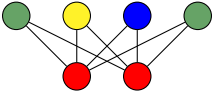


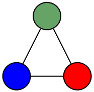
Extremal questions for H-colorings

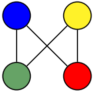
John Engbers

Department of Mathematics
University of Notre Dame

Graph Theory Seminar — Western Michigan University, Kalamazoo, MI

November 14, 2012

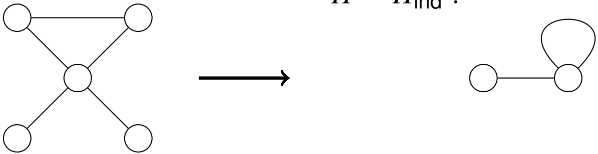




H-colorings

Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

$G:$ $\qquad$ $H = H_{\text{ind}}:$



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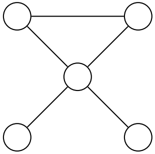

$\longrightarrow$

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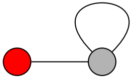

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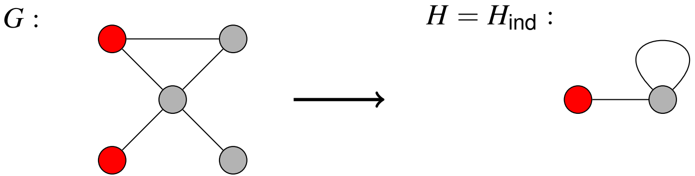

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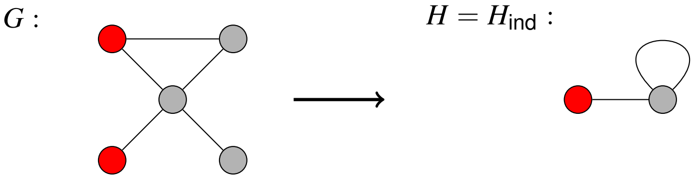

Examples: independent sets,

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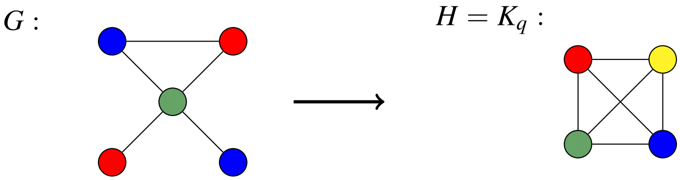

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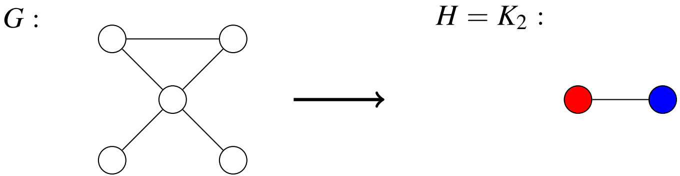

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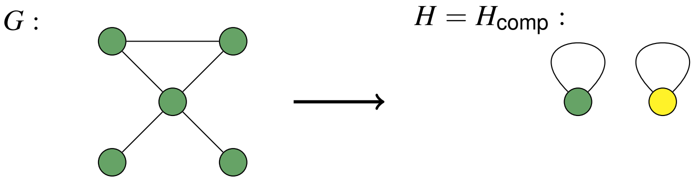

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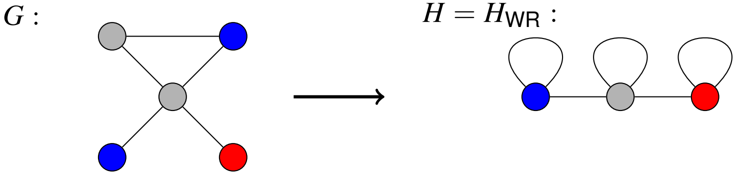

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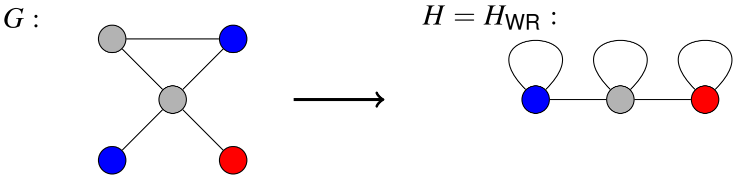

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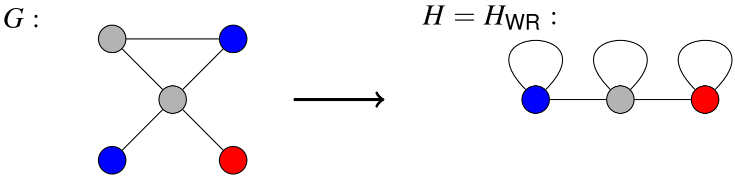

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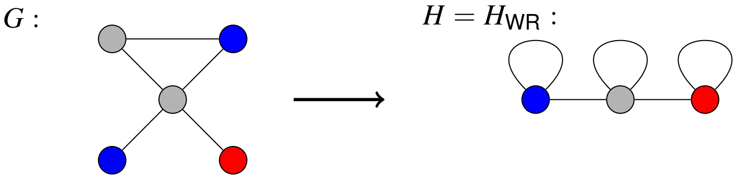

Examples: independent sets, proper q-colorings, bipartite, components, Widom-Rowlinson

- Terminology: map/color the vertices of G
- H is a 'blueprint'; it encodes the coloring scheme
- Natural for H to have loops

Notation and conventions

Notations:

$\mathrm{Hom}(G, H) = \{H\text{-colorings of } G\}$

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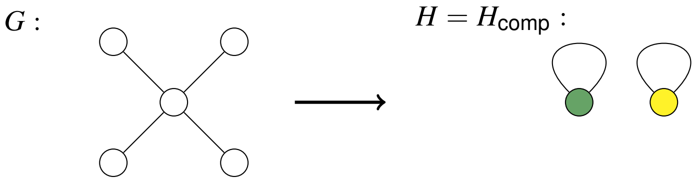

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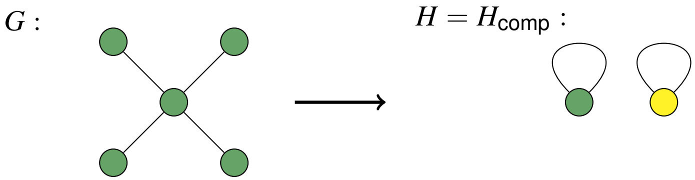

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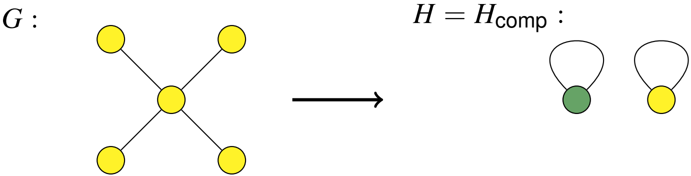

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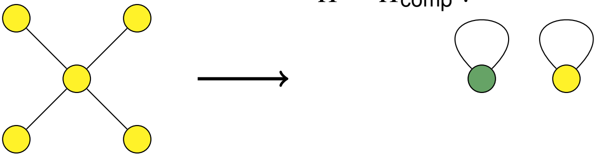

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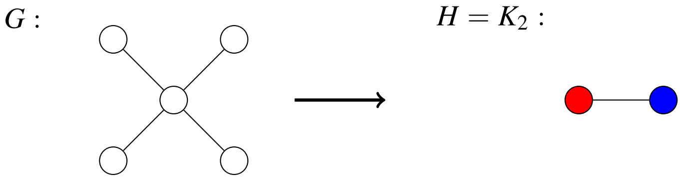

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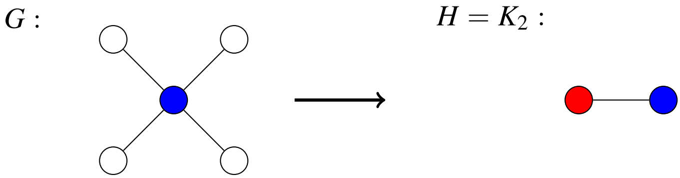

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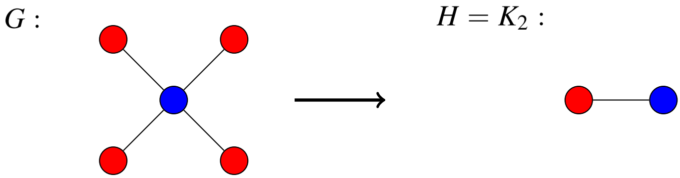

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- $\text{hom}(G, K_2) = \mathbf{1}_{\{G \text{ bipartite}\}} 2^{\#\text{ bipartite components of } G}$

Notation and conventions

Also: $d(v)$ is the degree of v (where loops count *once*)

Why?

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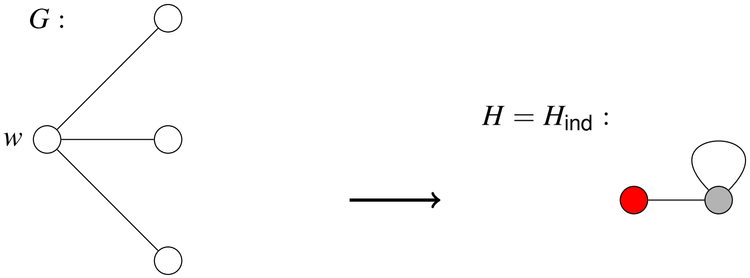

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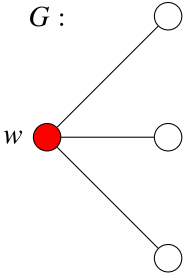

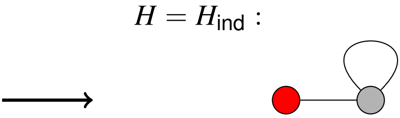

- w is red

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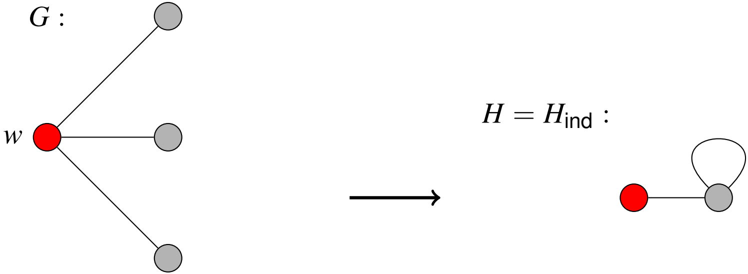

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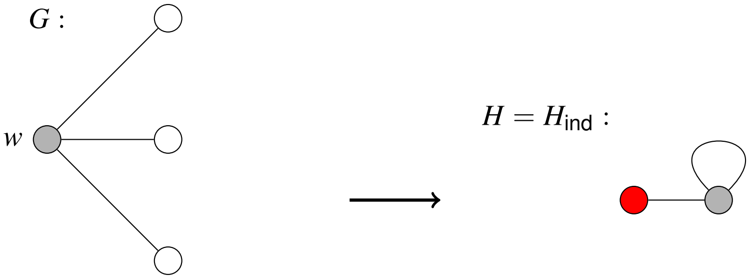

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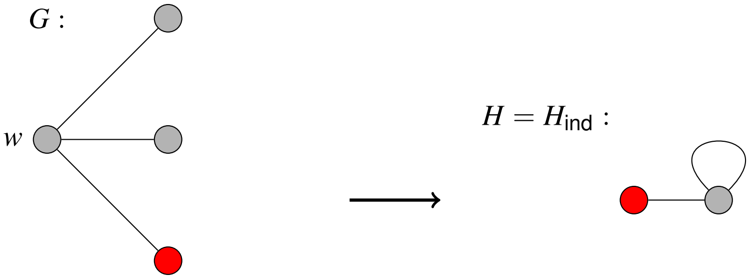

- w is red $\implies$ each neighbor of w has 1 choice ($d(\text{red}) = 1$)
- w is gray $\implies$ each neighbor of w has 2 choices ($d(\text{gray}) = 2$)

Statistical physics interpretation

Hard constraint spin systems:

Imagine $V(G)$ = particles, $E(G)$ = adjacency (e.g. spatial proximity)

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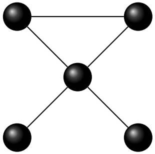

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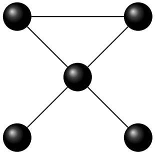

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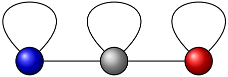

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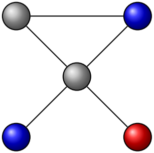

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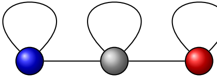

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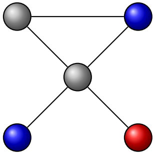

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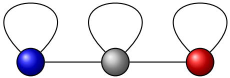

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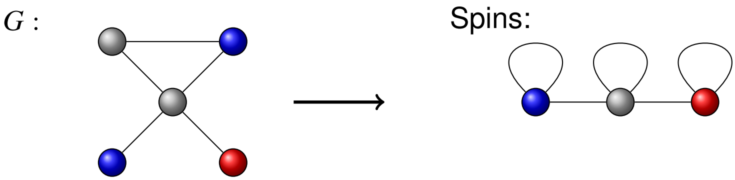

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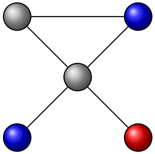

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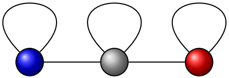

- Spins = colors; a spin configuration is an H-coloring
- Can put weights on the spins
- This idea generalizes to putting objects (with relationships) into classes with hard rules

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Extremal

- Rest of this talk...

An extremal question

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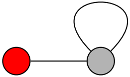

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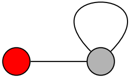

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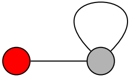

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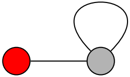

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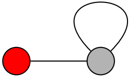

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- Perspective switch: Consider $\mathcal{G}$, answer for H_1, then H_2, ...
- Hope: A small list of graphs G maximize $\hom(G,H)$ for *every* H.

Various families

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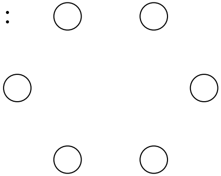

$$\hom(E_n, H) = |V(H)|^n$$

Various families

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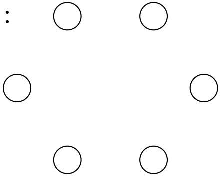
  - For any H, $\hom(G,H)$ is maximized when $G = E_n$

E_n :



$$\hom(E_n, H) = |V(H)|^n$$

- Interesting families force each graph G to have a large number of edges.

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

Various families

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- $\mathcal{G}$ = n-vertex m-edge graphs

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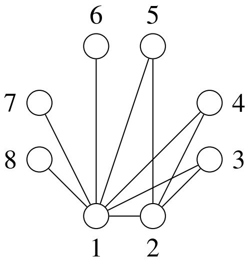
  - $H = H_{\text{ind}}$



$\text{Lex}(8, 11)$

Various families

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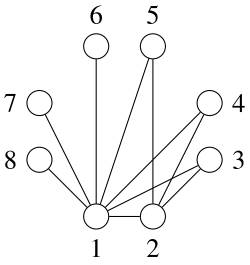
  - $H = H_{\text{ind}}$, $H = H_{\text{WR}}$, class of H (Cutler-Radcliffe)



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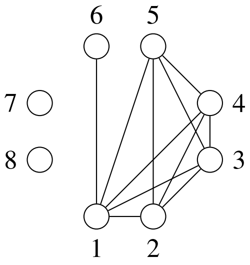

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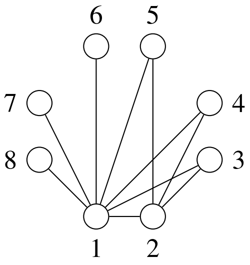

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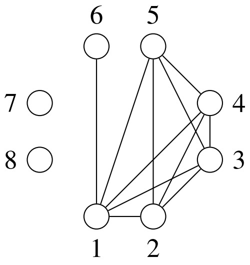

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  - $H = K_q$: various results, still open in general

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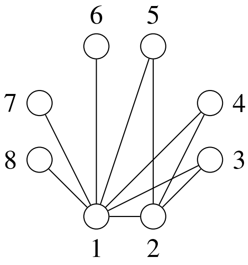

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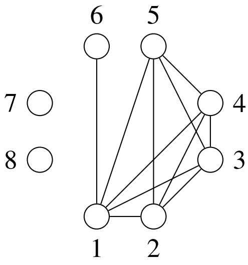

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  - $H = K_q$: various results, still open in general
- Extremal graphs can be non-homogeneous

Various families

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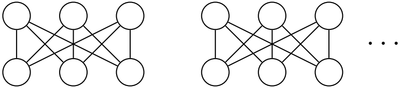
$\frac{n}{2d} K_{d,d}$:



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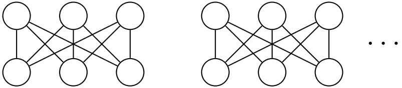

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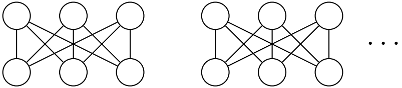

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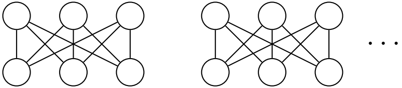

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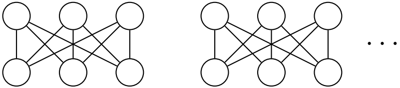

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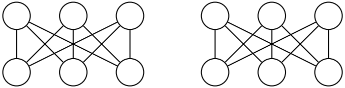

. . .

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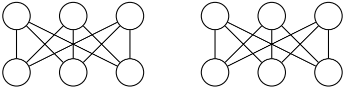

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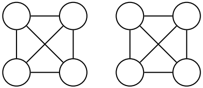

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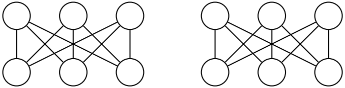

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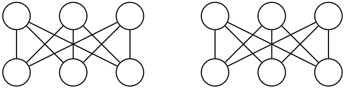

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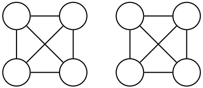

$\ldots$

Open Conjecture

Fix H. For $\mathcal{G} = n$-vertex d-regular graphs, $\hom(G,H)$ is maximized when $G = \frac{n}{2d} K_{d,d}$ or $\frac{n}{d+1} K_{d+1}$.

Today's family

$\mathcal{G} = \mathcal{G}(n, \delta)$ = n-vertex graphs with minimum degree δ

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Intuition: Maximizing graph is δ-regular (so likely either $\frac{n}{2\delta} K_{\delta,\delta}$ or $\frac{n}{\delta+1} K_{\delta+1}$).

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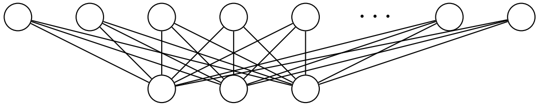
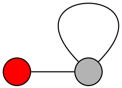
Theorem (Galvin, 2011)

For all $G \in \mathcal{G}(n, \delta)$ and $n \geq 8\delta^2$, $\hom(G, H_{ind})$ is maximized when $G = K_{\delta,n-\delta}$.





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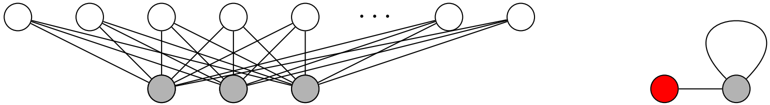

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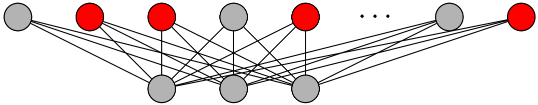

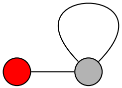

Note: $\hom(K_{\delta,n-\delta}, H_{\text{ind}}) \geq 2^{n-\delta}$.

Today's family

Conjecture

Fix H. For all $G \in \mathcal{G}(n,\delta)$ and n large enough, $\hom(G,H)$ is maximized when $G = K_{\delta,n-\delta}$, $\frac{n}{2\delta}K_{\delta,\delta}$, or $\frac{n}{\delta+1}K_{\delta+1}$.

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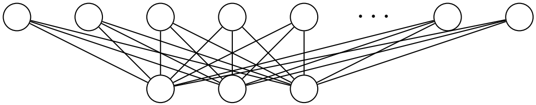
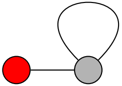
Sharpness:

- $H = H_{\text{ind}}$ maximized by $K_{\delta,n-\delta}$





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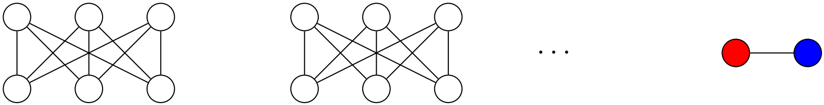

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Emphasis: Infinite collection of H, small # of maximizing graphs

Today's family

Progress:

Today's family

Progress:

Theorem (E., 2012)

- *Conjecture is true for* $\delta = 1$, $\delta = 2$.

Today's family

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Theorem (E., 2012)

- *Conjecture is true for* $\delta = 1$, $\delta = 2$.
- *Suppose that* H *satisfies* $\sum_{v \in V(H)} d(v) < (\Delta_H)^2$. *Then, for* $n > c^\delta$ *and* $G \in \mathcal{G}(n, \delta)$, $\hom(G, H)$ *is maximized when* $G = K_{\delta, n-\delta}$.

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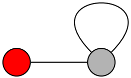
Examples:

- $H_{\text{ind}} : \sum d(v) = 3; (\Delta_H)^2 = 4$ ✓



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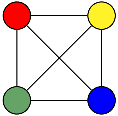

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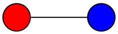

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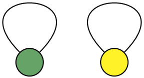

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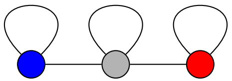
- $H_{\text{WR}} : \sum d(v) = 7; (\Delta_H)^2 = 9$ ✓



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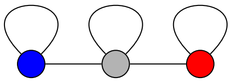
- $H_{\text{comp}} : \sum d(v) = 2; (\Delta_H)^2 = 1$ ✘
- $H_{\text{WR}} : \sum d(v) = 7; (\Delta_H)^2 = 9$ ✓
- Any* H with looped dominating vertex



Today's family

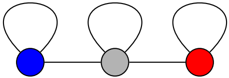
Progress:

Theorem (E., 2012)

- *Conjecture is true for* $\delta = 1$, $\delta = 2$.
- *Suppose that* H *satisfies* $\sum_{v \in V(H)} d(v) < (\Delta_H)^2$. *Then, for* $n > c^\delta$ *and* $G \in \mathcal{G}(n, \delta)$, $\hom(G, H)$ *is maximized when* $G = K_{\delta, n-\delta}$.

Examples:

- $H_{\text{ind}} : \sum d(v) = 3; (\Delta_H)^2 = 4$ ✓
- $K_q : \sum d(v) = q(q-1); (\Delta_H)^2 = (q-1)^2$ ✘
- $K_2 : \sum d(v) = 2; (\Delta_H)^2 = 1$ ✘
- $H_{\text{comp}} : \sum d(v) = 2; (\Delta_H)^2 = 1$ ✘
- $H_{\text{WR}} : \sum d(v) = 7; (\Delta_H)^2 = 9$ ✓
- Any* H with looped dominating vertex



Blue condition is combination of local (Δ_H) and global ($\sum_{v \in V(H)} d(v)$).

Idea of proof for $H = H_{\text{WR}}$

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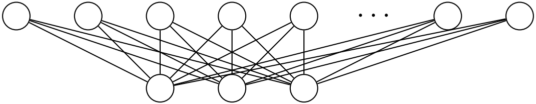

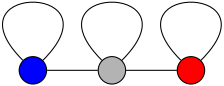

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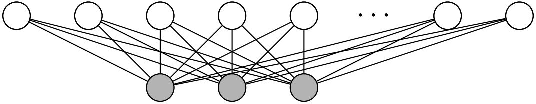

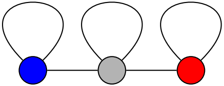

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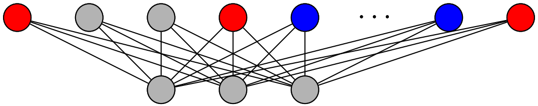

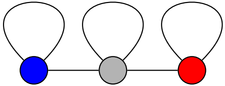

$$\text{hom}(K_{\delta,n-\delta}, H_{\text{WR}}) \geq 3^{n-\delta}$$

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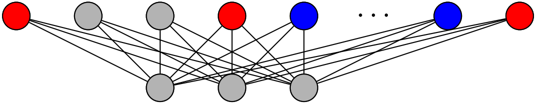

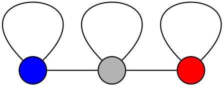

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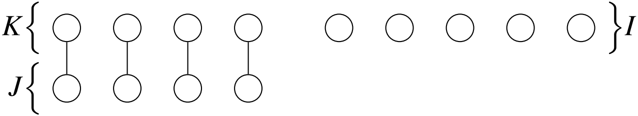
Idea: Partition $\mathcal{G}(n,\delta)$ by the size of maximum matching M.



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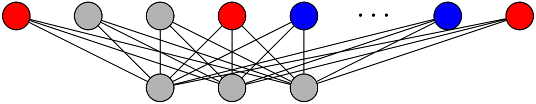

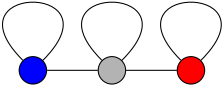

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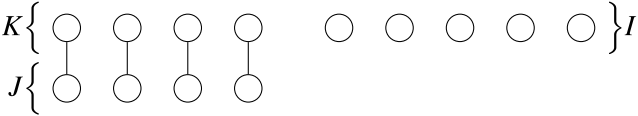

$$\text{hom}(G, H_{\text{WR}}) \leq 7^{|M|} 3^{n-2|M|} = \left(\frac{7}{3^2}\right)^{|M|} 3^n$$

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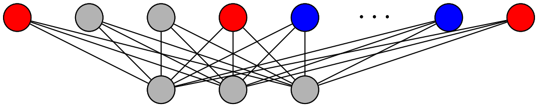

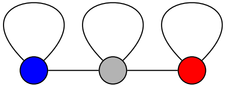

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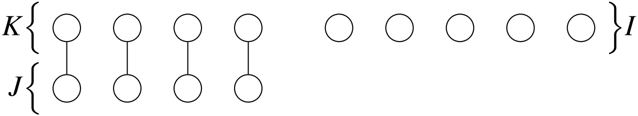

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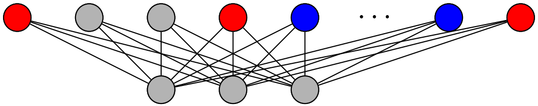

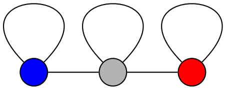

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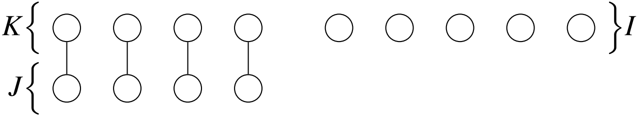

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Any maximizing graph G has $|M| \leq c\delta$

Idea of proof for $H = H_{\text{WR}}$

Graphs with $|M| \leq \delta$: Short argument gives $K_{\delta,n-\delta}$ maximizes

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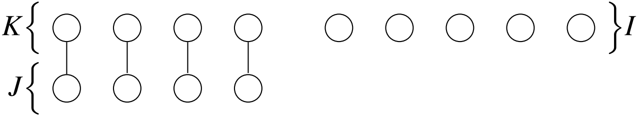

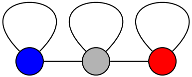

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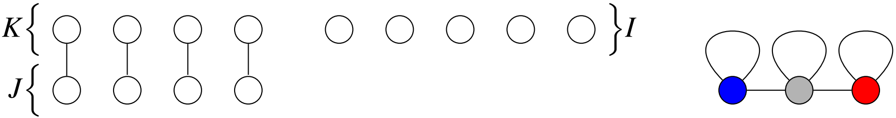

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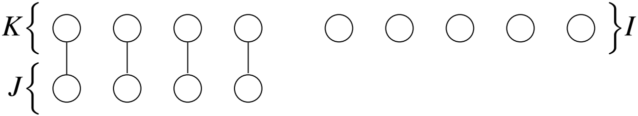

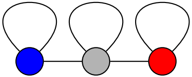

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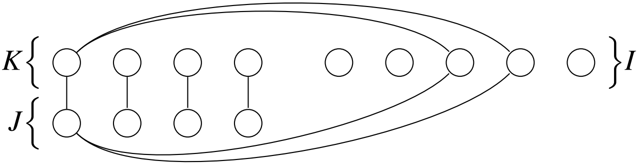

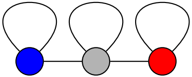

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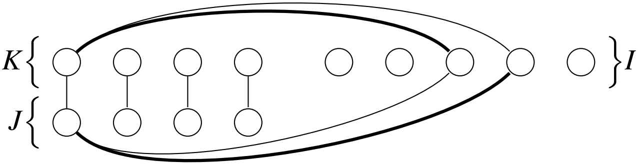

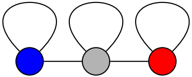

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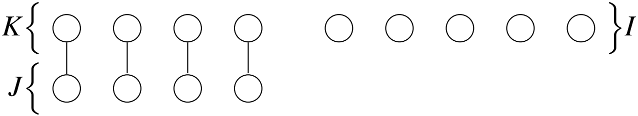

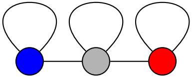

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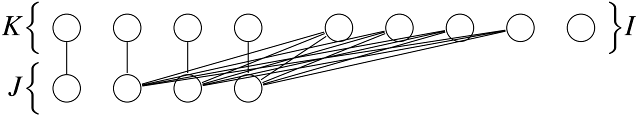

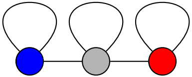

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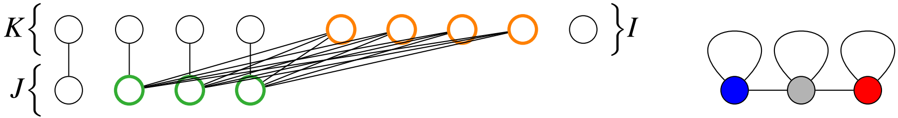

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- Some set of δ vertices in J has $\approx \frac{n-2|M|}{\binom{|M|}{\delta}} = \Omega(n)$ neighbors in I.

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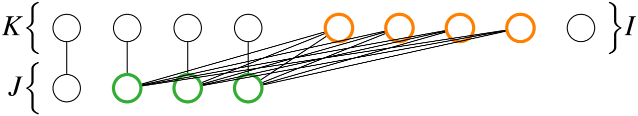

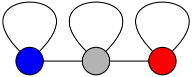

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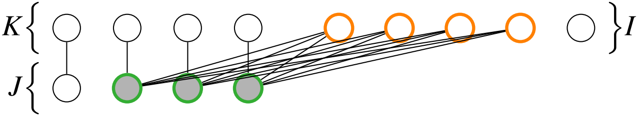

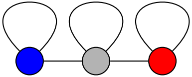

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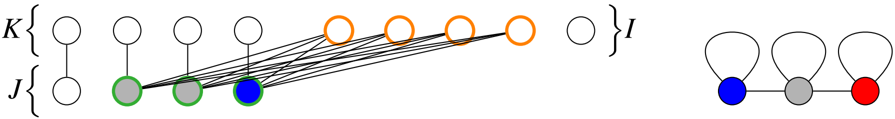

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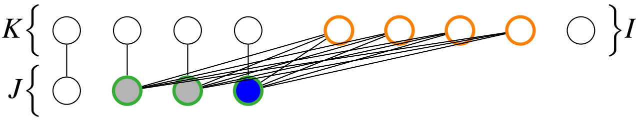

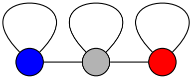

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And:

$$\left(\frac{7}{9}\right) 3^{n-\delta} + \left(\frac{2}{3}\right)^{\Omega(n)} 3^n < 3^{n-\delta} \leq \hom(K_{\delta,n-\delta}, H_{\mathrm{WR}})$$

Concluding remarks

Result for $\delta = 1, 2$:

- Analyze structural properties of *edge-critical* graphs G (remove any edge $\implies$ minimum degree drops)

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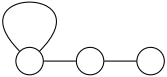

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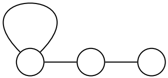

$$\sum_{v \in V(H)} d(v) = 5; (\Delta_H)^2 = 4 \quad | \quad \text{Maximized in } \mathcal{G}(n, 2) \text{ by } K_{2,n-2}$$

Sufficient ($K_{\delta,n-\delta}$): $\hom(K_{\delta,\delta}, H)^{\frac{1}{2\delta}} < \Delta_H$ & $\hom(K_{\delta+1}, H)^{\frac{1}{\delta+1}} < \Delta_H$?

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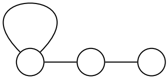

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- $\delta = 3$? Other small values of δ?

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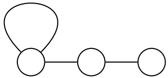
- Notice:

H : 

$$\sum_{v \in V(H)} d(v) = 5; (\Delta_H)^2 = 4 \quad | \quad \text{Maximized in } \mathcal{G}(n,2) \text{ by } K_{2,n-2}$$

Sufficient ($K_{\delta,n-\delta}$): $\hom(K_{\delta,\delta}, H)^{\frac{1}{2\delta}} < \Delta_H$ & $\hom(K_{\delta+1}, H)^{\frac{1}{\delta+1}} < \Delta_H$?

- $\delta = 3$? Other small values of δ?
- Meaningful structural properties of edge-critical graphs ($\delta \geq 3$)?

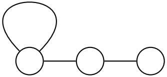
Concluding remarks

Result for $\delta = 1, 2$:

- Analyze structural properties of *edge-critical* graphs G (remove any edge $\implies$ minimum degree drops)

Future directions:

- Notice:

H :


$$\sum_{v \in V(H)} d(v) = 5; (\Delta_H)^2 = 4 \quad | \quad \text{Maximized in } \mathcal{G}(n,2) \text{ by } K_{2,n-2}$$

Sufficient ($K_{\delta,n-\delta}$): $\hom(K_{\delta,\delta}, H)^{\frac{1}{2\delta}} < \Delta_H$ & $\hom(K_{\delta+1}, H)^{\frac{1}{\delta+1}} < \Delta_H$?

- $\delta = 3$? Other small values of δ?
- Meaningful structural properties of edge-critical graphs ($\delta \geq 3$)?
- Results for $\mathcal{G}$ = n-vertex graphs with min degree δ, max degree at most Δ?

Thanks

Thank you!