Entropy and Counting

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Outline

1 Basics
   - Definition
   - Properties

2 Matchings

3 Homomorphisms

4 Independent Sets
Definition

The *entropy* of a discrete random variable $X$ is

$$H(X) = \sum_x p(x) \log_2 \frac{1}{p(x)},$$

where $p(x) = P(X = x)$. 

Think of entropy as the amount of uncertainty/randomness/surprise in $X$. For example, if $p(x) = 1$ for some $x$, then $H(X) = 0$. All random variables will be discrete, and $\log = \log_2$. 
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Note: $H(p) := H(X)$
Basic Properties

If $Q$ is an event, we define $H(X|Q) = \sum p(x|Q) \log \frac{1}{p(x|Q)}$. 
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**Definition**

The *conditional entropy* of $X$ given $Y$ is

$$H(X|Y) = E[H(X|\{Y = y\})] = \sum_y p(y) \sum_x p(x|y) \log \frac{1}{p(x|y)}.$$
Basic Properties

(Chain Rule)

\[ H(X_1, \ldots, X_n) = H(X_1) + H(X_2 | X_1) + \cdots + H(X_n | X_{n-1}, \ldots, X_1) \]
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- **(Subadditivity)** \( H(X_1, \ldots, X_n) \leq \sum H(X_i) \)
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**Lemma (Shearer’s Lemma)**

*Let $X = (X_1, \ldots, X_n)$ be a random vector and $A$ a collection of subsets (possibly with repeats) of $[n]$, with each element of $[n]$ contained in at least $t$ members of $A$. Then*

$$H(X) \leq \frac{1}{t} \sum_{A \in A} H(X_A).$$
Brégman’s Theorem

Suppose that we have a bipartite, $N$-vertex, $d$-regular graph $G$. How many perfect matchings are there?
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Proof of Brégman’s Theorem

Proof: (Radhakrishnan) Choose \( \sigma \) from \( \mathcal{M} \) uniformly, so

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H(\sigma) = \log(|\mathcal{M}|).
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Label the vertices on the left as 1, 2, \ldots, \( N/2 \);
so \( \sigma = (\sigma(1), \sigma(2), \ldots, \sigma(N/2)) \).
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Fix $i$; define $N_i(\sigma, \tau)$ be the neighbors of $i$ that are NOT already matched for the given $\sigma$ and $\tau$. 

For fixed $\sigma$, $P_{\tau}(|N_i(\sigma, \tau)| = j) = \frac{1}{d}$ for $j = 1, \ldots, d$. Then $P_{\sigma, \tau}(|N_i(\sigma, \tau)| = j) = \frac{1}{d}$. 

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\[ \leq \sum_{i=1}^{N/2} E_{\tau} \left[ \sum_{j=1}^{d} P_{\sigma}(|N_i(\sigma, \tau)| = j) \log j \right] \]

\[ \leq \frac{N}{2} \sum_{i=1}^{d} \log d \]

\[ \leq \log(d!) \]
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= \log(d!)^{N/2d}
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Question: Is this result true if we remove the word ‘perfect’?

Conjecture

In an \( N \)-vertex, \( d \)-regular bipartite graph \( G \), let \( \mathcal{M}_{\text{tot}}(G) \) be the set of all possible matchings of \( G \). Then

\[
|\mathcal{M}_{\text{tot}}(G)| \leq |\mathcal{M}_{\text{tot}}(K_{d,d})|^{N/2d} = \left( \sum_{i=0}^{d} \binom{d}{i}^2 \frac{1}{i!} \right)^{N/2d}.
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Conjecture (Friedland)

In a $N$-vertex, $d$-regular bipartite graph $G$, let $\mathcal{M}_t(G)$ be the set of all matchings of size $t$, $t \in \{0, 1, \ldots, N/2\}$ in $G$. Then

$$|\mathcal{M}_t(G)| \leq |\mathcal{M}_t(\frac{N}{2d}K_{d,d})|.$$
Definition

Given graphs $G$ and $H$ ($H$ possibly with loops), a function $f : V(G) \rightarrow V(H)$ is a graph homomorphism if $x \sim y$ implies $f(x) \sim f(y)$ for all $x, y \in V(G)$. Denote by $Hom(G, H)$ the set of all graph homomorphisms from $G$ to $H$. 
Graph Homomorphisms

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Example:

Given graphs $G$ and two $H$’s:

```
G
```

```
two H’s:
```

[Diagram showing a graph $G$ and two distinct graphs $H$ with arrows indicating a homomorphism from $G$ to $H$.]
Let $\mathcal{I}(G)$ denote the set of all independent sets in a graph $G$.

**Theorem (Kahn)**

*For any $N$-vertex, $d$-regular bipartite graph $G$,*

$$|\mathcal{I}(G)| \leq |\mathcal{I}(K_{d,d})|^N/2^d.$$
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**Theorem (Galvin, Tetali)**

For any $N$-vertex, $d$-regular bipartite graph $G$ and any $H$ (possibly with loops),

$$|\text{Hom}(G, H)| \leq |\text{Hom}(K_{d,d}, H)|^{N/2d}.$$
Proof

Choose $f$ uniformly from $\text{Hom}(G, H)$, so

$$\log |\text{Hom}(G, H)| = H(f)$$
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\leq \frac{1}{d} \sum_{v \in E} H(N_v) + \sum_{v \in E} H(f_v N_v) \\
= \frac{1}{d} \sum_{v \in E} [H(N_v) + dH(f_v N_v)]
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From the definitions, the uniform bound, and an application of Jensen’s formula, we have:

$$H(N_v) + dH(f_v|N_v) \leq \log |\text{Hom}(K_{d,d}, H)|$$

which completes the proof.
Theorem (Zhao, 2009)

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$$|\mathcal{I}(G)| \leq |\mathcal{I}(K_{d,d})|^{N/2d}.$$
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### Conjecture

For any $N$-vertex, $d$-regular graph $G$ and any $H$ (possibly with loops),

$$|\text{Hom}(G, H)| \leq |\text{Hom}(K_{d,d}, H)|^N/2d$$

This conjecture is FALSE! See $H$ being two disjoint loops and $G = K_3$. An interesting question is: For what $H$'s does this extension to general $d$-regular graphs hold?
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An interesting question is: For what $H$’s does this extension to general $d$-regular graphs hold?
We now put a probability distribution on the set of all independent sets of $G$. 

**Definition** 
For a finite graph $G$ and $\lambda > 0$, the hard-core distribution with activity $\lambda$ on $I(G)$ is given by 
$$p_\lambda(I) = \frac{\lambda |I|}{\sum \{\lambda |I'| : I' \in I(G)\}}$$ 
for $I \in I(G)$. 

Note: $\lambda = 1$ gives the uniform distribution on $I(G)$. 

We'll restrict our $G$ to be $N$-vertex, $d$-regular, and bipartite.
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If $I$ is an independent set chosen according to $p_\lambda$, let $p(v) := P(v \in I)$, and $\bar{p} = \sum_v p(v) \ (= E[|I|]/N)$. 

Theorem (Kahn)

Fix $\lambda > 0$, and let $I$ be chosen according to $p_\lambda$ on $G$. Then

$$\bar{p} \approx \alpha_\lambda$$

and, furthermore, most independent sets have size close to $\alpha_\lambda N$. 
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- Example: $\lambda = 1$ is the uniform case, where $\alpha_\lambda = 1/4$.
- Entropy allows us to count independent sets of a fixed size.
Theorem (E., Galvin)

Given any $N$-vertex, $d$-regular bipartite $G$ and a random (uniform) $q$ coloring of $G$, the fraction of vertices with any given color doesn’t differ far from

a) $1/q$ ($q$ even)

b) being in $[1/(q + 1), 1/(q - 1)]$ ($q$ odd).
Extension

Theorem (E., Galvin)

Given any $N$-vertex, $d$-regular bipartite $G$ and a random (uniform) $q$ coloring of $G$, the fraction of vertices with any given color doesn’t differ far from

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Why the even/odd difference?
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Why the even/odd difference?

Can the odd case be improved?
This idea can be extended to a weighted version:

**Theorem (E., Galvin)**

Given a fixed $H$ and weights $\Lambda = \{\lambda_h\}_{h \in V(H)}$ on $V(H)$, and any $N$-vertex, $d$-regular bipartite graph $G$ with some technical conditions, the number of vertices mapping to a fixed vertex of $H$ is close to an ideal value.
Thank you!