Combinatorial Proofs of Identities of Alzer and Prodinger and Some Generalizations

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Abstract

We provide combinatorial proofs of identities published by Alzer and Prodinger. These identities include that for integers b, n, and r with $b \geq 1$ and $n-1 \geq r \geq 0$ we have

$$\sum_{k=r}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1} = \sum_{k=0}^{r} \binom{r}{k}^2 \binom{n+k}{2r+b}$$

and for integers b, n, and r with $b \geq 0$ and $n-1 \geq r \geq 0$ we have

$$\sum_{k=0}^{b} (-1)^{b-k} \binom{n+k}{r}^2 \binom{b}{k} = \sum_{k=0}^{r} \binom{r}{k}^2 \binom{n+k}{2r-b}.$$

Our combinatorial proofs generalize squares to sth powers, and involve generalized Eulerian numbers and generalized Delannoy numbers.

1 Introduction and Statement of Results

In this note, we consider several identities involving sums of powers of binomial coefficients. Previously in [4] we proved two identities for $\sum_{k=r}^{n-1} \binom{k}{r}^s$. These proofs were combinatorial, and the two identities involved generalized Eulerian and generalized Delannoy numbers, respectively. Using the special case $s = 2$ (where the generalized Eulerian and generalized Delannoy numbers can be expressed in terms of binomial coefficients, see [4]), we showed that

$$\sum_{k=r}^{n-1} \binom{k}{r}^2 = \sum_{k=0}^{r} \binom{r}{k}^2 \binom{n+k}{2r+1}$$

and

$$\sum_{k=r}^{n-1} \binom{k}{r}^2 = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k+1}$$

hold for integers $n-1 \geq r \geq 0$.

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Alzer and Prodinger [1] then generalized these identities for $s = 2$. Specifically, they used generating function techniques to show that for integers b, n, and r with $b \geq 1$ and $n-1 \geq r \geq 0$, we have

$$\sum_{k=r}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1} = \sum_{k=0}^{r} \binom{r}{k}^2 \binom{n+k}{2r+b}$$

and

$$\sum_{k=r}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1} = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k+b}.$$

Notice that taking $b = 1$ produces the $s = 2$ results from [4]. For integers b, n, and r with $b \geq 0$ and $n-1 \geq r \geq 0$, they also showed

$$\sum_{k=0}^{b} (-1)^{b-k} \binom{n+k}{r}^2 \binom{b}{k} = \sum_{k=0}^{r} \binom{r}{k}^2 \binom{n+k}{2r-b}$$

and

$$\sum_{k=0}^{b} (-1)^{b-k} \binom{n+k}{r}^2 \binom{b}{k} = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k-b}.$$

We will provide combinatorial proofs of each of these identities, as well as generalize them to sth powers (and so $s = 2$ is the special case involving squares, as is done in [4]). In order to state these results, we require a few definitions, which also were used in the combinatorial proofs from [4].

Definition 1. Suppose $\mathbf{m} = \{1, ..., 1, 2, ..., 2, ..., s, ..., s\}$ is a multiset which contains r copies of each element. Let $\left\langle {\mathbf{m} \atop k} \right\rangle$ be the number of permutations of this multiset that have exactly k ascents, meaning that there are exactly k places where entry i is smaller than entry $i+1$.

The numbers $\left\langle {\mathbf{m} \atop k} \right\rangle$ are called *generalized Eulerian numbers*, and are investigated in e.g. [3]. In [4], it is shown that when $s = 2$ we obtain $\left\langle {\mathbf{m} \atop k} \right\rangle = \binom{r}{k}^2$. We also remark that when $s = 2$ we have $\sum_k \left\langle {\mathbf{m} \atop k} \right\rangle = \sum_k \binom{r}{k}^2 = \binom{2r}{r}$, which counts all orderings of the $2r$ elements of the multiset by grouping them according to the number of ascents present. This idea of grouping will be useful in part of our proof.

Definition 2. Fix integers $k, r, s \geq 0$. A *Delannoy path to* $(r, r, \ldots, r)$ in the s-dimensional integer lattice is a path from $(0, 0, \ldots, 0)$ to $(r, r, \ldots, r)$ so that each step in the path increases some non-empty set of coordinates by 1. The number of Delannoy paths to $(r, r, \ldots, r)$ in the s-dimensional integer lattice that use exactly k steps is denoted $d_k^s(r)$.

The numbers $\sum_k d_k^s(r)$ are *generalized Delannoy numbers.* When $s = 2$, it is shown in [4] that $d_k^2(r) = \binom{2(k-r)}{k-r}\binom{k}{2r-k}$. For $s > 2$, Inclusion-Exclusion can be utilized to calculate $d_k^s(r)$ [2, Theorem 11]. Another formula for $d_k^s(r)$ can be found in [5].

With these two definitions, we are now ready to state our main theorem.

Theorem 3. *Let b, n, r, and s be integers with $n-1 \geq r \geq 0$ and $s \geq 1$. Then for $b \geq 1$ we have*

$$\sum_{k=r}^{n-1} \binom{k}{r}^s \binom{n-k-1}{b-1} = \sum_{k \geq 0} \left\langle {\boldsymbol{m} \atop k} \right\rangle \binom{n+k}{sr+b} \tag{1}$$

and

$$\sum_{k=r}^{n-1}\binom{k}{r}^s\binom{n-k-1}{b-1}=\sum_{k=r}^{sr}d_k^s(r)\binom{n}{k+b}. \tag{2}$$

For $b \geq 0$ we also have

$$\sum_{k=0}^{b}(-1)^{b-k}\binom{n+k}{r}^s\binom{b}{k}=\sum_{k\geq 0}\left\langle{\boldsymbol{m} \atop k}\right\rangle\binom{n+k}{sr-b} \tag{3}$$

and

$$\sum_{k=0}^{b}(-1)^{b-k}\binom{n+k}{r}^s\binom{b}{k}=\sum_{k=r}^{sr}d_k^s(r)\binom{n}{k-b}. \tag{4}$$

Using the interpretations of $s=2$ for $\left\langle{\mathbf{m} \atop k}\right\rangle$ and $d_k^2(r)$ provided above, we have the following corollary, which was proved using generating functions in [1].

Corollary 4. *Let b, n, and r be integers with $n-1 \geq r \geq 0$. Then for $b \geq 1$ we have*

$$\sum_{k=r}^{n-1}\binom{k}{r}^2\binom{n-k-1}{b-1}=\sum_{k=0}^{r}\binom{r}{k}^2\binom{n+k}{2r+b}$$

and

$$\sum_{k=r}^{n-1}\binom{k}{r}^2\binom{n-k-1}{b-1}=\sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k+b}.$$

For $b \geq 0$ we also have

$$\sum_{k=0}^{b}(-1)^{b-k}\binom{n+k}{r}^2\binom{b}{k}=\sum_{k=0}^{r}\binom{r}{k}^2\binom{n+k}{2r-b}$$

and

$$\sum_{k=0}^{b}(-1)^{b-k}\binom{n+k}{r}^2\binom{b}{k}=\sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k-b}.$$

While $s=2$ is an appealing special case, we note that the combinatorial proof of Theorem 3 is no more difficult for $s>2$.

We also mention that our results provide combinatorial proofs of the following identities, corresponding to $b=0$ and $b=1$ in the latter two identities of Corollary 4:

$$\binom{n}{r}^2=\sum_{k=0}^{r}\binom{r}{k}^2\binom{n+k}{2r},$$

$$\binom{n}{r}^2=\sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k},$$

$$\binom{n+1}{r}^2-\binom{n}{r}^2=\sum_{k=r}^{2r}\binom{2(k-r)}{k-r}\binom{k}{2r-k}\binom{n}{k-1},$$

and

$$\binom{n+1}{r}^2-\binom{n}{r}^2=\sum_{k=0}^{r}\binom{r}{k}^2\binom{n+k}{2r-1}.$$

In Section 2 we provide a combinatorial proof of (1) and (2), while in Section 3 we provide a combinatorial proof of (3) and (4).

2 Proof of Theorem 3: (1) and (2)

We consider a set of n ordered families that each have s ordered members with different types. We will let $a_{i,j}$ represent the person with family label i and type j (with the natural order on each). We imagine the people in a matrix where the columns increase in i for a fixed j, and the rows increase in j for a fixed i. A fixed family forms a row, and a fixed type forms a column. See Figure 1.

$$\begin{array}{rcccc}
 & \text{(Type 1)} & & & \\
 & \downarrow & & & \\
 & a_{n,1} & a_{n,2} & \cdots & a_{n,s} \\
 & \vdots & \vdots & & \vdots \\
\text{(Family } k+1) \rightarrow & \boxed{a_{k+1,1}} & a_{k+1,2} & \cdots & a_{k+1,s} \\
 & a_{k,1} & a_{k,2} & \cdots & a_{k,s} \\
 & \vdots & \vdots & & \vdots \\
 & a_{3,1} & a_{3,2} & \cdots & a_{3,s} \\
 & a_{2,1} & a_{2,2} & \cdots & a_{2,s} \\
 & a_{1,1} & a_{1,2} & \cdots & a_{1,s}
\end{array}$$

Figure 1: The matrix of people $a_{i,j}$ for Theorem 3 equations (1) and (2), with host highlighted.

What to count: We count the number of parties thrown by a host $a_{k+1,1}$ so that:

(a) (GUESTS) for each type j there are exactly r people $a_{i,j}$ with $i \leq k$ that attend (i.e. r values in each column below the row of the host); and

(b) (HONORED GUESTS) exactly $b-1$ of the people $a_{i,1}$ with $i > k+1$ also attend (i.e. $b-1$ members in the first column above the host).

First way to count: We condition on the *family label of the host being* $k+1$. There are $\binom{k}{r}$ ways to choose r guests of each type, and there are $\binom{n-(k+1)}{b-1}$ ways to choose the $b-1$ honored guests. This gives $\sum_{k=r}^{n-1} \binom{k}{r}^s \binom{n-k-1}{b-1}$ such parties.

Second way to count: For each party, we have the people at the party line-up by ascending family label, and if two people have the same family label we require them to be in ascending order by type. In other words, a_{i_1,j_1} comes before a_{i_2,j_2} if $i_1 < i_2$, or if $i_1 = i_2$ and $j_1 < j_2$. In terms of the matrix from Figure 1, our ordering starts at the bottom row, read left-to-right, then reads the second row left-to-right, and continues until the top row is read left-to-right.

We then focus on the types of the guests, which gives an ordering of the multiset $\mathbf{m} = \{1, \ldots, 1, 2, \ldots, 2, \ldots, s, \ldots, s\}$ (which contains r copies of each element). Each ordering of this multiset $\mathbf{m}$ has some number of ascents. We remark that some ascents may occur within a family, while others may occur between members of different families. We group the parties together that *have exactly* k *ascents in this ordering.*

How many different parties are in one grouping? Given a multiset with exactly k ascents, we need to count the possible family labels that give rise to this multiset. This is the number of ways of assigning one of the n family labels to each of the $sr+b$ people at the party. By the ordering, we know that the family labels must be weakly increasing, and two members that are assigned the same family label must be part of an ascent. So we choose $sr+b$ things from $n+k$ possibilities: one for each of the n family labels, and k extra 'ascent boxes.' The selected family labels are assigned in increasing order with the rule that if the ith ascent box is chosen, then the people in the ith ascent share the same family label. The largest b family label choices determine the host family label and the $b-1$ honored guests.

Now, each party falls into one particular group (based on number of ascents of the ordered guests) and is counted once by an appropriate choice of family labels and/or ascent boxes. But furthermore any ordering of the multiset $\mathbf{m}$ with k ascents and any choice of $sr+b$ of the $n+k$ possibilities for family labels and/or ascent boxes forms a unique party. This gives $\sum_{k\geq 0} \left\langle {\mathbf{m} \atop k} \right\rangle \binom{n+k}{sr+b}$ such parties.

Third way to count: We condition on the *number of family labels that appear.* For each party, we let k denote the number of distinct guest family labels present, which implies that there are $k+b$ total family labels at the party. The largest b family labels determine the honored guests and the host. We consider the remaining k family labels in increasing order, and for each of these k family labels we know that some number of family members are at the party. Each family label i corresponds to taking a Delannoy step by letting $a_{i,j}$ correspond to the jth coordinate in the s-dimensional lattice. Then the set of types with family label i at the party corresponds to the coordinates to change when making the next Delannoy step. After all k steps are taken, there are exactly r guests of each of the s types.

We choose the family labels in $\binom{n}{k+b}$ ways, which determines the host and the honored guests. Then a Delannoy path determines which guests show up with these family labels present. This gives $\sum_{k=r}^{sr} d_k^s(r)\binom{n}{k+b}$ such parties.

3 Proof of Theorem 3: (3) and (4)

We consider a set of $n+b$ ordered families that each have s ordered members with different types. We again let $a_{i,j}$ represent the person with family label i and type j (with the natural order on each). We imagine the people in a matrix, as before. See Figure 2.

$$
\begin{array}{cccc}
a_{n+b,1} & a_{n+b,2} & \cdots & a_{n+b,s} \\
\vdots & \vdots & & \vdots \\
a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,s} \\
\hline
a_{n,1} & a_{n,2} & \cdots & a_{n,s} \\
\vdots & \vdots & & \vdots \\
a_{3,1} & a_{3,2} & \cdots & a_{3,s} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,s} \\
a_{1,1} & a_{1,2} & \cdots & a_{1,s}
\end{array}
$$

Figure 2: The matrix of people $a_{i,j}$ for Theorem 3 equations (3) and (4).

What to count: We count the number of parties that have exactly r people of each type (column) attending so that

(c) at least one person attends from each family with label larger than n, i.e., for all $i \geq n+1$ there exists a j so that $a_{i,j}$ attends the party.

First way to count: We use *inclusion-exclusion* to count the number of parties meeting condition (c). By conditioning on i family labels that were missed from the b family labels in

$\{n+1, n+2, \ldots, n+b\}$ (which happens in $\binom{b}{i}$ ways), we obtain

$$\begin{aligned}\binom{n+b}{r}^s - \binom{b}{1}\binom{n+b-1}{r}^s + \binom{b}{2}\binom{n+b-2}{r}^s - \quad \cdots \quad &+(-1)^b\binom{n}{r}^s \\ = \quad &\sum_{k=0}^{b}(-1)^{b-k}\binom{k+n}{r}^s\binom{b}{k}\end{aligned}$$

such parties.

Second way to count: We order all of the attendees as before: increasing in family label and increasing by type for fixed family label. We again group the parties based on *having exactly* k *ascents in this ordering*; note that here all attendees are included in the ordering, not just the guests.

How many different parties are in one such grouping? Given an ordering of the multiset $\mathbf{m} = \{1, \ldots, 1, 2, \ldots, 2, \ldots, s, \ldots, s\}$ (which contains r copies of each element), we need to count the possible family labels that give rise to this ordering. There are $n+b$ possible family labels, which must appear in weakly increasing order, but we know that the b largest family labels must appear at the party by (c). This gives n other family labels and k ascent boxes. We must choose $sr-b$ elements from these $n+k$ possibilities. As before, the selected family labels are assigned in increasing order with the rule that if the ith ascent box is chosen, then the people in the ith ascent share the same family label. So we have $sr-b$ family labels and/or ascent boxes in addition to the family labels $n+1, \ldots, n+b$, and as the family labels must be weakly increasing this determines the party.

Again, each party falls into one particular group and is counted once by an appropriate choice of family labels and/or ascent boxes. Furthermore, any ordering of the multiset $\mathbf{m}$ with k ascents and any choice of $sr-b$ of the $n+k$ possibilities forms a unique party. This gives $\sum_{k\geq 0}\left\langle {\mathbf{m} \atop k} \right\rangle\binom{n+k}{sr-b}$ such parties.

Third way to count: We again condition on *the number of family labels that appear.* Suppose that there are exactly k family labels present. Since the largest b family labels must appear, we need to choose $k-b$ of the remaining n family labels to appear. We then need to choose which types (columns) from that family label are at the party. This corresponds, as before, to the coordinates that increase in a given Delannoy step. This gives $\sum_{k=r}^{sr} d_k^s(r)\binom{n}{k-b}$ such parties.

References

[1] H. Alzer and H. Prodinger, On combinatorial identities of Engbers and Stocker, *Integers* **17** (2017), #A13.

[2] J. Caughman, C. Dunn, N. Neudauer, and C. Starr, Counting lattice chains and Delannoy paths in higher dimensions, *Discrete Math.* **311** (2011), 1803-1812.

[3] J. F. Dillon and D. P. Roselle, Simon Newcomb's Problem, *SIAM J. Appl. Math.* **17** (1969), 1086-1093.

[4] J. Engbers and C. Stocker, Two combinatorial proofs of identities involving sums of powers of binomial coefficients, *Integers* **16** (2016), #A58.

[5] M. Tărnăuceanu, The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers, *European J. Combin.* **30** (2009), 283-287.