $*$-bands.) The structure of the lattice of varieties of $*$-bands was found by Adair [2].

The work [16] was actually in the context of 'bifree' orthodox semigroups; however in Section 7 of that paper the connection with orthodox $*$-semigroups was made explicit. Their description of $F\mathbf{CW}_X$ generalized the common descriptions found by Szendrei in [28] for the cases $\mathbf{W} = \mathbf{B}, \mathbf{NB}, \mathbf{SL}$, (where $\mathbf{NB}$ denotes the (self-dual) variety of normal bands) that is for the free orthodox $*$-semigroups, free generalized inverse $*$-semigroups and free inverse semigroups respectively. In turn, the free orthodox $*$-semigroups and free generalized inverse semigroups were originally described, in somewhat more complicated fashion, by Szendrei [27] and Scheiblich [26], respectively. As is well known, the first description of free inverse semigroups was found by Scheiblich [24], and many alternative descriptions may be found in the literature.

We sketch the description of $F\mathbf{CW}_X$ found in [16]. Let Γ_X denote the Cayley graph of $F\mathbf{G}_X$. Its vertex set is the underlying set of $F\mathbf{G}_X$ and its edge set is $E_X = F\mathbf{G}_X \times X$, where $(g, x) : g \to gx$. Informally, such an edge is labelled by x. By adding the set $F\mathbf{G}_X \times X^{-1}$ of reverse edges, where $(gx, x^{-1}) : gx \to g$ is labelled x^{-1}, we may regard Γ_X as an involuted graph.

There is a natural mapping π of FI_X into the free involutory semigroup FI_{E_X} on the edge set of Γ_X: for $y = y_1 \cdots y_k \in FI_X$, $y\pi$ is the (string associated with the) *path* $e_1 \cdots e_k$, where $e_1 = (1, y_1)$ and for $i = 2, \ldots, k$, $e_i = (\overline{y_1 \cdots y_{i-1}}, y_i)$. Now $F\mathbf{G}_X$ acts on Γ_X by $g \cdot (h, x) = (gh, x)$, and this action extends to an action of $F\mathbf{G}_X$ on FI_{E_X}. Moreover, for any variety $\mathbf{W}$ of $*$-bands containing $\mathbf{SL}$, this extended action induces an action of $F\mathbf{G}_X$ on $F\mathbf{W}_{E_X}$. Thus the semidirect product $F\mathbf{W}_{E_X} \star F\mathbf{G}_X$ is well defined.

RESULT 3.5 *[16] Let* $\mathbf{W}$ *be a variety of $*$-bands that contains* $\mathbf{SL}$. *Then the map* $w\rho_{\mathbf{CW}} \mapsto ((w\pi)\rho_{(\mathbf{W},E_X)}, \overline{w})$ *is an isomorphism of* $F\mathbf{CW}_X$ *into the semidirect product* $F\mathbf{W}_{E_X} \star F\mathbf{G}_X$.

Thus the solution to the (involutory) word problem on $F\mathbf{CW}_X$ is reduced to that on the relatively free bands in $\mathbf{W}$ (and that on the free group) and is therefore solvable (by [23], cf the final remarks in [16, Section 2]).

4 Varieties of P-restriction semigroups

Denote by $\mathbf{PR}$ the variety of all P-restriction semigroups, under the set $\{\cdot,^+,^*\}$ of operations. It is defined by the identities (1) – (6) and their duals.

Given a variety $\mathbf{V}$ of $*$-regular semigroups, let $\mathbf{P}C\mathbf{V}$ consist of those P-restriction semigroups $(S,\cdot,^+,^*)$ for which C_S belongs to $\mathbf{V}$. (Thus $\mathbf{P}C\mathbf{V} = \mathbf{P}C(C\mathbf{V})$, in the notation of the previous section.) By Proposition 2.1, $\mathbf{P}C(\mathbf{RS}) = \mathbf{PR}$ itself. Denote by $\mathbf{R}$ the variety $\mathbf{P}C\mathbf{SL}$, which consists of the P-restriction semigroups S whose projections generate a semilattice: according to §1, these are just the restriction semigroups, which may be defined within $\mathbf{PR}$ by the identity $x^+y^+ = y^+x^+$. The variety $\mathbf{P}C\mathbf{T}$ consists of the (P-) restriction semigroups S for which $|P_S| = 1$, that is, the *reduced* semigroups defined in §1.

PROPOSITION 4.1 *For any variety* $\mathbf{V}$ *of regular* $*$*-semigroups,* $\mathbf{P}C\mathbf{V}$ *is a variety of* P*-restriction semigroups that is defined by identities of the form* $s(x_1^+,\dots,x_n^+) = t(x_1^+,\dots,x_n^+)$ *where* $s(x_1,\cdots,x_n) = t(x_1,\cdots,x_n)$ *is a semigroup identity. Conversely, any variety of* P*-restriction semigroups that is definable by such identities has this form.*

An analogous result holds for the variety $C\mathbf{V}$ *of regular* $*$*-semigroups, each* x_i^+ *being interpreted as* $x_ix_i^{-1}$.

Proof. Let $\mathbf{V}$ be a variety of regular $*$-semigroups and $u(x_1,\cdots,x_m) = v(x_1,\dots,x_m)$ an identity satisfied in $\mathbf{V}$, where $x_1,\dots,x_m$ belong to a countably infinite alphabet X. The involutory words u and v may be regarded as semigroup words over the letters $x_1, x_1^{-1},\dots,x_m,x_m^{-1}$.

Choose a new alphabet, disjoint from X, that contains for each x_i an infinite sequence $x_{i1}, x_{i2},\dots$. Consider all the semigroup identities $s = t$ obtained from $u(x_1,\cdots,x_m) = v(x_1,\dots,x_m)$ by replacing each x_i and x_i^{-1} by $x_{i1}\cdots x_{ik_i}$ and $x_{ik_i}\cdots x_{i1}$, respectively, for some $k_i \geq 1$. In view of Proposition 2.1, if $(S,\cdot,^+,^*)$ is a P-restriction semigroup and $e_{i1},\dots,e_{ik_i} \in P_S$, then $(e_{i1}\cdots e_{ik_i})^{-1} = e_{ik_i}\cdots e_{i1}$ in C_S. Thus satisfaction of the original identity by C_S is equivalent to satisfaction of the set of all equations obtained from such identities $s = t$ by replacing each x_{ij} by a projection e_{ij}. This is then equivalent to satisfaction by S of the set of identities obtained by replacing each x_{ij} by x_{ij}^+.

Conversely, given a P-restriction semigroup S, then since $x^+ = (x^+)^+$ for all $x \in S$, an identity $s(x_1^+,\dots,x_m^+) = t(x_1^+,\dots,x_m^+)$ of the form described is satisfied in S if and only if it is satisfied in C_S, where the latter is regarded as a P-restriction semigroup, and thus if and only if the identity $s(x_1x_1^{-1},\dots,x_mx_m^{-1}) = t(x_1x_1^{-1},\dots,x_mx_m^{-1})$ is satisfied in C_S, regarded as a regular $*$-semigroup.

The last statement follows similarly. □

In view of this proposition, if $s(x_1,\cdots,x_n) = t(x_1,\cdots,x_n)$ is a semigroup identity, then the statement "$s(e_1,\dots,e_n) = t(e_1,\dots,e_n)$, $e \in P_S$" represents the bi-unary identity $s(x_1^+,\dots,x_n^+) =$

$t(x_1^+, \ldots, x_n^+)$, or the corresponding identity for regular $*$-semigroups, according to context. For example, "$ef = fe,\ e, f \in P_S$" represents the identity that defines the variety of restriction semigroups, within the variety of P-restriction semigroup, or that defines the variety of inverse semigroups, within the variety of regular $*$-semigroups. We may simplify expressions such as $s(e_1, \ldots, e_n)$ by use of the operations $^+$, * and/or $^{-1}$, as appropriate. For instance, the identity $efe = e,\ e, f \in P_S$ may also be expressed as $(ef)^+ = e$, or $(fe)^* = e,\ e, f \in P_S$.

There is a natural alternative way to associate a variety of P-restriction semigroups with a given variety $\mathbf{V}$ of regular $*$-semigroups: since the $(^+, ^*)$-reduct of any member of $\mathbf{V}$ is a P-restriction semigroup, the class of all such reducts generates a variety of P-restriction semigroups, which we shall denote by $\mathcal{P}(\mathbf{V})$. Note that, since the class of all such reducts is clearly closed for direct products, $\mathcal{P}(\mathbf{V})$ consists of all the P-restriction semigroups that bi-unarily divide — that is, are bi-unary homomorphic images of bi-unary subsemigroups of — members of $\mathbf{V}$.

LEMMA 4.2 *For any variety* $\mathbf{V}$ *of regular $*$-semigroups,* $\mathcal{P}(C\mathbf{V}) \subseteq \mathbf{P}C\mathbf{V}$*. In particular,* $\mathcal{P}(\mathbf{RS}) \subseteq \mathbf{PR}$.

Proof. This follows from $C\mathbf{V} \subseteq \mathbf{P}C\mathbf{V}$. □

The major questions in this paper fall within the scope of the following.

QUESTION 4.3 *For which varieties* $\mathbf{V}$ *of regular $*$-semigroups is it true that* $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$*?*

Equality holds in case $\mathbf{V} = \mathbf{T}$, since any reduced P-restriction semigroup is a bi-unary homomorphic image of some free monoid, regarded as a semigroup of the same type, and every free monoid, again regarded as a (reduced) restriction semigroup, embeds in the corresponding free group.

PROPOSITION 4.4 *Let* $\mathbf{V}$ *be a variety of regular $*$-semigroups. If* $S \in \mathbf{P}C\mathbf{V}$*, then* $T_{P_S} \in C\mathbf{V}$*. Thus any P-fundamental member of* $\mathbf{P}C\mathbf{V}$ *belongs to* $\mathcal{P}(C\mathbf{V})$*.*

If $F\mathbf{P}C\mathbf{V}_X$ *is P-fundamental for a countably infinite set X, then* $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$*.*

Proof. Let $S \in \mathbf{P}C\mathbf{V}$ and put $P = P_S$. The image of S in T_P under the representation θ' described in Result 1.3 is a full subsemigroup and so contains C_{T_P}. Thus θ' restricts to a homomorphism of C_S upon C_{T_P}. By assumption, $C_S \in \mathbf{V}$, so $C_{T_P} \in \mathbf{V}$, that is, $T_P \in C\mathbf{V}$. If S is P-fundamental, then θ' is an embedding and so $S \in \mathcal{P}(C\mathbf{V})$.

Recalling that any variety of algebras is generated by its free members of countably infinite rank, the final statement follows immediately. □

The final assertion of this proposition provides an important sufficient condition for equality to hold. It will be shown in Proposition 4.8 that this condition is also necessary in some key cases. It is not in general necessary since, for example, in the case $\mathbf{V} = \mathbf{T}$ above, free monoids are not P-fundamental, μ being the universal relation. In Proposition 4.6, we will demonstrate a more general criterion that also provides much more information about $F\mathbf{P}C\mathbf{V}_X$. We first prove a very useful property of regular $*$-semigroups.

PROPOSITION 4.5 *Let $(S, \cdot, ^{-1})$ be a regular $*$-semigroup that is generated, as such, by the subset X. Let U_X be the P-restriction subsemigroup of the induced P-restriction semigroup $(S, \cdot, ^+, ^*)$ that is generated by X. Then U_X is a full subsemigroup of S. Further, U_X is generated as a subsemigroup by $X \cup P_S$.*

Proof. The projections of S have the form $w^+ = ww^{-1}$, where w may be regarded as a word in FI_X, so a proof that they belong to U_X may be based on induction on the length of w. If the length is 1, then $w^+ = xx^{-1} = x^+$ or $w^+ = x^{-1}x = x^*$. In either case, $w^+ \in U_X$. Now supposing that $w^+ \in U_X$ for all w of length at most n, suppose u has length $n+1$. Either $u = xw$ or $u = x^{-1}w$, where w has length n. In the former case, $u^+ = (xw)^+ = (xw^+)^+ \in U_X$, since $xw^+ \in U_X$; in the latter case, $u^+ = (x^{-1}w)^+$. In S, $(x^{-1}w)^+ = x^{-1}ww^{-1}x = (ww^{-1}x)^* = (w^+x)^* \in U_X$. Hence U_X contains P_S. However, by Lemma 1.1, $E_S = P_S^2$ in any regular $*$-semigroup, so U_X is full.

The last statement is a consequence of Lemma 1.4. □

PROPOSITION 4.6 *Let $\mathbf{V}$ be a variety of regular $*$-semigroups and let X be a nonempty set. Then $F\mathcal{P}(\mathbf{V})_X \cong U_X$, the P-restriction subsemigroup of $F\mathbf{V}_X$ generated by X, as described in Proposition 4.5. Thus $F\mathcal{P}(\mathbf{V})_X$ is P-full.*

Proof. This result is a specialization of a general one from universal algebra. Clearly, $U_X \in \mathcal{P}(\mathbf{V})$ and so there is a bi-unary homomorphism α, say, of $F\mathcal{P}(\mathbf{V})_X$ upon U_X such that $x\alpha = x$ for all $x \in X$. Now since $F\mathcal{P}(\mathbf{V})_X \in \mathcal{P}(\mathbf{V})$, there exist $R \in \mathbf{V}$, a P-restriction subsemigroup T of R, and a bi-unary surjective homomorphism $\beta : T \longrightarrow F\mathcal{P}(\mathbf{V})_X$. For each $x \in X$, choose an inverse image $\bar{x}$, say, in T. By freedom, there is a $^{-1}$-preserving homomorphism $\gamma : F\mathbf{V}_X \longrightarrow R$ such that $x\gamma = \bar{x}$, for all $x \in X$, the restriction of which to U_X is a bi-unary homomorphism into T, with the same property. Now the composition $\alpha\gamma\beta : F\mathcal{P}(\mathbf{V})_X \longrightarrow F\mathcal{P}(\mathbf{V})_X$ fixes the elements of X. By freedom, this composition is the identity map. Therefore α is an isomorphism.

The final statement follows from the remarks on fundamentality in §1. □

THEOREM 4.7 *Let $\mathbf{V}$ be a variety of regular $*$-semigroups and X be a nonempty set. There is a P-separating bi-unary homomorphism ψ of $F\mathbf{P}C\mathbf{V}_X$ upon the P-restriction subsemigroup U_X of $FC\mathbf{V}_X$ generated by X. Hence the projection algebra of $F\mathbf{P}C\mathbf{V}_X$ is isomorphic to that of $FC\mathbf{V}_X$.*

The equality $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ holds if and only if ψ is one-one for every X [for any countably infinite X].

Proof. Put $F = F\mathbf{P}C\mathbf{V}_X$. Clearly, $U_X \in \mathcal{P}(C\mathbf{V}) \subseteq \mathbf{P}C\mathbf{V}$. Thus there is a surjective bi-unary homomorphism ψ, say, from F to U_X, taking x to x for all $x \in X$.

Put $P = P_F$. By freedom, the P-separating homomorphism θ' is the unique bi-unary homomorphism $F \longrightarrow T_P$ that maps x to θ'_x, for all $x \in X$. As noted in Proposition 4.4, $T_P \in C\mathbf{V}$. Now the map $x \rightarrow \theta'_x$ induces a ${}^{-1}$-homomorphism $\omega : F\mathbf{RS}_X \longrightarrow T_P$. By uniqueness, $\theta' = \psi\omega$ and so ψ is P-separating.

Since U_X is full in $F\mathbf{RS}_X$, ψ induces an algebra isomorphism between the respective projection algebras. (The restriction is a bijection and the operation $\star$ is defined by $e \star f = fef$ in each case.)

Turning to the last statement, observe first that the alternative reading stems from the well-known fact that any variety of algebras is generated by a free member on a countably infinite set. The statement is then clear from the previous proposition. $\square$

We now prove the promised partial converse to the final assertion of Proposition 4.4.

PROPOSITION 4.8 *Suppose a variety* $\mathbf{V}$ *of regular* $*$*-semigroups is closed under ideal extensions of rectangular* $*$*-bands by members of* $\mathbf{V}$*. If* $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$*, then* $F\mathbf{P}C\mathbf{V}_X$ *is necessarily fundamental for any countably infinite set* X*.*

Proof. If $\mathbf{V}$ has the stipulated property then, by Corollary 3.4, $FC\mathbf{V}_X$ is fundamental for any countably infinite set. From the paragraphs on fundamentality in §1, $FC\mathbf{V}_X$ is P-fundamental and so U_X also has this property. But $F\mathbf{P}C\mathbf{V}_X \cong U_X$, by Theorem 4.7. $\square$

The conjecture $\mathcal{P}(\mathbf{RS}) = \mathbf{PR}$ remains open at this point. We summarize.

PROPOSITION 4.9 *The following are equivalent:*

1. $\mathcal{P}(\mathbf{RS}) = \mathbf{PR}$*, that is, the conjecture* $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ *holds for* $\mathbf{V} = \mathbf{RS}$*;*
2. *every* P*-restriction semigroup divides a regular* $*$*-semigroup;*
3. $F\mathbf{PR}_X$ *is isomorphic with the bi-unary subsemigroup* U_X *of* $F\mathbf{RS}_X$ *generated by* X*, for any set [any countably infinite set]* X*;*
4. $F\mathbf{PR}_X$ *is* P*-fundamental for any countably infinite set* X*.*

Section 7 provides some scanty supporting evidence for the conjecture. Apart from semantic arguments, a possible route to confirmation is to show that the word problem for free P-restriction semigroups parallels that for free regular $*$-semigroups [22], inducing an isomorphism of the former with the subsemigroup U_X of the latter.

Finally, observe that by virtue of Proposition 2.1, we may also consider varieties of P-Ehresmann semigroups in a similar, albeit more limited, fashion. Let $\mathbf{PE}$ denote the $({}^+,{}^*)$-variety of all such semigroups. Once more, for any variety $\mathbf{V}$ of regular $*$-semigroups, let $\mathbf{P}C\mathbf{V}$ be the subvariety of $\mathbf{PE}$ comprising those S for which $C_S \in \mathbf{V}$. This subvariety is defined by the same additional identities that may be used in the case of P-restriction semigroups. For instance, the E-Ehresmann semigroups, originally defined in [17] (and sometimes just termed

‘Ehresmann’) comprise the subvariety **PCSL** of **PE**, in this more general context, defined by the identity $ef = fe$, $e, f \in P_S$. While some structural results may be found, the tight connection that is exhibited between regular $*$-semigroups and P-restriction semigroups has no direct analogue in this context.

5 Orthodox P-restriction semigroups.

The variety $\mathbf{O} = C\mathbf{B}$ of orthodox $*$-semigroups was introduced in §3. It was first considered by Nordahl and Scheiblich [19] under the name $*$-*orthodox* semigroups. In that paper, the authors provided the following characterization via identities, which we re-interpret in the form described following Proposition 4.1.

RESULT 5.1 *[19] A regular $*$-semigroup $(S, \cdot, ^{-1})$ is orthodox if and only if it satisfies the identity that may be expressed as $(efg)^2 = efg$, $e, f, g \in P_S$.*

By analogy, we denote the variety $\mathbf{P}C\mathbf{B}$ of P-restriction semigroups by **PO** and also term its members *orthodox*. Thus, a P-restriction semigroup is orthodox if the product of any number of *projections* is idempotent. (Recall that the product of any *two* projections is always idempotent.) We may also use this terminology in the more general context of P-Ehresmann semigroups, as in the next result (see the final remarks in the preceding section).

PROPOSITION 5.2 *A P-Ehresmann semigroup $(S, \cdot, ^+, ^*)$ is orthodox if and only if it satisfies $efg = (efg)^+(efg)^*$, $e, f, g \in P_S$, in which case $C_S = P_S^2$.*

Proof. Suppose S is orthodox. Let $e, f, g \in P_S$ and let $B = \langle e, f, g \rangle$, a subband of C_S. Now, by the equation (8) and its dual, the equation $efg = (efg)^+(efg)^*$ may be restated as $efg = (efgfe)(gfefg)$, which is satisfied in any band, either by an argument involving content in the free band on $\{e, f, g\}$, or by an argument involving Green's relations: $efgfe \ \mathcal{R} \ efg \ \mathcal{L} \ gfefg$ and the $\mathcal{D}$-class of efg is a rectangular band.

Conversely, the identity implies that $P_S^3 = P_S^2$ and so $P_S^n = P_S^2 \subseteq E_S$ for all $n \geq 3$. $\square$

In the following, $S = (S, \cdot, ^+, ^*)$ will always be an orthodox P-restriction semigroup. Denote the band C_S by B_S, or simply B. It is decomposable in the usual way (see [11]) as a semilattice of rectangular bands, where the semilattice decomposition is provided by Green's relation $\mathcal{J}_B = \mathcal{D}_B$, which in this case is a congruence that is described as follows: $e \ \mathcal{D}_B \ f$ if and only if $efe = e$ and $fef = f$. Let γ be the relation on S defined by

$$\gamma = \{(a, b) : a^+ \ \mathcal{D}_B \ b^+, \ a^* \ \mathcal{D}_B \ b^*, \text{ and } b^+a = ba^*\}.$$

Observe that if $a \ \gamma \ b$, then $a^+ = a^+b^+a^+$, so $a = a^+b^+a$. Similarly, $a = ab^*a^*$ and $b = b^+a^+b = ba^*b^*$. From $b^+a = ba^*$ it then follows that $ab^* = a^+b^+ab^* = a^+ba^*b^* = a^+b$. That is, γ is symmetric. Also observe that for $e, f \in P$, $e \ \gamma \ f$ if and only if $e \ \mathcal{D}_B \ f$. Thus if $a \ \gamma \ b$, then $a^+ \ \gamma \ b^+$ and $a^* \ \gamma \ b^*$.

LEMMA 5.3 *If $a, b \in S$, then $a\ \gamma\ b$ if and only if $a = a^+ba^*$ and $b = b^+ab^*$.*

Proof. Suppose $a\ \gamma\ b$. As noted above, $a = a^+b^+a$ and so $a = a^+ba^*$, the other relation being similarly proved. Conversely, if $a = a^+ba^*$ and $b = b^+ab^*$, then $a = (a^+b^+)ba^*$ implies $a = a^+b^+a$, since $a^+b^+ \in E_S$, and, similarly, $b = b^+a^+b$. Thus $b^+a = b^+(a^+ba^*) = ba^*$. Also, applying (2), (3) and (4), $a^+ = (a^+b^+a)^+ = (a^+b^+a^+)^+ = a^+b^+a^+$; similarly, $b^+ = b^+a^+b^+$, that is, $a^+\ \mathcal{D}_B\ b^+$; similarly, $a^*\ \mathcal{D}_B\ b^*$. Therefore $a\ \gamma\ b$. □

PROPOSITION 5.4 *The relation γ is the least bi-unary congruence on S whose quotient is a restriction semigroup.*

If $(S, \cdot, ^{-1})$ is an orthodox $*$-semigroup, then γ is the least congruence on S whose quotient is an inverse semigroup (cf [11, Theorem 6.2.5] for orthodox semigroups in general).

Proof. Note first that the definition of γ is self-dual, in the sense used in this paper. Clearly it is symmetric and reflexive. Suppose $a\ \gamma\ b$ and $b\ \gamma\ c$. Then $a^+\ \mathcal{D}_B\ c^+$ and $a^*\ \mathcal{D}_B\ c^*$. Now the $\mathcal{D}_B$-class containing a^+, b^+, c^+ is a rectangular band, so $a^+b^+c^+ = a^+c^+$ and $c^*b^*a^* = c^*a^*$. Applying the lemma above, $a = a^+(b^+cb^*)a^* = (a^+b^+c^+)c(c^*b^*a^*) = a^+ca^*$. Similarly, $c = c^+ac^*$. So γ is transitive.

As a preliminary step toward proving compatibility, we prove that if $a\ \gamma\ b$ and $g \in P_S$, then $ga\ \gamma\ gb$. By (12), $(ga)^* \leq a^*$. Thus $gb(ga)^* = g(ba^*)(ga)^* = g(b^+a)(ga)^* = gb^+(a^+ga) = (gb^+)a^+(ga^+)a$, where the penultimate equality follows from the generalized ample condition (6). Now $a^+\ \mathcal{D}_B\ b^+$ and $\mathcal{D}_B$ is a congruence on B, so $gb^+\ \mathcal{D}_B\ ga^+$. The associated $\mathcal{D}_B$-class is a rectangular band, so $(gb^+)a^+(ga^+) = (gb^+)(ga^+)$. Again applying (6), $gb^+g = (gb)^+g$. Thus $gb(ga)^* = (gb)^+ga^+a = (gb)^+(ga)$, as required.

Now suppose $a\ \gamma\ b$ and $x \in S$. Note that $(xa)^* = (x^*a)^*$ and $(xb)^* = (x^*b)^*$. Since $x^* \in P_S$, the previous step of the proof implies that $(xa)^*\ \mathcal{D}_B\ (xb)^*$ and $x^*a(x^*b)^* = (x^*a)^+x^*b$. Thus $(xa)(xb)^* = x(x^*a)(x^*b)^* = x(x^*a)^+x^*b = x(x^*a^+x^*)x^*b = xa^+x^*b$, where we have applied (2) and (3); applying (6) then yields $xa^+x^*b = (xa)^+xb$. It remains to prove that $(xa)^+\ \mathcal{D}_B\ (xb)^+$. Here, by (2), $(xa)^+ = (xa^+)^+$ and $(xb)^+ = (xb^+)^+$, so it suffices to prove that if $e, f \in P_S$ and $e\ \mathcal{D}_B\ f$, then $(xe)^+\ \mathcal{D}_B\ (xf)^+$. Now since $\mathcal{D}_B$ is a congruence on B, $x^*e\ \mathcal{D}_B\ x^*f$, that is, $x^*ex^*fx^*e = x^*e$. Multiplying on the left by x and on the right by x^*, this yields $xex^*fx^*ex^* = xex^*$. Applying (6) once on the right hand side and three times on the left, we obtain $(xe)^+(xf)^+(xe)^+x = (xe)^+x$. Finally, applying the $^+$ operation to both sides, using (2) and noting that $(xe)^+ \leq x^+$, the required equation $(xe)^+(xf)^+(xe)^+ = (xe)^+$ ensues.

Therefore γ is left compatible. By the self-duality noted above, it is right compatible. As also noted earlier, it respects the unary operations. So it is a bi-unary congruence on S, which restricts to $\mathcal{D}_B$ on B. Such a congruence clearly maps P_S onto $P_{S/\gamma}$ and so $P_{S/\gamma}$ is a semilattice, that is, S/γ is a restriction semigroup.

If ρ is any bi-unary congruence on S such that S/ρ is a restriction semigroup, suppose $a\ \gamma\ b$ in S. Then $a^+\rho$ and $b^+\rho$ are $\mathcal{D}$-related projections in the semilattice $P_{S/\rho}$ and so are equal. Similarly, $a^*\rho = b^*\rho$. Thus $b\rho = (bb^*)\rho = (ba^*)\rho = (b^+a)\rho = a\rho$ and $\gamma \subseteq \rho$, completing the proof of the first assertion of the proposition.

Finally, if S is actually a regular $*$-semigroup, then as noted in §1, any bi-unary congruence also respects $^{-1}$, so γ is the least congruence on S whose quotient is a (regular $*$-semigroup that is also a) restriction semigroup, that is, an inverse semigroup. $\square$

PROPOSITION 5.5 *The congruences γ and μ are disjoint. Hence S embeds in the product of the restriction semigroup S/γ and the fundamental orthodox $*$-semigroup T_{P_S}.*

Proof. Since $\mu \subseteq \widetilde{\mathcal{H}}_P$, it suffices to prove that $\gamma \cap \widetilde{\mathcal{H}}_P = \iota$. But this is obvious from Lemma 5.3: if $a \ \gamma \ b$ then $a = a^+ba^*$ and from $a \ \widetilde{\mathcal{H}}_P \ b$, $a = b^+bb^* = b$. It follows that S is isomorphic to a subdirect product of the restriction semigroup S/γ and the P-fundamental, orthodox P-restriction semigroup S/μ. But, as in the proof of Proposition 4.4, T_{P_S} is an orthodox $*$-semigroup in which S/μ embeds. $\square$

We return briefly to the congruence σ: the least semigroup congruence that identifies all the projections, equivalently, the least bi-unary congruence whose quotient is reduced. On any P-restriction semigroup, σ is a bi-unary congruence whose quotient is a restriction semigroup (with the identity as its only projection). On an orthodox P-restriction semigroup, then, $\gamma \subseteq \sigma$. Thus σ is induced by the analogous congruence on the greatest quotient that is a restriction semigroup. A description of the latter was given by Gould [9, Lemma 8.1]: $(a, b) \in \sigma$ if and only if $ea = eb$ for some projection e.

PROPOSITION 5.6 *Let $a, b \in S$. The following are equivalent:*

(i) $a \ \sigma \ b$;

(ii) $eaf = ebf$ *for some* $e, f \in P_S$;

(iii) $ea = bf$ *for some* $e, f \in P_S^2$.

If S is also P-full, that is, $E_S = P_S^2$, then $a \ \sigma \ b$ if and only if $ea = bf$ for some $e, f \in E_S$.

Proof. If either (ii) or (iii) holds, then since σ identifies all members of P_S, $a \ \sigma \ b$.

If (i) holds, then, as noted above, $a\gamma \ \sigma \ b\gamma$ in S/γ and thus $g(a\gamma) = g(b\gamma)$ for some $g \in P_{S/\gamma}$. Since γ is a bi-unary congruence, $g = k\gamma$ for some $k \in P_S$. Applying Lemma 5.3, $ka = (ka)^+(kb)(ka)^*$. By (12), $(ka)^+ = (ka)^+k$, so $eaf = ebf$, where $e = (ka)^+, f = (ka)^* \in P_S$. Thus (ii) holds.

To deduce (iii) from (i), note first that since, in a restriction semigroup, if e is any projection then $eb = b(eb)^*$, the equation $ea = eb$ may be rewritten as $ea = bf$, where f is also a projection. Thus from $a\gamma \ \sigma \ b\gamma$ we instead obtain $g(a\gamma) = (b\gamma)h$, where $g = k\gamma$ and $h = \ell\gamma$, for some $k, \ell \in P_S$. This time applying the definition of γ, $ea = bf$, where $e = (b\ell)^+k, f = \ell(ka)^* \in P_S^2$. $\square$

The last assertion of the proposition applies, in particular, to the least group congruence σ on an orthodox $*$-semigroup S. In fact the least group congruence on any orthodox semigroup has this form (attributed to Saito in [29]).

Without the P-full property, that conclusion may fail. For example, in a nontrivial monoid S with zero, regarded as a reduced P-restriction semigroup, $0a = b0$ for all $a, b \in S$, but σ is the identical relation.

6 Varieties of orthodox P-restriction semigroups and their free objects.

We first show that $\mathcal{P}(C\mathbf{W}) = \mathbf{P}C\mathbf{W}$ for any variety $\mathbf{W}$ of $*$-bands that contains $\mathbf{SL}$, starting with what turns out to be the exceptional case $\mathbf{V} = \mathbf{SL}$. As observed earlier, $C\mathbf{SL} = \mathbf{I}$ and $\mathbf{P}C\mathbf{SL} = \mathbf{R}$. That $\mathcal{P}(\mathbf{I}) = \mathbf{R}$ is already known, as a result of the description of the free restriction semigroups given by Fountain, Gomes and Gould in [8], as subsemigroups of the corresponding free inverse semigroups. (From another perspective, the mere consequence that the free restriction semigroups are P-combinatorial by itself implies the stated equality, by virtue of Proposition 4.4.) However, we show here that in fact there is a relatively elementary proof of this equality. In turn, this provides a similarly elementary independent route to an explicit description of the free restriction semigroups.

The key to the proof is the existence, for any restriction semigroup S, of a proper restriction semigroup T and a homomorphism of T upon S. A restriction semigroup is *proper* if $\widetilde{\mathcal{R}}_P \cap \sigma = \widetilde{\mathcal{L}}_P \cap \sigma = \iota$. In §9 we will provide a simple proof of both this covering theorem (first proved in [8]) and its one-sided analogues – in fact, of the stronger one found in [8, Proposition 6.5 and Lemma 6.6].

PROPOSITION 6.1 *The equality $\mathcal{P}(\mathbf{I}) = \mathbf{R}$ holds. In other words, $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ holds for $\mathbf{V} = \mathbf{SL}$.*

Proof. Let S be a restriction semigroup. By Theorem 9.1, S is a homomorphic image of a proper restriction semigroup T, say. Then, on T, $\mu \cap \sigma \subseteq \widetilde{\mathcal{H}}_P \cap \sigma \subseteq \widetilde{\mathcal{R}}_P \cap \sigma = \iota$, so T is isomorphic to a subdirect product of T/μ and T/σ. But $T/\mu \in \mathcal{P}(C\mathbf{SL})$, by Proposition 4.4; and $T/\sigma \in \mathbf{P}C\mathbf{T} = \mathcal{P}(C\mathbf{T})$. Thus $T \in \mathcal{P}(C\mathbf{SL}) = \mathcal{P}(\mathbf{I})$, whence the same is true of S, as claimed. □

THEOREM 6.2 *The equation $\mathcal{P}(C\mathbf{W}) = \mathbf{P}C\mathbf{W}$ holds for any variety $\mathbf{W}$ of $*$-bands. In particular, the variety $\mathbf{O}$ of orthodox $*$-semigroups generates the variety $\mathbf{PO}$ of orthodox P-restriction semigroups.*

Proof. Let $S \in \mathbf{P}C\mathbf{W}$. By Proposition 5.5, S is isomorphic to a (bi-unary) subdirect product of the restriction semigroup S/γ and the fundamental orthodox P-restriction semigroup S/μ. On the one hand, by Proposition 4.4, $S/\mu \in \mathcal{P}(C\mathbf{W})$. On the other, $S/\gamma \in \mathbf{P}C\mathbf{SL} \cap \mathbf{P}C\mathbf{W} \subseteq \mathbf{P}C(\mathbf{SL} \cap \mathbf{W})$. Now it is easy to see [2] that either $\mathbf{SL} \subseteq \mathbf{W}$ or $\mathbf{SL} \cap \mathbf{W} = \mathbf{T}$. In the case $\mathbf{SL} \subseteq \mathbf{W}$, then $S/\gamma \in \mathbf{P}C\mathbf{SL} = \mathcal{P}(C\mathbf{SL})$, by the last proposition, whence $S \in \mathcal{P}(C\mathbf{W})$, as required; otherwise, $S/\gamma \in \mathbf{P}C\mathbf{T} = \mathcal{P}(C\mathbf{T})$, as noted in §4, whence $S \in \mathcal{P}(C\mathbf{W})$ once more.□

As a consequence of §4, we obtain an explicit description of the free orthodox P-restriction semigroups $F\mathbf{P}C\mathbf{W}_X$. Recall from Proposition 4.6 that $F\mathcal{P}(C\mathbf{W})_X \cong U_X$, the bi-unary subsemigroup of $FC\mathbf{W}_X$ generated by X, and that by Proposition 4.5, U_X is the subsemigroup generated by X together with the projections of $FC\mathbf{W}_X$. In the general situation $FC\mathbf{V}_X$, membership in U_X may not be easy to determine, even should an explicit description of the latter be known. In the orthodox case, the following more explicitly useful characterization is available.

PROPOSITION 6.3 *Let* $\mathbf{V}$ *be a variety of orthodox* $*$*-semigroups that contains* $\mathbf{G}$*. Then* $U_X = X^*\sigma^{-1}$.

Proof. It suffices to prove the statement for the variety $\mathbf{O}$. The conclusion is implicit in the description(s) of $F\mathbf{O}_X$ in [27, 28] but we present a direct proof. First, by Proposition 4.5, $U_X = \langle X \cup P \rangle$, where $P = P_{F\mathbf{O}_X}$, and so its image in $F\mathbf{G}_X$ is simply X^*. Let $w \in FI_X$ and suppose w maps to $x_1 \cdots x_n \in X^+$ under the congruence σ on FI_X. Based on any of the well-known descriptions of free inverse semigroups, we may express the image of w in the free inverse semigroup on X as $ww^{-1}x_1 \cdots x_n$. In $F\mathbf{O}_X$, then, $w \;\gamma\; w^+x_1 \cdots x_n$ and so, by Lemma 5.3, $w = w^+(w^+x_1 \cdots x_n)w^* \in \langle X \cup P \rangle$. □

THEOREM 6.4 *Let* $\mathbf{W}$ *be any variety of* $*$*-bands. For any nonempty set* X*,* $F\mathbf{P}C\mathbf{W}_X$ *is isomorphic to the* P*-restriction subsemigroup* U_X *of the free orthodox* $*$*-semigroup* $FC\mathbf{W}_X$ *over* $C\mathbf{W}$ *that is generated by* X*. Moreover,* U_X *is the complete inverse image of the free monoid on* X *under the map* $FC\mathbf{W}_X \to F\mathbf{G}_X$*.*

Proof. The first assertion was essentially proven in the paragraph before Proposition 6.3, applying Theorem 6.2. The second follows from that proposition, noting that since $\mathbf{T} \subseteq \mathbf{W}$, $\mathbf{G} = C\mathbf{T} \subseteq C\mathbf{W}$. □

When Theorem 6.4 is applied to the special case $\mathbf{W} = \mathbf{SL}$, each of the many explicit descriptions of the free inverse semigroups yields an explicit description of the free restriction semigroups. For instance, if one chooses that used in [11], based on the original of Scheiblich [24], the result is precisely the (first) description in [8, Theorem 4.4].

COROLLARY 6.5 *If* $\mathbf{SL} \subseteq \mathbf{W}$*, then in terms of the representation of* $FC\mathbf{W}_X$ *in* $F\mathbf{W}_{E_X} \star F\mathbf{G}_X$ *stated in §3, the image of* $F\mathbf{P}C\mathbf{W}_X$ *in* $FC\mathbf{W}_X$ *under the representation obtained in Theorem 6.4 comprises the pairs* (u, g) *for which* $g \in X^*$*.*

Finally, if $\mathbf{SL} \cap \mathbf{W} = \mathbf{T}$, then either $\mathbf{W} = \mathbf{RB}$ or $\mathbf{W} = \mathbf{T}$ (in which case $F\mathbf{P}C\mathbf{W}_X \cong X^*$). In the case $\mathbf{W} = \mathbf{RB}$, the following facts are easily established. The members of $C\mathbf{RB}$ are the 'rectangular $*$-groups', that is, the rectangular groups over rectangular $*$-bands, and so are direct products of rectangular $*$-bands with groups. (The rectangular $*$-bands are essentially just the 'square' rectangular bands $X \times X$, where $(i, j)^{-1} = (j, i)$, and each of these is actually free on X.) Thus $FC\mathbf{RB}_X \cong (X \times X) \times F\mathbf{G}_X$. From the theorem, $F\mathbf{P}C\mathbf{RB}_X \cong (X \times X) \times X^*$.

Clearly the rectangular $*$-groups are the orthodox completely simple $*$-semigroups. Extending our consideration outside the orthodox milieu, it is clear that the class $\mathbf{V} = \mathbf{CS}$ of completely simple $*$-semigroups is closed under the operation $\mathbf{V} \to C\mathbf{V}$ and one may ask if the equation $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ also holds true in this case. We have proven that this is indeed the case, but since the arguments used also apply to classes of general completely simple semigroups, this topic will be treated in a future work. The case of $\mathbf{CS}$ will also demonstrate that Proposition 6.3 may no longer hold when $\mathbf{V}$ is nonorthodox.

7 Monogenic P-restriction semigroups

A P-restriction semigroup $(S, \cdot, ^+, ^*)$ is *monogenic* if it is generated, as a bi-unary semigroup, by a single element. The main result of this section is that every such semigroup is orthodox. It should be noted that, should the conjectured equation $\mathcal{P}(\mathbf{RS}) = \mathbf{PR}$ hold, then this theorem would in fact follow from that of Yamada and Imaoka [30], that every monogenic regular $*$-semigroup is orthodox. Instead, we deduce that theorem from ours.

In the following, $(S, \cdot, ^+, ^*)$ is a P-restriction semigroup generated by x, say. For $m \geq 1$, put $g_m = (x^m)^+$ and $h_m = (x^m)^*$. Set $G_S = \{g_m, h_m : m \geq 1\}$ and $\langle G_S \rangle$ the subsemigroup it generates. By virtue of (8), $\langle G_S \rangle$ is also closed under the two unary operations.

LEMMA 7.1 *The following hold in S.*

(i) $g_1 \geq g_2 \geq \cdots \geq g_n \geq \cdots$ *and* $h_1 \geq h_2 \geq \cdots \geq h_n \geq \cdots$*;*

(ii) $(xg_m)^+ = g_{m+1}$ *and* $xg_m x^* = g_{m+1}x$*, for all* $m \geq 1$*;*

(iii) $xh_1 = x$ *and* $xh_n = g_1 h_{n-1} x$ *for all* $n > 1$.

Proof (i) These follow from the equation $(ab)^+ \leq a^+$ and its dual.

(ii) $(xg_m)^+ = (x(x^m)^+)^+ = (xx^m)^+ = g_{m+1}$; $xg_m x^* = x(x^m)^+ x^* = (xx^m)^+ x = g_{m+1}x$, applying the generalized ample identity $ab^+a^* = (ab)^+a$ to obtain the middle equality.

(iii) The first equation is clear; the second is essentially the dual of the second statement in (ii). $\square$

It follows from (i) that the members of $\langle G_S \rangle$ may be written as alternating products of g_m's and h_m's.

LEMMA 7.2 *Every projection in S is a product of projections from G_S. Thus $P_S \subseteq \langle G_S \rangle$ and $C_S = \langle G_S \rangle$.*

Proof We follow the method of Lemma 1.4. So F denotes the free bi-unary semigroup generated by x, namely the subsemigroup of the free semigroup, on the set consisting of x itself and the four symbols $($, $)^+$, $($ and $)^*$, that is generated recursively as follows: $x \in F$; if $u \in F$, then $(u)^+, (u)^* \in F$; if $u, v \in F$, then $uv \in F$. We will use the symbol $\equiv$ to denote equality of words in F, so the usual equality symbol denotes their equality (more precisely, the equality of their images) in S.

Proceed by induction on the minimum number of steps required to build a projection e of S in the above fashion. The base cases, where $e = x^+$ or $e = x^*$, are clear. In general, we may assume that $e = (w)^+$, where w is built in one fewer step (the case $e = (w)^*$ being similar). If (the image of) w is a projection in S, then the induction hypothesis implies that $e \in G_S$. Otherwise, $w \equiv w_1 w_2$ for some $w_1, w_2 \in F$, whence, in S, $e = (w_1 w_2^+)^+$, the induction hypothesis then applying to w_2^+. Now if (the image of) w_1 is a projection in S, then once again the induction hypothesis applies and, applying (3), $e = w_1(w_2)^+ w_1 \in G_S$.

Applying this argument repeatedly, it remains to consider the situation whereby $w \equiv w_1 w_2 \cdots w_k$, the first term being neither factoring in F nor representing a projection in S. Thus $w_1 = x$.

Put $v = w_2 \cdots w_n$, so that, as above, $e = (xv^+)^+$ and $v^+ \in \langle G_S \rangle$. Write v^+ as an alternating product of g_n's and h_m's and induct on the length of this expression for v^+. We use the preceding lemma without explicit reference. First suppose $v^+ = g_m u$, where either $u = 1$ or u starts with some h_n. If $u = 1$ then $e = (xg_m)^+ = g_{m+1}$; otherwise, then since $h_n = x^* h_n$, $e = (xg_m x^* u)^+ = (g_{m+1} xu)^+ = g_{m+1}(xu)^+ g_{m+1}$. By the induction hypothesis on v, $(xu)^+ \in \langle G_S \rangle$, whence the same is true for e. Alternatively, $v^+ = h_n u$, where either $u = 1$ or u begins with some g_m. In any case, if $n = 1$, then $e = (xh_1 u)^+ = (xu)^+$ and either $e = x^+$ or the previous case applies. If $n > 1$ then $e = (xh_n u)^+ = (g_1 h_{n-1} xu)^+ = g_1 h_{n-1}(xu)^+ h_{n-1} g_1$ and, again, the previous case applies. □

As a consequence of Lemma 7.1, G_S is generated as a semigroup by two chains (totally ordered semilattices), each consisting of projections of S.

PROPOSITION 7.3 *Let B be a semigroup that is generated by two chains G and H of idempotents and has the property that $gh, hg \in E_B$ for all $g \in G, h \in H$. Then B is a band.*

Proof. We first prove a technical result. Consider a product $e_1e_2e_3e_4$, where the terms alternate between G and H, $e_1 \leq e_3$ and $e_2 \geq e_4$. Then $e_1e_2e_3e_4 = e_1(e_3e_2e_3e_2)e_4 = e_1(e_3e_2)e_4 = e_1e_4$. Now consider a product $e_1 \cdots e_{2n}$, where the terms alternate between G and H, $e_1 \leq e_k$ for each odd k with $1 < k < 2n$, and $e_k \geq e_{2n}$ for each even k with $1 < k < 2n$. Then $e_1 \cdots e_{2n} = e_1 e_{2n}$ may be proven by induction on n. For the case $n = 2$ has just been proven. Then consider products of four successive terms, beginning from the left. The product $e_1e_2e_3e_4$ may be shortened unless $e_2 < e_4$, in which event $e_2e_3e_4e_5$ may be shortened unless $e_3 < e_5$, and so on. But $e_{2n-2} \geq e_{2n}$, so at least one of the four-fold products can be shortened and the induction hypothesis applied.

Now let $b \in B$, $b \notin G \cup H$. Express b as an alternating product of members of G and of H, of length at least two. Then, without loss of generality, $b = ugvhw$, where $g \in G$ is a term in the product that is least among all such terms, h is chosen likewise from H, and $u, v, w \in B^1$. From the technical step above, $gvh = gh$ and $(ghw)(ugh) = gh$, whence $b^2 = (ughw)(ughw) = ughw = ugvhw = b$. □

We remark that in fact the band B in the proposition is a *regular* band, in the sense that it satisfies the identity $axaya = ayaxa$. This follows from [12], where it was shown that any band that is generated by two semilattices is regular.

COROLLARY 7.4 *Any monogenic P-restriction semigroup is orthodox. Hence ([30]) any monogenic regular $*$-semigroup is orthodox.*

Proof. Combining Lemma 7.2 with Proposition 7.3, if the P-restriction semigroup S is generated by x, then C_S is a $*$-band. Now consider the homomorphism $\theta : F\mathbf{PR}_x \to F\mathbf{RS}_x$

given by Theorem 4.7. The image U_x is also orthodox. But U_x is a full subsemigroup of $F\mathbf{RS}_x$ and so the latter semigroup is again also orthodox. Therefore every monogenic regular $*$-semigroup is orthodox. □

THEOREM 7.5 *The free monogenic P-restriction semigroup $F\mathbf{PR}_x$ is orthodox and so it is also the free monogenic orthodox P-restriction semigroup $F\mathbf{PO}_x$, as described in §6. The free monogenic regular $*$-semigroup $F\mathbf{RS}_x$ is orthodox and so it is also the free monogenic orthodox $*$-semigroup $F\mathbf{O}_x$. Hence the P-separating homomorphism $\psi : F\mathbf{PR}_x \to F\mathbf{RS}_x$ of Theorem 4.7 is an isomorphism upon U_X.*

Proof. For each of the first two sentences, the first statement follows from the previous corollary and the second is then clear. That ψ is one-one now follows from Theorem 6.4. □

8 Left restriction semigroups

In principle, all the generalities of this paper may be repeated in the one-sided situation(s). That is, given a regular $*$-semigroup $(S, \cdot, ^{-1})$, we may instead consider the reducts $(S, \cdot, ^{+})$ and $(S, \cdot, ^{*})$ separately. Respectively, these fall within the realms of left and right P-Ehresmann semigroups, which were studied extensively in [13]. However, as in the two-sided case, the respective varieties of unary semigroups generated by the regular $*$-semigroups cannot consist of all such P-Ehresmann semigroups (since it is not the case even for inverse semigroups).

At the present time, in the one-sided situation there is no known analogue of the P-restriction semigroups and thus no prospect of addressing the one-sided versions of Question 4.3 in general. However, we can use the analogues of the techniques developed above in order to provide a quick derivation of the structure theorem for free left (similarly for right) restriction semigroups. In view of Corollary 9.2, the free left restriction semigroup on a set coincides with the free left ample semigroup, the structure of which was first determined by J. Fountain [7]. C. Cornock [5] has already constructed the free left restriction semigroups by different methods.

The left-handed versions (and dually, the right-handed versions) of the general definitions will use the same notation, without elaboration, unless otherwise stated. In particular, $\mathcal{P}(\mathbf{V})$ now refers to the variety of left P-Ehresmann semigroups generated by the variety $\mathbf{V}$ of regular $*$-semigroups.

PROPOSITION 8.1 *(cf Proposition 4.6, Theorem 4.7) Let $\mathbf{V}$ be a variety of regular $*$-semigroups and let X be a nonempty set. Then $F\mathcal{P}(\mathbf{V})_X \cong U_X$, the $^{+}$-subsemigroup of $F\mathbf{V}_X$ generated by X. The equality $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ holds if and only if $F\mathbf{PV}_X \cong F\mathcal{P}(\mathbf{V})_X$, for every X [for any countable set X].*

Observe that although the analogue of the homomorphism ψ in Theorem 4.7 exists, it need not be P-separating, since the analogue of Result 4.4 no longer holds. However, the analogue of Lemma 1.4 holds. Thus U_X is generated, as a semigroup, by X, together with its projections (which will no longer comprise all the projections of $F\mathbf{V}_X$).

We will prove the analogue of Proposition 6.1. For our purposes, we may use as our definition of the left ample property the characterization given in [10, Lemma 5.5]: a left restriction semigroup S is *left ample* if $\widetilde{\mathcal{R}}_P$ coincides with the generalized $\mathcal{R}$-relation $\mathcal{R}^*$, which is defined by $a\ \mathcal{R}^*\ b$ if $xa = ya$ if and only if $xb = yb$, for all $x, y \in S^1$. Only the following fact is needed, in addition to the covering result in the final section. References to its origin may be found in the cited paper.

RESULT 8.2 *[10, Theorem 4.12, Lemma 5.6] Let $(S, \cdot, ^+)$ be a left ample semigroup. Then there is a $^+$-preserving embedding in the symmetric inverse semigroup on the set S, regarded as a left restriction semigroup.*

PROPOSITION 8.3 *The variety of left restriction semigroups is generated by the $((\cdot, ^+)$-reducts of the members of) the variety of inverse semigroups. That is, in its one-sided interpretation, the equality $\mathcal{P}(C\mathbf{SL}) = \mathbf{P}C\mathbf{SL}$ holds.*

Proof. As in the proof of the Proposition 6.1, it suffices to show that any left restriction semigroup $(S, \cdot, ^+)$ unarily divides an inverse semigroup. But by Corollary 9.2, S is a (unary) homomorphic image of a left ample, left restriction semigroup, which by Result 8.2 unarily embeds in an inverse semigroup. □

COROLLARY 8.4 *The free left restriction semigroup on a nonempty set X is isomorphic to the $(\cdot, ^+)$-subsemigroup U_X of the free inverse semigroup on X that is generated by X.*

Given any concrete description of the free inverse semigroups, it is then not difficult to identify the members of U_X.

One may also view the reduction from inverse semigroups to left restriction semigroups as a two-step process, via restriction semigroups (the last step entailing 'forgetting' the unary operation *). Clearly the variety of restriction semigroups generates the variety of left restriction semigroups and so, by the same universal algebraic principles as above, the free left restriction semigroup is necessarily embeddable into the free restriction semigroup (as witnessed by the ultimate description cited in the previous paragraph).

9 An elementary proof of the existence of proper covers

A *proper cover* of a restriction semigroup S consists of a proper restriction semigroup T and a surjective $(^+, ^*)$-homomorphism $T \to S$ that separates P_T. The one-sided definitions are entirely analogous. (The definition of 'proper' in the left-handed case requires only that $\widetilde{\mathcal{R}}_P \cap \sigma = \iota$.)

We provide a remarkably elementary simultaneous proof of (1) the existence of proper covers of restriction semigroups, which was at the heart of the proof of Proposition 6.1, and (2) the one-sided version of this result, which was at the heart of the proof of Proposition 8.3. The former statement was first proved in [8, Proposition 6.5 and Lemma 6.6], the proof of which can easily be adapted to yield a proof of the latter. That the cover can be chosen to be ample was not needed in the proof of Proposition 6.1 but, in contrast, was the essential ingredient in the proof of Proposition 8.3.

It is well known (e.g. see [10]) that the relation on a left restriction semigroup S defined by $a \leq b$ if $a = eb$ for some $e \in P_S$ is a partial order that is compatible with both operations, and that $a \leq b$ if and only if $a = a^+b$. If S is a restriction semigroup, then by virtue of the ample conditions, this relation and its dual coincide, so that also $a \leq b$ if and only if $a = ba^*$.

THEOREM 9.1 *Let S be a [left] restriction semigroup. Let $M = S^1$, regarded as a reduced monoid. Then $T = \{(s,m) \in S \times M : s \leq m \text{ in } S^1\}$ is a proper [left] restriction cover for S, which is finite if S is finite.*

Proof. It is immediate from the properties of the partial order that T is a [left] restriction subsemigroup of the [left] restriction semigroup $S \times M$. Clearly $P_T = P_S \times \{1\}$, T contains $\{(s,s) : s \in S\}$, and projection onto the first factor is a covering of S.

Suppose S is left restriction and that $((s,m),(t,n)) \in \widetilde{\mathcal{R}}_P \cap \sigma_T$ in T. Then $s^+ = t^+$ and $m \,\sigma_M\, n$ in M, that is, $m = n$, whence $s = s^+m = t^+n = t$. So T is proper. The restriction case follows by duality. □

The parameters in the theorem can be modified so as to strengthen it, albeit at the cost of preserving finiteness. The left ample property was defined in the previous section. The 'ample' property is defined in the obvious way.

COROLLARY 9.2 *(to the proof) Let M be any reduced monoid that maps homomorphically upon the reduced monoid S^1, via ϕ, say. Then $T = \{(s,m) \in S \times M : s \leq m\phi \text{ in } S^1\}$ is again a proper [left] restriction cover for S. If M is [right] cancellative, then T is [left] ample. In particular, choosing M to be a free monoid that maps onto S^1 yields a proper [left]* ample *cover of S.*

Proof. As in the first paragraph of the proof of the theorem, T is a [left] restriction semigroup (surjectivity of ϕ ensuring that T projects onto S). That T is proper also follows similarly, the only modification being to replace m and n by $m\phi$ and $n\phi$ in the equation $s = s^+m = t^+n = t$.

Now suppose M is right cancellative, the two-sided case then following easily. Since $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_P$ always holds, we must show the reverse inclusion. Let $(s,m),(t,n) \in T$ and suppose $(s,m)^+ = (t,n)^+$, so that $s^+ = t^+$. Also let $(u,p),(v,q) \in T$.

Assume that $(u,p)(s,m) = (v,q)(s,m)$. Thus $us = vs$ and $pm = qm$. We must show that $(u,p)(t,n) = (v,q)(t,n)$. From $s^+ = t^+$, we have $(ut)^+ = (ut^+)^+ = (us^+)^+ = (us)^+$ and, similarly, $(vt)^+ = (vs)^+$. The equation $us = vs$ therefore yields $(ut)^+ = (vt)^+$. Now since $(ut,pn),(vt,qn) \in T$, $ut = (ut)^+(pn)\phi$ and $vt = (vt)^+(qn)\phi$. By right cancellativity of M, $p = q$ and therefore $(ut,pn) = (vt,qn)$, as required.

A similar argument proves that $(u,p)(s,m) = (s,m)$ implies $(v,q)(t,n) = (t,n)$. Therefore $(s,m)\ \mathcal{R}^*\ (t,n)$.
eop

The existence of a proper ample cover was proven in [8, Proposition 6.5 and Lemma 6.6] in the two-sided case and, in essence, in [7] in the one-sided case. Note that finiteness cannot in general be preserved in this result, as a result of the following.

LEMMA 9.3 *Suppose that the finite monoid M, regarded as a (reduced) restriction semigroup, has a finite, left ample cover. Then M is a group.*

Proof. Let T be the cover. Then M is a quotient of T/σ. But it is well known, and easily demonstrated, that if a left restriction semigroup T is left ample, then necessarily $P_T = E_T$ and so T/σ is a *unipotent* monoid, that is, the identity element is its only idempotent. If, further, T/σ is finite, then it must be a group (since its completely simple kernel must be a group, which in turn must be the entire semigroup). Thus M itself is a group. □

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