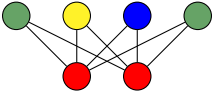


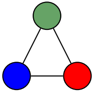
Counting graph homomorphisms

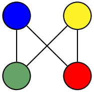
John Engbers

Department of Mathematics
University of Notre Dame

MIGHTY LIII — Iowa State University, Ames, IA

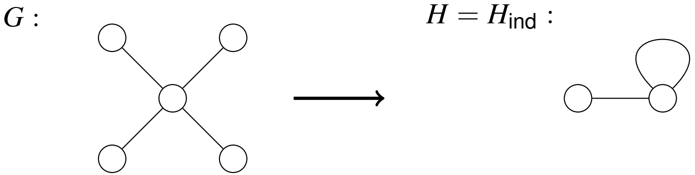
September 22, 2012





An extremal question

Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.



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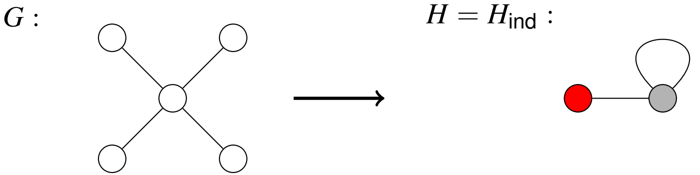

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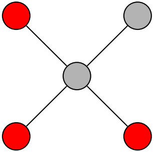
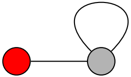
G :



$\longrightarrow$

$H = H_{\text{ind}}$:



Examples: independent sets,

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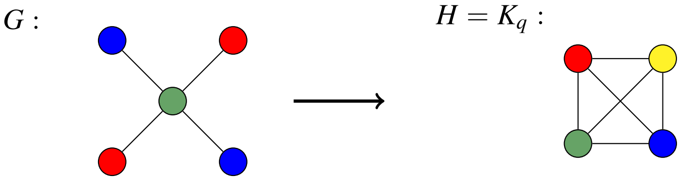

Examples: independent sets, proper q-colorings,

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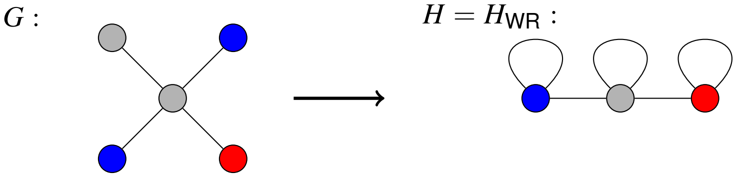

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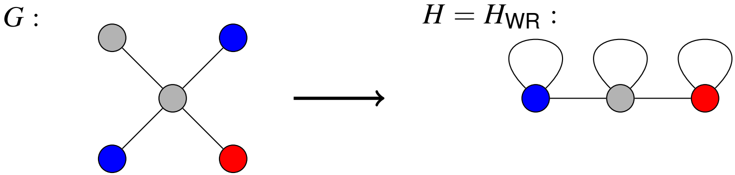

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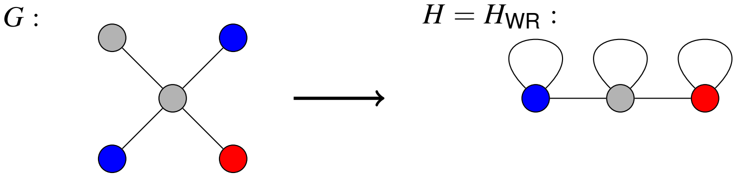

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Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

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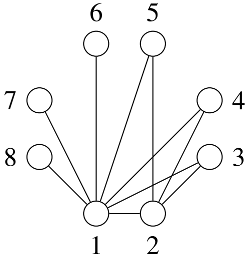
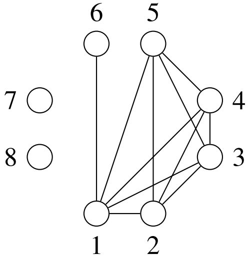


Lex$(8, 11)$



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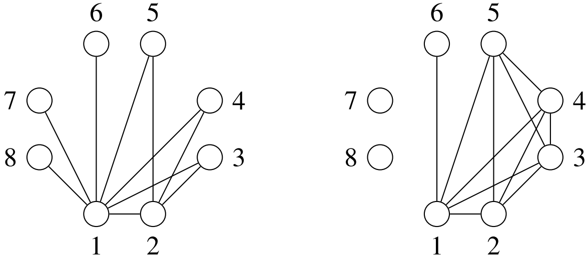

Lex$(8, 11)$ Colex$(8, 11)$

  - $H = K_q$: various results, still open

Various families

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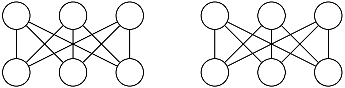

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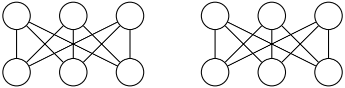

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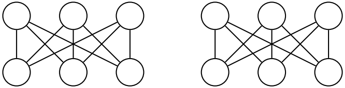

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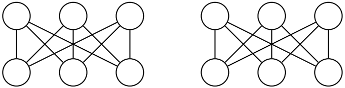

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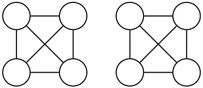

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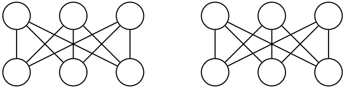

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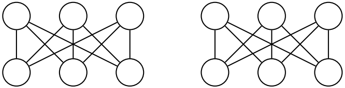

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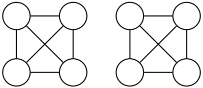

. . .

Open Conjecture

Fix H. For $\mathcal{G} = n$-vertex d-regular graphs, $\hom(G,H)$ is maximized when $G = \frac{n}{2d}K_{d,d}$ or $\frac{n}{d+1}K_{d+1}$.

Today's family

$\mathcal{G} = \mathcal{G}(n, \delta)$ = n-vertex graphs with minimum degree δ

Today's Question

Fix H. Which $G \in \mathcal{G}(n, \delta)$ maximizes $\hom(G, H)$?

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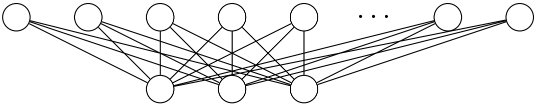
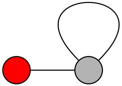
Theorem (Galvin, 2011)

For all $G \in \mathcal{G}(n,\delta)$ and $n \geq 8\delta^2$, $\hom(G,H_{ind})$ is maximized when $G = K_{\delta,n-\delta}$.





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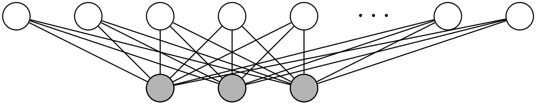

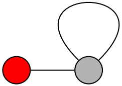

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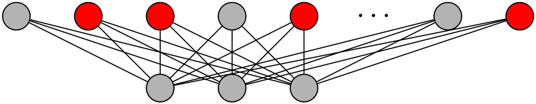

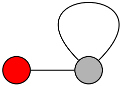

Note: $\hom(K_{\delta,n-\delta},H_{\text{ind}}) \geq 2^{n-\delta}$.

Convention: Loops add 1 to degree

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Fix H. For all $G \in \mathcal{G}(n,\delta)$ and n large enough, $\hom(G,H)$ is maximized when $G = K_{\delta,n-\delta}$, $\frac{n}{2\delta}K_{\delta,\delta}$, or $\frac{n}{\delta+1}K_{\delta+1}$.

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Progress:

Theorem (E., 2012+)

- *Conjecture is true for $\delta = 1$, $\delta = 2$.*
- *Suppose that H satisfies $\sum_{v \in V(H)} d(v) < (\Delta_H)^2$. Then, for $n > c^\delta$ and $G \in \mathcal{G}(n,\delta)$, $\hom(G,H)$ is maximized when $G = K_{\delta,n-\delta}$.*

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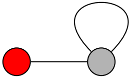
Examples:

- $H_{\text{ind}} : \sum d(v) = 3; (\Delta_H)^2 = 4$ ✓



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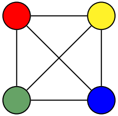

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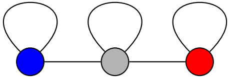
- $K_q : \sum d(v) = q(q-1); (\Delta_H)^2 = (q-1)^2$ ✘
- $H_{\text{WR}} : \sum d(v) = 7; (\Delta_H)^2 = 9$ ✓



Idea of proof for $H = H_{\text{WR}}$

Goal: $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta, n-\delta}$

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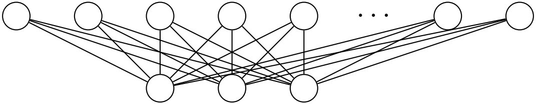

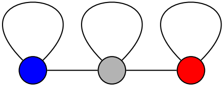

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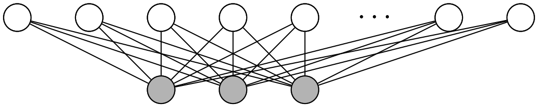

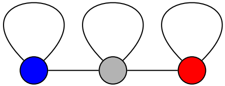

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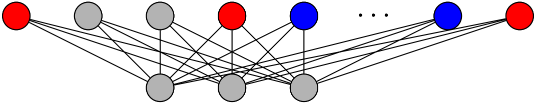

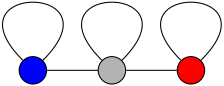

$$\hom(K_{\delta, n-\delta}, H_{\text{WR}}) \geq 3^{n-\delta}$$

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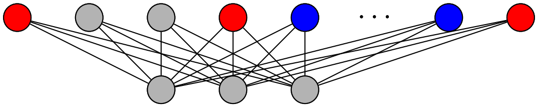

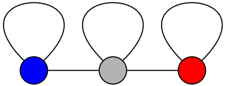

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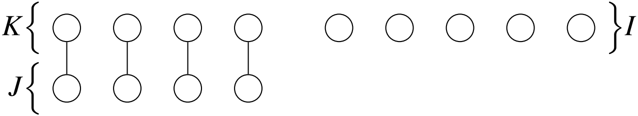
Idea: Partition $\mathcal{G}(n, \delta)$ by the size of maximum matching M.



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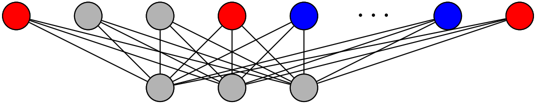

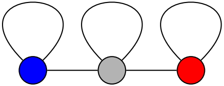

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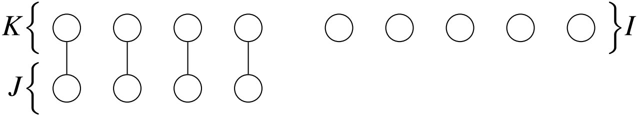

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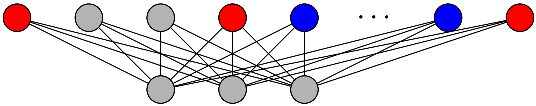

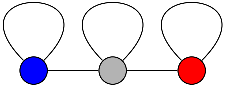

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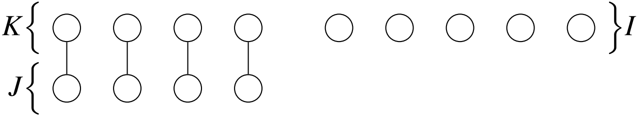

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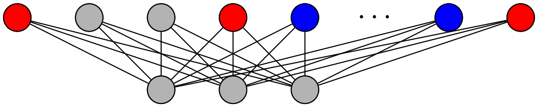

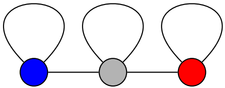

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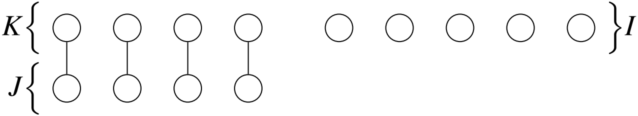

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Any maximizing graph G has $|M| \leq c\delta$

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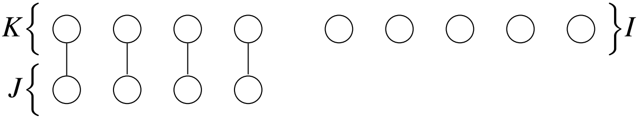

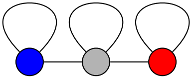

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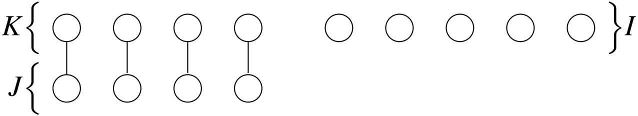

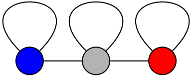

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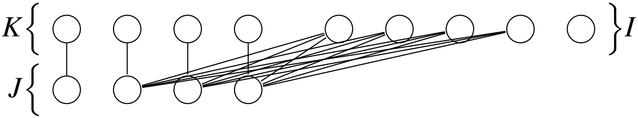

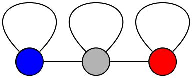

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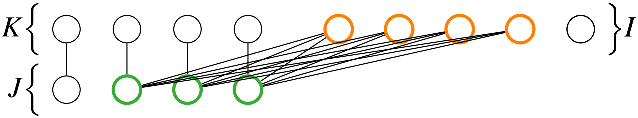

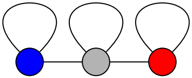

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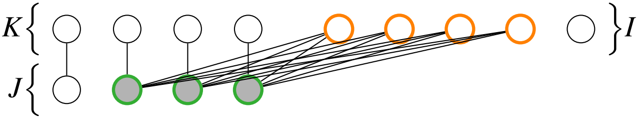

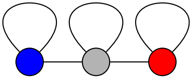

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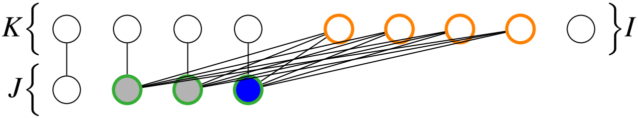

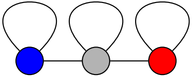

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Concluding remarks

Result for $\delta = 1, 2$:

- Analyze structural properties of *edge-critical* graphs G (remove any edge $\implies$ minimum degree drops)

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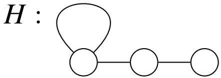

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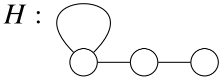

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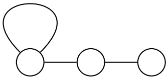

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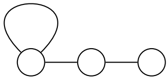
H : 

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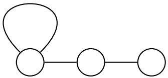

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