TWO COMBINATORIAL PROOFS OF IDENTITIES INVOLVING SUMS OF POWERS OF BINOMIAL COEFFICIENTS

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Abstract

Let r be a fixed non-negative integer. We provide a combinatorial proof of the identity

$$\sum_{i=0}^{n}\binom{i}{r}^2=\sum_{i=0}^{r}\binom{r}{i}^2\binom{n+1+i}{2r+1}.$$

We do this by generalizing to two identities involving $\sum_{i=0}^{n}\binom{i}{r}^s$, for which we provide combinatorial proofs. These two identities involve the generalized Eulerian numbers and the generalized Delannoy numbers respectively.

In this note, we consider sums of powers of binomial coefficients. Recall the classical identity $\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$, which can be seen by partitioning lattice paths from $(0,0)$ to (n,n) using right and up steps based on which element $(i,n-i)$ they pass through. Our first result is an identity involving the sum of the squares of the binomial coefficients where the index of summation is over the *top* of the binomial coefficient.

Theorem 1. *Let $r\geq 0$ be fixed. Then for all $n\geq r$, we have*

$$\sum_{i=0}^{n}\binom{i}{r}^2=\sum_{k=0}^{r}\binom{r}{k}^2\binom{n+1+k}{2r+1}.$$

Note that, in particular, we recover the well-known formulas (from using $r=0$

and $r = 1$, respectively)

$$\sum_{i=0}^{n} 1 = \sum_{i=0}^{n} \binom{i}{0}^2 = n + 1,$$

and

$$\sum_{i=0}^{n} i^2 = \sum_{i=0}^{n} \binom{i}{1}^2 = \binom{n+1}{3} + \binom{n+2}{3}.$$

When $r = 2$ we obtain the following:

$$\sum_{i=0}^{n} \binom{i}{2}^2 = \binom{n+1}{5} + 4\binom{n+2}{5} + \binom{n+3}{5}.$$

The cases $r = 3$, 4, 5, 6, and 7 appear as sequences A086020, A086023, A086025, A086027, and A086029 (respectively) in OEIS [6].

Instead of proving Theorem 1 directly, we generalize to sums of higher powers of binomial coefficients, leaving Theorem 1 as a special case. This requires the following definition.

Definition 2. Consider the multiset $\mathbf{m} = \{1, ..., 1, 2, ..., 2, ..., s, ..., s\}$ which contains r copies of each element. Let $\left\langle {\mathbf{m} \atop k} \right\rangle$ denote the number of permutations of this multiset that has exactly k descents, meaning that there are exactly k places where entry i is larger than entry $i + 1$.

The numbers $\left\langle {\mathbf{m} \atop k} \right\rangle$ are explored in [2]. When $s = 2$, we can calculate $\left\langle {\mathbf{m} \atop k} \right\rangle$ fairly easily. Here we are considering the multiset $\{1, 1, ..., 1, 2, 2, ..., 2\}$. In this case, we have $\left\langle {\mathbf{m} \atop k} \right\rangle = \binom{r}{k}^2$. Indeed, we need to identify the k 1s and k 2s that will form the descents. We then list the 1s until the first 1 in a descent, then the 2s through the first descent, then the 1 in the first descent until prior to the second 1 chosen, and so on. For example, if we take $s = 4$ and identify the second and third 1 and the second and fourth 2, we have the multipermutation 12212211; notice that the second and third 1 form descents with the second and fourth 2, respectively.

Theorem 3. *Let $r \geq 0$ and $s \geq 0$ be fixed integers, and let $\boldsymbol{m}$ denote the multiset $\{1, \ldots, 1, 2, \ldots, 2, \ldots, s, \ldots, s\}$ that contains r copies of each element. Then for all integers $n \geq r$, we have*

$$\sum_{i=0}^{n} \binom{i}{r}^s = \sum_{k \geq 0} \left\langle {\boldsymbol{m} \atop k} \right\rangle \binom{n+1+k}{rs+1}.$$

For the case $r = 1$ (and so $\left\langle {s \atop k} \right\rangle = \left\langle {\mathbf{m} \atop k} \right\rangle$), Worpitzky's identity (see e.g. [5]), which states that

$$i^s = \sum_{k=0}^{s} \left\langle {s \atop k} \right\rangle \binom{i+k}{s},$$

can be used to obtain Theorem 3 in this special case. For a generalization of Worpitzky's identity for multipermutations, see [3]. Theorem 3 appears as a consequence of the multipermutation version in [4]; our first goal is to provide a short combinatorial proof of this result.

Proof of Theorem 3. The result holds when $r = 0$ or $s = 0$ by inspection (note that we have $\left\langle {\mathbf{m} \atop 0} \right\rangle = 1$ and $\left\langle {\mathbf{m} \atop k} \right\rangle = 0$ for all $k > 0$ when $\mathbf{m}$ is the empty set). We consider $s(n+1)$ people organized into $n+1$ families of size s. Label the families from the set $\{1, 2, \ldots, n+1\}$ and the members of a family with types $a_1, \ldots, a_s$; in particular, each person has both a family label i and a member type a_j. We count the number of dinner parties that have a host family and include r members of each type, each with smaller family label than that of the host family. The left-hand side conditions on the *label of the host family* (being $i + 1$).

Next, we show that the right-hand side also counts the number of these dinner parties. Here, we will consider dinner parties with attendees knowing their *type*, and count the ways of assigning family labels to these attendees. To do this, we consider rs guests (r of each type), and we will assign family labels for these rs guests plus the label of the host family.

Put the rs people in a fixed order, and then count the *number of descents* present in this ordering, in other words, the number of times that the mth person is of type a_j and the $(m + 1)$st person is of type a_i for some $i < j$. Since we know that there are exactly r people of each type, there are $\left\langle {\mathbf{m} \atop k} \right\rangle$ orderings that have exactly k descents. To the $n + 1$ possible family labels we add k "descent" boxes, and choose $rs + 1$ of these $n + 1 + k$ things: the largest family label corresponds to the label of the host family, the remaining labels are given to the attendees in increasing order, and if the jth descent box is chosen, the people involved in the jth descent receive the same family label.

Each such dinner party will be counted exactly once, since starting with a dinner party we can simply have guests line up by increasing label (where members with the same family label stand in decreasing order by member type), and each such labeling of the $rs + 1$ ordered guests plus host family label corresponds to a dinner party. Therefore the right-hand side also counts the number of such dinner parties. $\square$

Example 4. Suppose $r = 4$ and $s = 2$. When we consider the case of having 2 descents involving the second and third 1 with the second and fourth 2, we have the multipermutation 12212211. Using only the first descent box leaves seven distinct family labels, where the third smallest label is given to the second 1 and second 2. If only the second descent box is used, then the third 1 and fourth 2 receive the same label. If both descent boxes are used, there are six distinct labels remaining: the second 1 and second 2 both receive the same label, and the third 1 and fourth 2 also receive the same label.

We now move to our second identity. To mirror Theorem 1 and Theorem 3, we first include the result for the sum of squares of binomial coefficients, but then prove a generalized statement involving sth powers.

Theorem 5. *Let $r \geq 0$ be fixed. Then for all $n \geq r$, we have*

$$\sum_{i=0}^{n} \binom{i}{r}^2 = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n+1}{k+1}.$$

The generalized statement requires the following definition.

Definition 6. Fix integers $k, r, s \geq 0$. A *Delannoy path to* $(r, r, \ldots, r)$ in the s-dimensional integer lattice is a path from $(0, 0, \ldots, 0)$ to $(r, r, \ldots, r)$ so that each step in the path increases some non-empty set of coordinates by 1. The number of Delannoy paths to $(r, r, \ldots, r)$ in the s-dimensional integer lattice that use exactly k steps is denoted $d_k^s(r)$.

Theorem 7. *Let $r \geq 0$ and $s \geq 1$ be fixed. Then for all $n \geq 0$, we have*

$$\sum_{i=0}^{n} \binom{i}{r}^s = \sum_{k=r}^{sr} d_k^s(r) \binom{n+1}{k+1}.$$

When $s = 2$, we can find $d_k^2(r)$ by noting that there must be $2r - k$ steps that increase both coordinates (diagonal steps), and then among the $2k - 2r$ steps that remain we choose $k - r$ of them to correspond to increasing the first coordinate only (horizontal steps). This shows that Theorem 7 indeed generalizes Theorem 5. For larger values of s, the value $d_k^s(r)$ can be computed using Inclusion-Exclusion [1, Theorem 11]:

$$d_k^s(r) = \binom{k}{r} \sum_{i=0}^{k-r} (-1)^i \binom{k-r}{i} \binom{k-i}{r}^{s-1}.$$

Another formula for $d_k^s(r)$ can be found in [7].

We now present a combinatorial proof of Theorem 7 that follows the same lines as the proof of Theorem 3.

Proof of Theorem 7. As before, we consider $s(n+1)$ people organized into $n+1$ families of size s. Label the families from the set $\{1, 2, \ldots, n+1\}$ and the members of a family with types $a_1, \ldots, a_s$; in particular, each person has a family label i and a member type a_j. We count the number of dinner parties that have a host family and include r members of each type, each with smaller family label than that of the host family.

In light of the proof of Theorem 3, we only need to show that the right-hand side counts the number of these dinner parties. In this direction, first choose the *labels that will be present* at the party. The largest family label is clearly that of the host family. Then we condition on seeing exactly k labels smaller than the label of the host family.

Each label corresponds to taking a Delannoy step in the following way. First, let member type a_i correspond to the ith coordinate in the s-dimensional lattice. Then the set of member types with label ℓ at the party corresponds to the coordinates to change when making the next Delannoy step. The k steps taken correspond to the k family labels, and after all k steps are taken there are exactly r guests of each member type. □

For example, when $s = 2$ and $r = 2$, we have

$$\sum_{i=0}^{n} \binom{i}{2}^2 = 6\binom{n+1}{5} + 6\binom{n+1}{4} + \binom{n+1}{3}.$$

Notice that we may obtain the previous formula for $\sum_{i=0}^{n} \binom{i}{2}^2$ by iteratively applying Pascal's identity.

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