Extremal questions for $H$-colorings

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**H-colorings**

**Graph homomorphism (H-coloring):** A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

$G :$ 

\[ \begin{array}{c}
\text{Graph G}
\end{array} \]

\[ \begin{array}{c}
\text{Graph H}
\end{array} \]

$H = H_{\text{ind}} :$

\[ \begin{array}{c}
\text{Induced graph H}
\end{array} \]
$H$-colorings

Graph homomorphism (**$H$-coloring**): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

- **Examples:** independent sets, proper $q$-colorings, bipartite, components, Widom-Rowlinson

**Terminology:** map/color the vertices of $G$; $H$ is a 'blueprint'; it encodes the coloring scheme; natural for $H$ to have loops
Graph homomorphism (**H-coloring**): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.
**$H$-colorings**

**Graph homomorphism ($H$-coloring):** A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

**Examples:** independent sets,
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$H$-colorings

Graph homomorphism (\textit{$H$-coloring}): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

\begin{figure}
    \begin{center}
        \begin{tikzpicture}
            \node[fill=black!20, circle] (A) at (0,0) {};
            \node[fill=blue, circle] (B) at (1,1) {};
            \node[fill=red, circle] (C) at (1,-1) {};
            \node[fill=black!20, circle] (D) at (2,0) {};
            \draw (A) -- (B);
            \draw (A) -- (C);
            \draw (A) -- (D);
            \draw (B) -- (D);
            \draw (C) -- (D);
            \node[fill=black!20, circle] (E) at (5,0) {};
            \node[fill=blue, circle] (F) at (6,1) {};
            \node[fill=red, circle] (G) at (6,-1) {};
            \draw (E) -- (F);
            \draw (E) -- (G);
        \end{tikzpicture}
    \end{center}
\end{figure}

Examples: independent sets, proper $q$-colorings, bipartite, components, Widom-Rowlinson

- Terminology: map/color the vertices of $G$
**$H$-colorings**

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- Terminology: map/color the vertices of $G$
- $H$ is a ‘blueprint’; it encodes the coloring scheme
- Natural for $H$ to have loops
Notation and conventions

Notations:

Hom\((G, H)\) = \{\text{\(H\)-colorings of \(G\)}\}

\text{Note:} \hom\((G, H) \comp \) = 2

\text{# components of } G

\hom\((G, K_2)\) = 1

\text{# bipartite components of } G

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\text{Extremal } H\text{-colorings}  

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\[ \text{hom}(G, H_{\text{comp}}) = 2^\# \text{ components of } G \]
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Note:

- \( \text{hom}(G, H_{\text{comp}}) = 2^\# \text{ components of } G \)
- \( \text{hom}(G, K_2) = 1_{\{ G \text{ bipartite} \}} 2^\# \text{ bipartite components of } G \)
Notation and conventions

Also: $d(v)$ is the degree of $v$ (where loops count \textit{once})

Why?
Notation and conventions

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Why?

\[ G : \]

\[ w \]

\[ H = H_{\text{ind}} : \]
Notation and conventions

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$G:$

$w$ is red

$H = H_{\text{ind}}:$
Notation and conventions

Also: $d(v)$ is the degree of $v$ (where loops count once)

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$G$:

$H = H_{\text{ind}}$:

- $w$ is red $\implies$ each neighbor of $w$ has 1 choice ($d(\text{red}) = 1$)
Notation and conventions

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\[ G : \]

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\[ H \]

- \( w \) is red \( \implies \) each neighbor of \( w \) has 1 choice (\( d(\text{red}) = 1 \))
- \( w \) is gray
Notation and conventions

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Why?

\[ G : \]

- \( w \) is red \( \implies \) each neighbor of \( w \) has 1 choice \( (d(\text{red}) = 1) \)
- \( w \) is gray \( \implies \) each neighbor of \( w \) has 2 choices \( (d(\text{gray}) = 2) \)

\[ H = H_{\text{ind}} : \]

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Statistical physics interpretation

Hard constraint spin systems:

Imagine $V(G)$ = particles, $E(G)$ = adjacency (e.g. spatial proximity)
**Statistical physics interpretation**

**Hard constraint spin systems:**

Imagine $V(G) = \text{particles}$, $E(G) = \text{adjacency (e.g. spatial proximity)}$

Place spins on those particles so that adjacent particles receive ‘compatible’ spins
Statistical physics interpretation

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\[ \text{Spins:} \]

\[ \text{Spins = colors; a spin configuration is an } H\text{-coloring} \]

Can put weights on the spins

This idea generalizes to putting objects (with relationships) into classes with hard rules
Statistical physics interpretation

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Existential

- Given a $G$ and $H$, does an $H$-coloring of $G$ exist? [hard]
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Algorithmic
- Can we easily produce an $H$-coloring of $G$?
- Can we obtain a (uniform) random $H$-coloring of $G$?
- Can we quickly move from one $H$-coloring of $G$ to another via random local updating algorithms?
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**Structural**
- e.g. What does the typical $H$-coloring of $G$ look like?
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- What is $\text{hom}(G, H)$? [hard]
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- What is $\text{hom}(G, H)$? [hard]

Extremal
- Rest of this talk...
An extremal question

Question

Fix $H$. Given a family of graphs $G$, which $G \in G$ maximizes $\text{hom}(G, H)$?

$H = H_{\text{ind}}$:

![Graph Diagram]

Remarks:

Pick $G$ and $H$. Often: Consider $H$ (e.g. $H_{\text{ind}}$), answer for $G_1$, then $G_2$, ...

Perspective switch: Consider $G$, answer for $H_1$, then $H_2$, ...

Hope: A small list of graphs $G$ maximize $\text{hom}(G, H)$ for every $H$. 
An extremal question

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$$H = H_{\text{ind}} :$$

\begin{center}
\begin{tikzpicture}
\node[red] (A) at (0,0) {};
\node[black!50] (B) at (1,0) {};
\draw (A) -- (B);
\end{tikzpicture}
\end{center}
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Various families

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  - For any $H$, $\text{hom}(G, H)$ is maximized when $G =$
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- $\mathcal{G} = n$-vertex graphs
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\[ E_n : \quad \circ \quad \circ \]

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\[ \text{hom}(E_n, H) = |V(H)|^n \]
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    $E_n : \quad \bigcirc \quad \bigcirc$

    \[ \bigcirc \quad \bigcirc \quad \bigcirc \]

    $\text{hom}(E_n, H) = |V(H)|^n$

- Interesting families force each graph $G$ to have a large number of edges.
Various families

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  - $H = H_{\text{ind}}$

$\begin{align*}
\text{Lex}(8, 11) & \quad \text{Colex}(8, 11) \\
\end{align*}$

Extremal graphs can be non-homogeneous

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Various families

Question

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?

- $\mathcal{G} = n$-vertex $m$-edge graphs
  - $H = H_{\text{ind}}$, $H = H_{\text{WR}}$, class of $H$ (Cutler-Radcliffe)

\[\begin{array}{c}
6 & 5 \\
7 & 4 \\
8 & 3 \\
\end{array} \quad \begin{array}{c}
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  $H = K_q$ : various results, still open in general
Various families

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  - $H = H_{\text{ind}}$ (Kahn)

$$\frac{n}{2d}K_{d,d} : \quad \text{...}$$
Various families

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- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs
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$$\frac{n}{2d}K_{d,d} : \quad \begin{array}{ccc}
  \cdots
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  $\frac{n}{2d}K_{d,d}$:
  
  ![Diagram](image)

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    $\frac{n}{2d} K_{d,d}$:

- $\mathcal{G} = n$-vertex $d$-regular graphs
  - $H_{\text{ind}}$ (Zhao)

Open Conjecture

Fix $H$. For $G = n$-vertex $d$-regular graphs, $\text{hom}(G, H)$ is maximized when $G = n^2 K_{d,d}$, $d$ or $n^2 d + 1 K_{d+1}$. 

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Open Conjecture

Fix $H$. For $\mathcal{G} = n$-vertex $d$-regular graphs, $\text{hom}(G, H)$ is maximized when $G = \frac{n}{2d} K_{d,d}$ or $\frac{n}{d+1} K_{d+1}$. 

Today’s family

\[ \mathcal{G} = \mathcal{G}(n, \delta) = n\text{-vertex graphs with minimum degree } \delta \]

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Fix \( H \). Which \( G \in \mathcal{G}(n, \delta) \) maximizes \( \text{hom}(G, H) \)?
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*Fix* \( H \). Which \( G \in \mathcal{G}(n, \delta) \) maximizes \( \text{hom}(G, H) \)?

**Intuition:** Maximizing graph is \( \delta \)-regular (so likely either \( \frac{n}{2\delta} K_{\delta, \delta} \) or \( \frac{n}{\delta + 1} K_{\delta + 1} \)).
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Theorem (Galvin, 2011)

For all \( G \in G(n, \delta) \) and \( n \geq 8\delta^2 \), \( \text{hom}(G, H_{\text{ind}}) \) is maximized when \( G = K_{\delta, n-\delta} \).
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**Note:** \( \text{hom}(K_{\delta, n-\delta}, H_{\text{ind}}) \geq 2^{n-\delta} \).
Today’s family

Conjecture

Fix $H$. For all $G \in \mathcal{G}(n, \delta)$ and $n$ large enough, $\text{hom}(G, H)$ is maximized when $G = K_{\delta, n-\delta}, \frac{n}{2\delta}K_{\delta, \delta},$ or $\frac{n}{\delta+1}K_{\delta+1}$. 
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Conjecture

Fix $H$. For all $G \in \mathcal{G}(n, \delta)$ and $n$ large enough, $\text{hom}(G, H)$ is maximized when $G = K_{\delta, n-\delta}$, $\frac{n}{2\delta} K_{\delta, \delta}$, or $\frac{n}{\delta+1} K_{\delta+1}$.

Sharpness:

- $H = H_{\text{ind}}$ maximized by $K_{\delta, n-\delta}$
Today’s family

Conjecture

Fix $H$. For all $G \in \mathcal{G}(n, \delta)$ and $n$ large enough, $\text{hom}(G, H)$ is maximized when $G = K_{\delta, n-\delta}$, $\frac{n}{2\delta} K_{\delta, \delta}$, or $\frac{n}{\delta+1} K_{\delta+1}$.

Sharpness:

- $H = H_{\text{ind}}$ maximized by $K_{\delta, n-\delta}$
- $H = K_2$ maximized by $\frac{n}{2\delta} K_{\delta, \delta}$
Today’s family

Conjecture

Fix $H$. For all $G \in \mathcal{G}(n, \delta)$ and $n$ large enough, $\text{hom}(G, H)$ is maximized when $G = K_{\delta,n-\delta, \frac{n}{2\delta} K_{\delta,\delta}, \frac{n}{\delta+1} K_{\delta+1}}$.

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- $H = K_2$ maximized by $\frac{n}{2\delta} K_{\delta,\delta}$
- $H = H_{\text{comp}}$ maximized by $\frac{n}{\delta+1} K_{\delta+1}$

\[\begin{align*}
\text{\includegraphics[width=0.3\textwidth]{graph1}} & \quad \text{\includegraphics[width=0.3\textwidth]{graph2}} & \quad \ldots & \quad \text{\includegraphics[width=0.3\textwidth]{graph3}}
\end{align*}\]
Conjecture

Fix $H$. For all $G \in \mathcal{G}(n, \delta)$ and $n$ large enough, $\text{hom}(G, H)$ is maximized when $G = K_{\delta,n-\delta}$, $\frac{n}{2\delta}K_{\delta,\delta}$, or $\frac{n}{\delta+1}K_{\delta+1}$.

Sharpness:

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- $H = H_{\text{comp}}$ maximized by $\frac{n}{\delta+1}K_{\delta+1}$

**Emphasis:** Infinite collection of $H$, small # of maximizing graphs
Today’s family

Progress:

\[ \text{Theorem (E., 2012)} \]

\[ \text{Conjecture is true for } \delta = 1, \delta = 2. \]

Suppose that \( H \) satisfies
\[ \sum_{v \in V(H)} d(v) < (\Delta H)^2. \]

Then, for \( n > c \delta \) and \( G \in G(n, \delta) \),
\[ \text{hom}(G, H) \text{ is maximized when } G = K_{\delta, n-\delta}. \]

Examples:

\[ \text{Hind: } \sum d(v) = 3; (\Delta H)^2 = 4 \]

\[ \text{K}_q: \sum d(v) = q(q-1); (\Delta H)^2 = (q-1)^2 \]

\[ \text{K}_2: \sum d(v) = 2; (\Delta H)^2 = 1 \]

\[ \text{Hcomp: } \sum d(v) = 2; (\Delta H)^2 = 1 \]

\[ \text{HWR: } \sum d(v) = 7; (\Delta H)^2 = 9 \]

Any \( \ast H \) with looped dominating vertex is a combination of local \( (\Delta H) \) and global \( \sum_{v \in V(H)} d(v) \).
Today’s family

Progress:

Theorem (E., 2012)

- Conjecture is true for $\delta = 1$, $\delta = 2$. 

Examples:

- $H_{\text{ind}}$: $\sum d(v) = 3$; $(\Delta H)_2^2 = 4$
- $K_q$: $\sum d(v) = q(q-1)$; $(\Delta H)_2^2 = (q-1)^2$
- $K_2$: $\sum d(v) = 2$; $(\Delta H)_2^2 = 1$
- $H_{\text{WR}}$: $\sum d(v) = 7$; $(\Delta H)_2^2 = 9$
Today’s family

Progress:

Theorem (E., 2012)

- Conjecture is true for $\delta = 1$, $\delta = 2$.
- Suppose that $H$ satisfies $\sum_{v \in V(H)} d(v) < (\Delta_H)^2$. Then, for $n > c\delta$ and $G \in \mathcal{G}(n, \delta)$, $\text{hom}(G, H)$ is maximized when $G = K_{\delta, n-\delta}$. 

Examples:

- $\text{Ind}$: $\sum d(v) = 3$; $(\Delta_H)^2 = 4$.
- $K_q$: $\sum d(v) = q(q-1)$; $(\Delta_H)^2 = (q-1)^2$.
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- $\text{WR}$: $\sum d(v) = 7$; $(\Delta_H)^2 = 9$.
Theorem (E., 2012)

- Conjecture is true for $\delta = 1$, $\delta = 2$.
- Suppose that $H$ satisfies $\sum_{v \in V(H)} d(v) < (\Delta_H)^2$. Then, for $n > c^{\delta}$ and $G \in G(n, \delta)$, $\text{hom}(G, H)$ is maximized when $G = K_{\delta, n-\delta}$.

Examples:

- $H_{\text{ind}}$: $\sum d(v) = 3; (\Delta_H)^2 = 4 \checkmark$
Today's family

Progress:

**Theorem (E., 2012)**

- Conjecture is true for $\delta = 1$, $\delta = 2$.
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**Examples:**

- $H_{\text{ind}}: \sum d(v) = 3; (\Delta_H)^2 = 4 \checkmark$
- $K_q: \sum d(v) = q(q - 1); (\Delta_H)^2 = (q - 1)^2 \times$
Today’s family

Progress:

Theorem (E., 2012)

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Today’s family

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Today’s family

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Examples:

- $H_{\text{ind}} : \sum d(v) = 3; (\Delta_H)^2 = 4$ √
- $K_q : \sum d(v) = q(q - 1); (\Delta_H)^2 = (q - 1)^2$ ×
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- Any* $H$ with looped dominating vertex
Today’s family

Progress:

Theorem (E., 2012)

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- Any* $H$ with looped dominating vertex

Blue condition is combination of local $\Delta_H$ and global $\left(\sum_{v \in V(H)} d(v)\right)$. 

* Any*
Idea of proof for $H = H_{\text{WR}}$

Goal: $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta, n-\delta}$
Idea of proof for $H = H_{WR}$

Goal: $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta, n-\delta}$
Idea of proof for $H = H_{WR}$

**Goal:** $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta, n-\delta}$

Any maximizing graph $G$ has $|M| \leq c\delta$.
Idea of proof for $H = H_{WR}$

**Goal:** \( \sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H) \text{ maximized for } G = K_{\delta, n-\delta} \)

\[
\text{hom}(K_{\delta, n-\delta}, H_{WR}) \geq 3^{n-\delta}
\]
Idea of proof for $H = H_{WR}$

Goal: $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta, n-\delta}$

\[
\text{hom}(K_{\delta, n-\delta}, H_{WR}) \geq 3^{n-\delta}
\]

Idea: Partition $G(n, \delta)$ by the size of maximum matching $M$.

\[
K\left\{ \begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
J\end{array} \right\} I
\]
Idea of proof for $H = H_{WR}$

**Goal:** $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta,n-\delta}$

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**Idea:** Partition $G(n, \delta)$ by the size of maximum matching $M$.

$$\text{hom}(G, H_{WR}) \leq 7|M|3^{n-2|M|} = \left(\frac{7}{3^2}\right)^{|M|} 3^n$$
Idea of proof for $H = H_{WR}$

Goal: $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta,n-\delta}$

\[
\text{hom}(K_{\delta,n-\delta}, H_{WR}) \geq 3^{n-\delta}
\]

**Idea:** Partition $G(n, \delta)$ by the size of maximum matching $M$.

\[
\begin{align*}
K &\overset{\{\}}{\left\{ \begin{array}{c}
\circ \circ \circ \circ \\
\circ \circ \circ \circ \\
\circ \circ \circ \circ
\end{array} \right\}} I \\
J &\overset{\{\}}{\left\{ \begin{array}{c}
\circ \circ \circ \circ \\
\circ \circ \circ \circ \\
\circ \circ \circ \circ
\end{array} \right\}}
\end{align*}
\]

\[
\text{hom}(G, H_{WR}) \leq 7^{|M|} 3^{n-2|M|} = \left( \frac{7}{3^2} \right)^{|M|} 3^n = \left( \frac{\sum d(v)}{(\Delta_H)^2} \right)^{|M|} 3^n
\]
Idea of proof for $H = H_{WR}$

Goal: $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \text{hom}(G, H)$ maximized for $G = K_{\delta,n-\delta}$

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Idea: Partition $\mathcal{G}(n, \delta)$ by the size of maximum matching $M$.

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Any maximizing graph $G$ has $|M| \leq c\delta$
Idea of proof for $H = H_{WR}$

**Graphs with $|M| \leq \delta$:** Short argument gives $K_{\delta, n-\delta}$ maximizes

---

John Engbers (Notre Dame)  
Extremal $H$-colorings  
November 2012
Idea of proof for $H = H_{WR}$

**Graphs with $|M| \leq \delta$:** Short argument gives $K_{\delta, n-\delta}$ maximizes

**Graphs with $\delta + 1 \leq |M| \leq c\delta$:**

\[
K\left\{ \begin{array}{c}
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc
\end{array} \right\} I
\]

\[
J\left\{ \begin{array}{c}
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc
\end{array} \right\}
\]

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Idea of proof for $H = H_{WR}$

Graphs with $|M| \leq \delta$: Short argument gives $K_{\delta, n-\delta}$ maximizes

Graphs with $\delta + 1 \leq |M| \leq c\delta$:

Facts:

- $I$ is an independent set
Idea of proof for $H = H_{WR}$

Graphs with $|M| \leq \delta$: Short argument gives $K_{\delta,n-\delta}$ maximizes

Graphs with $\delta + 1 \leq |M| \leq c\delta$:

Facts:
- $I$ is an independent set
  - There are at least $\delta(n - 2|M|)$ edges from $I$ to $J \cup K$. 

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Idea of proof for $H = H_{WR}$

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Facts:
- $I$ is an independent set
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- Both endpoints of edge in $M$ cannot have degree $\geq 2$ to $I$
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Graphs with $|M| \leq \delta$: Short argument gives $K_{\delta,n-\delta}$ maximizes

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Idea of proof for $H = H_{WR}$

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Idea of proof for $H = H_{WR}$

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- $\implies$ Most vertices in $I$ have all neighbors in $J$
Idea of proof for $H = H_{WR}$

Graphs with $|M| \leq \delta$: Short argument gives $K_{\delta,n-\delta}$ maximizes

Graphs with $\delta + 1 \leq |M| \leq c\delta$:

Facts:
- $I$ is an independent set
  - There are at least $\delta(n - 2|M|)$ edges from $I$ to $J \cup K$.
- Both endpoints of edge in $M$ cannot have degree $\geq 2$ to $I$
- Choose $K$ to be vertices with smallest degree to $I$
- $\implies$ Most vertices in $I$ have all neighbors in $J$
- Some set of $\delta$ vertices in $J$ has $\approx \frac{n - 2|M|}{\binom{|M|}{\delta}} = \Omega(n)$ neighbors in $I$. 
Idea of proof for $H = H_{WR}$

Graphs with $\delta + 1 \leq |M| \leq c\delta$:

- Some set of $\delta$ vertices in $J$ has $\approx \frac{n-2|M|}{|M|/\delta} = \Omega(n)$ neighbors in $I$

Then:
Idea of proof for $H = H_{WR}$

Graphs with $\delta + 1 \leq |M| \leq c\delta$:

- Some set of $\delta$ vertices in $J$ has $\approx \frac{n-2|M|}{(\frac{|M|}{\delta})} = \Omega(n)$ neighbors in $I$

Then:

- Case 1: All $\delta$ vertices get color gray ($\leq \left(\frac{7}{9}\right) 3^{n-\delta}$)

\[ K\{ \qquad \qquad \qquad \qquad \qquad \} \setminus \{ \text{gray vertices} \} \]

\[ J\{ \text{green vertices} \} \]

\[ I \]

\[ \text{blue, red, orange vertices} \]

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Idea of proof for $H = H_{WR}$

Graphs with $\delta + 1 \leq |M| \leq c\delta$:
- Some set of $\delta$ vertices in $J$ has $\approx \frac{n-2|M|}{(|M|/\delta)} = \Omega(n)$ neighbors in $I$

Then:
- Case 1: All $\delta$ vertices get color gray ($\leq (\frac{7}{9}) 3^{n-\delta}$)
- Case 2: At least 1 of $\delta$ vertices gets color blue/red ($\leq (\frac{2}{3})^{\Omega(n)} 3^n$)
Idea of proof for \( H = H_{WR} \)

**Graphs with** \( \delta + 1 \leq |M| \leq c\delta \):

- Some set of \( \delta \) vertices in \( J \) has \( \approx \frac{n-2|M|}{(|M|\delta)} = \Omega(n) \) neighbors in \( I \)

Then:

- **Case 1**: All \( \delta \) vertices get color gray \( (\leq \left(\frac{7}{9}\right) 3^{n-\delta}) \)
- **Case 2**: At least 1 of \( \delta \) vertices gets color blue/red \( (\leq \left(\frac{2}{3}\right) \Omega(n) 3^n) \)

And:

\[
\left(\frac{7}{9}\right) 3^{n-\delta} + \left(\frac{2}{3}\right)^{\Omega(n)} 3^n < 3^{n-\delta} \leq \text{hom}(K_{\delta,n-\delta}, H_{WR})
\]
Concluding remarks

Result for $\delta = 1, 2$:

- Analyze structural properties of *edge-critical* graphs $G$ (remove any edge $\implies$ minimum degree drops)
Concluding remarks

Result for $\delta = 1, 2$:
- Analyze structural properties of *edge-critical* graphs $G$ (remove any edge $\implies$ minimum degree drops)

Future directions:
- Notice:

$$H : \begin{array}{c}
\sum_{v \in V(H)} d(v) = 5; (\Delta_H)^2 = 4 \\
\text{Maximized in } G(n, 2) \text{ by } K_{2,n-2}
\end{array}$$
Concluding remarks

Result for $\delta = 1, 2$:
- Analyze structural properties of edge-critical graphs $G$ (remove any edge $\implies$ minimum degree drops)

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\end{array}$$

Sufficient ($K_{\delta, n-\delta}$): $\text{hom}(K_{\delta, \delta}, H)^{\frac{1}{2\delta}} < \Delta_H$ & $\text{hom}(K_{\delta+1}, H)^{\frac{1}{\delta+1}} < \Delta_H$?
Concluding remarks

**Result for \( \delta = 1, 2 \):**
- Analyze structural properties of *edge-critical* graphs \( G \) (remove any edge \( \Rightarrow \) minimum degree drops)

**Future directions:**
- Notice:

\[
H : \begin{array}{c}
\circ & \circ & \circ \\
\circ & \end{array}
\]

\[\sum_{v \in V(H)} d(v) = 5; (\Delta_H)^2 = 4 \quad \text{Maximized in} \ G(n, 2) \text{ by} \ K_{2,n-2}\]

Sufficient \((K_{\delta,n-\delta})\): \(\text{hom}(K_{\delta,\delta}, H)^{1/2\delta} < \Delta_H \) & \(\text{hom}(K_{\delta+1,\delta}, H)^{1/\delta+1} < \Delta_H\)?

- \(\delta = 3\) ? Other small values of \(\delta\)?
Concluding remarks

Result for $\delta = 1, 2$:
- Analyze structural properties of *edge-critical* graphs $G$ (remove any edge $\implies$ minimum degree drops)

Future directions:
- Notice:

$$H: \quad \sum_{v \in V(H)} d(v) = 5; (\Delta_H)^2 = 4 \quad | \quad \text{Maximized in } G(n, 2) \text{ by } K_{2,n-2}$$

Sufficient ($K_{\delta,n-\delta}$): $\text{hom}(K_{\delta,\delta}, H)^{\frac{1}{2\delta}} < \Delta_H$ & $\text{hom}(K_{\delta+1, H})^{\frac{1}{\delta+1}} < \Delta_H$?

$\delta = 3$? Other small values of $\delta$?
- Meaningful structural properties of edge-critical graphs ($\delta \geq 3$)?
Concluding remarks

**Result for $\delta = 1, 2$:**
- Analyze structural properties of *edge-critical* graphs $G$ (remove any edge $\implies$ minimum degree drops)

**Future directions:**
- Notice:

$$H : \quad \sum_{v \in V(H)} d(v) = 5; (\Delta_H)^2 = 4 \quad | \quad \text{Maximized in } \mathcal{G}(n, 2) \text{ by } K_{2,n-2}$$

Sufficient $(K_{\delta,n-\delta})$: $\text{hom}(K_{\delta,\delta}, H)^{\frac{1}{2\delta}} < \Delta_H$ & $\text{hom}(K_{\delta+1}, H)^{\frac{1}{\delta+1}} < \Delta_H$?

- $\delta = 3$? Other small values of $\delta$?
- Meaningful structural properties of edge-critical graphs $(\delta \geq 3)$?
- Results for $\mathcal{G} = n$-vertex graphs with min degree $\delta$, max degree at most $\Delta$?
Thanks

Thank you!