Maximizing H-colorings of connected graphs with fixed minimum degree

John Engbers*

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Abstract

For graphs G and H, an H-coloring of G is a map from the vertices of G to the vertices of H that preserves edge adjacency. We consider the following extremal enumerative question: for a given H, which connected n-vertex graph with minimum degree δ maximizes the number of H-colorings? We show that for non-regular H and sufficiently large n, the complete bipartite graph $K_{\delta,n-\delta}$ is the unique maximizer. As a corollary, for non-regular H and sufficiently large n the graph $K_{k,n-k}$ is the unique k-connected graph that maximizes the number of H-colorings among all k-connected graphs. Finally, we show that this conclusion does not hold for all regular H by exhibiting a connected n-vertex graph with minimum degree δ which has more K_q-colorings (for sufficiently large q and n) than $K_{\delta,n-\delta}$.

1 Introduction and statement of results

Given a simple loopless graph $G = (V(G), E(G))$ and a graph $H = (V(H), E(H))$ without multi-edges but possibly with loops, an *H-coloring* of G is an adjacency preserving map $f : V(G) \to V(H)$ (i.e., f satisfies $f(v) \sim_H f(w)$ whenever $v \sim_G w$). An H-coloring of G is also referred to as a *graph homomorphism* from G to H. When $H = H_{\text{ind}}$, the graph consisting of a single edge and a loop on one endvertex, an H_{ind}-coloring of a graph G may be identified with an independent set in G, and when $H = K_q$, the complete unlooped graph on q vertices, a K_q-coloring of G may be identified with a proper q-coloring of G. We let $\hom(G, H)$ denote the number of H-colorings of G.

Much recent work has been done on the following extremal enumerative question: given a family of graphs $\mathcal{G}$ and a graph H, which $G \in \mathcal{G}$ maximizes the quantity

*john.engbers@marquette.edu; Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201.

hom(G,H)? For a survey of results and conjectures surrounding questions of this type, see [2].

A family relevant to the current paper is the family of bipartite n-vertex d-regular graphs. Here, Galvin and Tetali [6] (extending work of Kahn [7] for independent sets) showed that the graph in this family with the maximum number of H-colorings is $\frac{n}{2d}K_{d,d}$, which consists of $\frac{n}{2d}$ disjoint copies of the complete bipartite graph $K_{d,d}$. (Note that throughout we assume n is large and d is fixed, and we will assume any necessary divisibility conditions between d and n.) When moving to the larger class of *all* n-vertex d-regular graphs, for some H the graph $\frac{n}{2d}K_{d,d}$ still maximizes the number of H-colorings (see [10, 11]) but for other graphs H the graph $\frac{n}{d+1}K_{d+1}$, which consists of $\frac{n}{d+1}$ disjoint copies of the complete graph K_{d+1}, maximizes the number of H-colorings (for example, take H to be the disjoint union of two looped vertices). Recently Sernau [8] showed that for certain H there are disjoint copies of other small graphs (in particular, not copies of $K_{d,d}$ or K_{d+1}) that maximize the number of H-colorings, and also gave new families of H for which copies of K_{d+1} are the maximizer and other H for which copies of $K_{d,d}$ are the maximizer.

What drives some of these results is that the maximizing graph G is obtained by taking the disjoint union of some number of copies of the same fixed small graph. In particular, the idea in [8] to find maximizing graphs other than copies of K_{d+1} or $K_{d,d}$ is to find an H and a graph with the smallest number of vertices (more than $d+1$ but less than $2d$) that has a non-zero number of H-colorings, and then to let H' be a large number of copies of H. In this way, the H'-coloring count is driven by having the largest number of components (each with a non-zero number of H-colorings of each component) in an n-vertex d-regular graph. It is worth noting that H' can be modified to be connected with the maximizing graph remaining the same [8]. While there is a single graph having the most number of H-colorings within the family of *bipartite* n-vertex d-regular graphs for each possible H, at this point it is unclear what the smallest list of n-vertex d-regular graphs is so that for each possible H, some graph in the list maximizes the number of H-colorings of graphs in the family.

Another natural and related family that has been studied is the family of n-vertex graphs with minimum degree δ. Here we have complete extremal results for H_{ind}-colorings [3]. The extremal question for all other H was investigated for $\delta = 1$ and $\delta = 2$ in [4]. For larger values of δ, a general answer is still unknown, but as shown in [4] there is a family of H for which $K_{\delta,n-\delta}$ maximizes the number of H-colorings for all δ when n is sufficiently large. However, it is not difficult to construct H for which some other graph maximizes the number of H-colorings for all δ. Indeed, as with regular graphs, there are H' for which having a large number of components with a non-zero number of H-colorings significantly increases the H'-coloring count, and so the connected graph $K_{\delta,n-\delta}$ does not maximize the number of H'-colorings.

Notice that an H-coloring of G requires, by definition, a component of G to be mapped to a component of H, and so it is natural to restrict our attention to n-vertex graphs with minimum degree δ which are connected. Doing this produces a much more coherent structure of the maximizing graphs for almost all H, even large H. To describe these results, we now state the degree convention that we will use.

Convention. The degree of a vertex v is $d(v) = |\{w : v \sim w\}|$. In particular, a loop on a vertex adds one to the degree. When considering H-colorings of G, we let Δ denote the maximum degree of a vertex in H.

It was shown in [9] (with a short proof given in [1]) that in the family of trees the star $K_{1,n-1}$ has the largest number of H-colorings for any H. In [5], this tree result is proved for sufficiently large n. The proof technique from [5] generalizes to a proof for the family of 2-connected graphs on n vertices, where for sufficiently large n the graph $K_{2,n-2}$ maximizes the number of H-colorings for any connected non-regular H. In [5], the authors ask what happens for k-connected graphs where $k > 2$. We answer this question for connected non-regular H and sufficiently large n.

Theorem 1.1. *Fix $\delta \geq 2$. Let H be connected and non-regular, and let G be an n-vertex connected graph with minimum degree at least δ. Then there exists a constant $c(\delta, H)$ so that for $n \geq c(\delta, H)$ we have*

$$\hom(G, H) \leq \hom(K_{\delta,n-\delta}, H),$$

with equality if and only if $G = K_{\delta,n-\delta}$.

When $\delta = 1$, the conclusion is still true (for all n) as a corollary to the result for trees [9]. Also, notice that a k-connected graph must be connected and have minimum degree at least k. Since $K_{k,n-k}$ is k-connected, the following is immediate.

Corollary 1.2. *Fix $k \geq 2$. Let H be connected and non-regular, and let G be an n-vertex k-connected graph. Then there exists a constant $c(k, H)$ so that for $n \geq c(k, H)$ we have*

$$\hom(G, H) \leq \hom(K_{k,n-k}, H),$$

with equality if and only if $G = K_{k,n-k}$.

The proof of Theorem 1.1 uses a stability technique similar to that in [4] and [5]. In particular, we first show that if a connected n-vertex graph with minimum degree δ isn't structurally close to $K_{\delta,n-\delta}$ (specifically, if it doesn't contain a subgraph isomorphic to $K_{\delta,bn}$ for some small constant b), then it admits fewer H-colorings than $K_{\delta,n-\delta}$. If it is structurally close to $K_{\delta,n-\delta}$ but isn't $K_{\delta,n-\delta}$, then a second argument will show that it again admits fewer H-colorings than $K_{\delta,n-\delta}$.

It is shown in [5] that when $H = K_4$, for example, $\hom(C_n, K_4) > \hom(K_{2,n-2}, K_4)$, showing that the conclusion of Theorem 1.1 doesn't hold in general when removing the non-regularity restriction on H, even for connected H. We also show here that when $H = K_q$ for large enough q, there are connected n-vertex graphs with minimum degree δ with more K_q-colorings than $K_{\delta,n-\delta}$. To do so, consider first the graph $\frac{n}{\delta+1}K_{\delta+1}$ and in this graph choose one specified vertex in each copy of $K_{\delta+1}$, form a star on those specified vertices, and let G_1 be the resulting graph.

Theorem 1.3. *Fix $\delta \geq 2$ and let $H = K_q$. For sufficiently large n (depending on δ, with $(\delta+1)|n$) and sufficiently large q (depending on n and δ) we have*

$$\hom(K_{\delta,n-\delta}, H) < \hom(G_1, H),$$

where G_1 is the graph in the preceding paragraph.

While the graph G_1 in Theorem 1.3 has more K_q-colorings than $K_{\delta,n-\delta}$, it isn't clear which connected n-vertex graph with minimum degree δ maximizes the number of K_q-colorings in this family.

2 Proofs of Theorems 1.1 and 1.3

Recall that a loop on a vertex adds one to the degree and that Δ refers to the maximum degree of a vertex in H. We begin with a few useful lemmas.

Lemma 2.1. *Fix δ. For non-regular connected H there exists a constant ℓ (depending on δ and H) such that if $k \geq \ell$, then $hom(P_k, H) < \Delta^{k-\delta}$.*

Proof. The proof of Lemma 3.1 in [5] shows that for non-regular connected H we have $\hom(P_k, H) \leq |V(H)|^2 c_1 \lambda^k$ for some $\lambda < \Delta$ (where here c_1 and λ are constants depending on H). This implies the result. □

For shorter paths, we have the following, which is Lemma 4.5 from [5].

Lemma 2.2 ([5])**.** *Suppose H is not the complete looped graph on Δ vertices or $K_{\Delta,\Delta}$. Then for any two vertices i, j of H and for $k \geq 4$ there are at most $(\Delta^2 - 1)\Delta^{k-4}$ H-colorings of P_k that map the initial vertex of that path to i and the terminal vertex to j.*

Notice that if H is a non-regular graph, then the conclusion of Lemma 2.2 applies. Furthermore, we remark that colorings of P_k where the endpoints receive the same color (i.e. $i = j$ in Lemma 2.2) are in bijection with colorings of C_{k-1} with one fixed vertex receiving that color. In other words, Lemma 2.2 may be applied on a cycle C_{k-1} that has one vertex already colored.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\delta \geq 2$ be fixed and H be a connected non-regular graph. Following a definition from [4], we let $S(\delta, H)$ denote the vectors in $V(H)^\delta$ with the property that the elements of the vector have Δ common neighbors in H, and let $s(\delta, H) = |S(\delta, H)|$. Notice that $S(\delta, H)$ is always non-empty, since if v is any vertex of degree Δ in H then $(v, v, \ldots, v)$ is in $S(\delta, H)$. Then we have the lower bound

$$\hom(K_{\delta,n-\delta}, H) \geq s(\delta, H)\Delta^{n-\delta} \geq \Delta^{n-\delta}, \tag{1}$$

which colors the size δ partition class of $K_{\delta,n-\delta}$ using an element of $S(\delta, H)$, and independently colors each vertex of the size $n-\delta$ partition class using one of the Δ common neighbors.

Now let G be a connected n-vertex graph with minimum degree δ. The proof will proceed by iteratively coloring parts of the graph G; throughout we use connectivity to always color vertices adjacent to a previously colored vertex (after an initial subset of vertices is colored). This iterative coloring procedure produces an upper bound on $\hom(G, H)$ which in most cases is smaller than the lower bound on $\hom(K_{\delta,n-\delta}, H)$ given in (1).

Suppose that G has a path of length $k \geq \ell$ (with ℓ as given in Lemma 2.1). Coloring the path, and then iteratively coloring the remaining vertices — which each have at most Δ possibilities for their color — we have

$$\hom(G, H) < \Delta^{k-\delta}\Delta^{n-k} = \Delta^{n-\delta} \leq \hom(K_{\delta,n-\delta}, H).$$

Therefore, from now on we assume that the graph G does not have any paths of length $k \geq \ell$.

Fix a vertex v and color that vertex from H; there are $|V(H)|$ ways that this can be done. Then run the following procedure.

1. Denote by $C \subseteq V(G)$ the set of vertices that have already received a color.

2. Search for a non-trivial component in the induced graph on vertex set $V(G) \setminus C$, i.e., a non-trivial component in the induced graph on the set of uncolored vertices.

   (a) If there is a non-trivial component that is a tree, find a maximal path P in that component and iteratively color the vertices of the path P (since $\delta(G) \geq 2$, the endpoints of P have a neighbor in C). Then proceed back to step 1.

   (b) If all non-trivial components are not trees, choose one such component and fix a cycle in the component. Let X be the graph consisting of the cycle and the shortest path from a vertex on the cycle to a vertex in C. Iteratively color the vertices of X by coloring the path first and then the cycle. Then proceed back to step 1.

   (c) If all components are trivial, stop.

Note that since we're assuming that there are no paths of length at least ℓ in G, at every iteration of this coloring scheme we color fewer than ℓ previously uncolored vertices. (In 2(b), removing one edge from the cycle that is adjacent to a vertex on the path makes X into a path.) Also, if we run step 2(a) and color k_1 new vertices, Lemma 2.2 shows that there are at most $(\Delta^2 - 1)\Delta^{k_1-2}$ ways this can be done. For step 2(b), each vertex in the path of X, including one vertex on the cycle C_{k_2}, has at most Δ possible colors. And once one vertex on the cycle has been colored, the discussion following the statement of Lemma 2.2 shows that there are at most $(\Delta^2 - 1)\Delta^{k_2-3}$ ways to color the remaining $k_2 - 1$ vertices of the cycle. In short, each iteration that colors k previously uncolored vertices can be done in at most $(1 - \frac{1}{\Delta^2})\Delta^k$ ways, regardless of the colors appearing on the vertices of C.

If there are a iterations of this procedure run, then by coloring any remaining vertices using at most Δ possible choices for each uncolored vertex, we have

$$\hom(G, H) \leq |V(H)|(1 - \frac{1}{\Delta^2})^a \Delta^n < \Delta^{n-\delta} \leq \hom(K_{\delta,n-\delta}, H)$$

whenever $a > \Delta^2 (\log |V(H)| + \delta \log \Delta)$; here $|V(H)|$ comes from the initial vertex, and $(1 - \frac{1}{\Delta^2})\Delta^k$ comes from each iteration that colors k new vertices.

So then suppose that $a \leq \Delta^2 (\log |V(H)| + \delta \log \Delta)$. In this case, the final output of colored vertices $C \subseteq V(G)$ upon termination has $|C|$ bounded above by a constant depending on δ and H, and therefore $n - |C|$ vertices remain uncolored. Since the procedure terminated, the vertices in $V(G) \setminus C$ form an independent set, and so in particular each vertex $v \in V(G) \setminus C$ has at least δ neighbors in the set C. By the pigeonhole principle, some set of δ vertices (among the set C) must have at least $(n - |C|)/\binom{|C|}{\delta}$ common neighbors. In other words, the graphs G for which we do not yet have $\hom(G, H) < \hom(K_{\delta,n-\delta}, H)$ are those which have a (not necessarily induced) subgraph isomorphic to $K_{\delta,bn}$ for some small constant b (depending on δ and H). Note that these graphs are "structurally close" to $K_{\delta,n-\delta}$.

So, finally, suppose that $G \neq K_{\delta,n-\delta}$ contains $K_{\delta,bn}$ for some constant b (depending on δ and H). We will consider two cases: when G does not have an isomorphic copy of $K_{\delta,n-\delta}$ as a subgraph, and finally when it does.

G contains no subgraph isomorphic to $K_{\delta,n-\delta}$: Suppose first that G does not contain a subgraph isomorphic to $K_{\delta,n-\delta}$. This means that the size δ partition class in $K_{\delta,bn}$ is not a dominating set. In particular, the induced subgraph on the vertices outside of the size δ partition class must have a non-trivial component. As before, if some component is a tree, let X be a maximal path in the tree; if all components are not trees, let X be the union of a cycle and a shortest path from a vertex on the cycle to a vertex in the size δ partition class. With X now defined, we partition the H-colorings of G based on whether the colors in the size δ partition class form a vector in $S(\delta, H)$ or not.

If they do form a vector in $S(\delta, H)$, we then color X, and then the rest of the graph. Using Lemma 2.2 and discussion following it on X, this gives at most

$$s(\delta, H)(\Delta^2 - 1)\Delta^{n-\delta-2}$$

H-colorings of G of this type.

If the colors on the size δ partition class do not form a vector in $S(\delta, H)$, then the at least bn neighbors have at most $\Delta - 1$ possible choices for their color. Using at most Δ choices for each of the remaining $n-\delta-bn$ vertices, there are at most

$$|V(H)|^\delta \cdot (\Delta-1)^{bn}\Delta^{n-\delta-bn}$$

H-colorings of G of this type.

Putting these together, we have

$$\begin{aligned} \hom(G,H) &\leq s(\delta,H)(\Delta^2-1)\Delta^{n-\delta-2} + |V(H)|^\delta \cdot (\Delta-1)^{bn}\Delta^{n-\delta-bn} \\ &\leq s(\delta,H)\Delta^{n-\delta} - s(\delta,H)\Delta^{n-\delta-2} + |V(H)|^\delta \cdot e^{-bn/\Delta}\Delta^{n-\delta} \\ &< s(\delta,H)\Delta^{n-\delta} \end{aligned}$$

where the final inequality holds for all sufficiently large n.

G **contains a subgraph isomorphic to** $K_{\delta,n-\delta}$**:** Lastly, suppose that G contains a subgraph isomorphic to $K_{\delta,n-\delta}$. Since $G \neq K_{\delta,n-\delta}$, it suffices to show that adding any edge to $K_{\delta,n-\delta}$ strictly decreases the number of H-colorings. To this end, we suppose that G is obtained from $K_{\delta,n-\delta}$ by the addition of a single edge, and we show that each H-coloring of $K_{\delta,n-\delta}$ is also an H-coloring of G only when H is the complete looped graph.

Suppose that i and j are distinct adjacent vertices of H. Then the mapping that sends the size δ partition class of $K_{\delta,n-\delta}$ to i and the other partition class to j is an H-coloring of $K_{\delta,n-\delta}$, and similarly the mapping that sends the size δ partition class to j and the other partition class to i is another H-coloring of $K_{\delta,n-\delta}$. This is only an H-coloring of G if i and j are both looped. By similar reasoning, if i and j are adjacent in H and j and k are also adjacent in H, then i and k must be adjacent. As H is connected, this implies that H is a fully looped complete graph. So if G is obtained from $K_{\delta,n-\delta}$ by adding an edge and H is non-regular, then $\hom(G,H) < \hom(K_{\delta,n-\delta},H)$. □

We conclude this section with the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $H = K_q$. First, we find an upper bound on $\hom(K_{\delta,n-\delta},K_q)$. There are

$$q(q-1)\cdots(q-\delta+1)(q-\delta)^{n-\delta}$$

colorings of $K_{\delta,n-\delta}$ that use distinct colors on the size δ partition class. In a similar way, there are at most

$$\delta^2 \cdot q \cdot q^{\delta-2} \cdot (q-1)^{n-\delta} \leq \delta^2 q^{n-1}$$

colorings of $K_{\delta,n-\delta}$ that have the same color on two or more vertices in the size δ partition class. This means that there are at most

$$q(q-1)\cdots(q-\delta+1)(q-\delta)^{n-\delta} + \delta^2 q^{n-1}$$

K_q-colorings of $K_{\delta,n-\delta}$.

For the graph G_1, we first color the $K_{\delta+1}$ containing the center of the star, and then the remaining copies of $K_{\delta+1}$. This gives

$$\hom(G_1, K_q) = q(q-1)\cdots(q-\delta)\left[(q-1)(q-1)(q-2)\cdots(q-\delta)\right]^{\frac{n}{\delta+1}-1}.$$

The coefficient of q^{n-1} in the upper bound on $\hom(K_{\delta,n-\delta}, H)$ is $-n\delta + \frac{3\delta^2+\delta}{2}$ and the coefficient of q^{n-1} in $\hom(G_1, K_q)$ is $-\frac{n}{\delta+1}+1-\frac{n\delta}{2}$. So $\hom(G_1, K_q)-\hom(K_{\delta,n-\delta}, K_q)$ is bounded below by a polynomial in q of degree $n-1$ with leading coefficient $\frac{n\delta}{2} - \frac{n}{\delta+1} + \frac{-3\delta^2-\delta+2}{2}$. This shows that for $\delta \geq 2$, sufficiently large n (depending on δ), and sufficiently large q (depending on n and δ) we have $\hom(K_{\delta,n-\delta}, K_q) < \hom(G_1, K_q)$. □

3 Concluding Remarks

Here we highlight a few open questions related to the contents of this paper. We showed that for sufficiently large n and connected non-regular H, the number of H-colorings of a connected n-vertex graph G with minimum degree δ is maximized uniquely when $G = K_{\delta,n-\delta}$.

First, it would be interesting to know if this behavior holds for all $n \geq 2\delta$, as it does for independent sets [3].

Question 3.1. *Fix a non-regular H and $n \geq 2\delta$. Does $K_{\delta,n-\delta}$ maximize the number of H-colorings over all connected n-vertex graphs with minimum degree δ?*

The behavior for non-trivial regular H is still unknown.

Question 3.2. *Fix a regular H and $n \geq 2\delta$. Which connected n-vertex graph with minimum degree δ maximizes the number of H-colorings?*

In particular, we have the following interesting extremal question for proper q-colorings.

Question 3.3. *For a given δ, n, and q, which connected n-vertex graph with minimum degree δ maximizes the number of proper q-colorings?*

It was shown [5] that in the case $\delta = 2$ we have $\hom(K_{2,n-2}, K_q) < \hom(C_n, K_q)$ for all fixed $q \geq 4$ and $n \geq 3$, and so in particular $K_{\delta,n-\delta}$ is not always the maximizing graph for fixed q, even for sufficiently large n. Note also that when $q = 2$, any bipartite graph maximizes the number of K_q-colorings.

Investigating what happens when $n < 2\delta$ would also be interesting; notice that in this range all n-vertex graphs with minimum degree at least δ are connected. The the following question has been answered in the special case of independent sets [3].

Question 3.4. *Fix any H, and let $n < 2\delta$. Which n-vertex graph with minimum degree δ maximizes the number of H-colorings?*

We can ask these questions for the family of all n-vertex graphs with minimum degree δ; answers for some H and sufficiently large n are given in [4].

Question 3.5. *Fix any H. Which n-vertex graph with minimum degree δ maximizes the number of H-colorings?*

Question 3.6. *For a given δ, n, and q, which n-vertex graph with minimum degree δ maximizes the number of proper q-colorings?*

Note that when $q = 2$, we seek the n-vertex graph with minimum degree δ that has the largest number of bipartite components, and so the maximizing graph is $\frac{n}{2\delta}K_{\delta,\delta}$.

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