Extremal $H$-colorings of trees

John Engbers*   David Galvin

Department of Mathematics, Statistics and Computer Science
Marquette University

2014 MathFest — Portland, OR

August 8, 2014
An extremal question

Graph homomorphism ($H$-coloring):

$G :$ 

$H = H_{\text{ind}} :$

$H = H_{\text{ind}}$
An extremal question
Graph homomorphism ($H$-coloring):

$$G: \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad H = H_{\text{ind}}:$$

Examples: independent sets, proper $q$-colorings, Widom-Rowlinson

Notation: $\text{hom}(G, H) = \text{number of } H\text{-colorings of } G.$
An extremal question

Graph homomorphism ($H$-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

**Examples:** independent sets,
An extremal question

Graph homomorphism ($H$-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

Example:

$G$:

$H = K_q$:

Examples: independent sets, proper $q$-colorings,
An extremal question

**Graph homomorphism (H-coloring):** A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

Examples: independent sets, proper $q$-colorings, Widom-Rowlinson
An extremal question

Graph homomorphism \((H\text{-coloring})\): A map from \(V(G)\) to \(V(H)\) that preserves edge adjacency.

\[ G : \quad H = H_{\text{WR}} : \]

Examples: independent sets, proper \(q\)-colorings, Widom-Rowlinson

Notation: \(\text{hom}(G, H) = \text{number of } H\text{-colorings of } G\).
An extremal question

Graph homomorphism (*$H$*-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

Examples: independent sets, proper $q$-colorings, Widom-Rowlinson

Notation: $\text{hom}(G, H) =$ number of *$H$*-colorings of $G$.

**Question**

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes/minimizes $\text{hom}(G, H)$?
Various families

Question

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?
Various families

**Question**

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?

**Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin; E.; (E., Galvin) Sidorenko)**

- $\mathcal{G} = n$-vertex $m$-edge graphs: **Few $H$**
Various families

Question

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali)

- $\mathcal{G} = n$-vertex $m$-edge graphs: Few $H$
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs: All $H$
Various families

Question

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin)

- $\mathcal{G} = n$-vertex $m$-edge graphs, few $H$
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs, all $H$
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs, lots of $H$
Various families

Question

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin; E.)

- $\mathcal{G} = n$-vertex $m$-edge graphs: Few $H$
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs: All $H$
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs: Lots of $H$
- $\mathcal{G} = n$-vertex graphs with minimum degree $\delta$: Several $H$
Various families

Question
Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin; E.)

- $\mathcal{G} = n$-vertex $m$-edge graphs Few $H$ ← Open questions
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs All $H$
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs Lots of $H$ ← Open questions
- $\mathcal{G} = n$-vertex graphs with minimum degree $\delta$ Several $H$ ← Open questions
Various families

Question

Fix $H$. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\text{hom}(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin; E.; (E., Galvin) Sidorenko)

- $\mathcal{G} = n$-vertex $m$-edge graphs: Few $H \leftarrow$ Open questions
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs: All $H$
- $\mathcal{G} = n$-vertex $d$-regular bipartite graphs: Lots of $H \leftarrow$ Open questions
- $\mathcal{G} = n$-vertex graphs with minimum degree $\delta$: Several $H \leftarrow$ Open questions
- $\mathcal{G} = n$-vertex trees: All $H$
Trees

Question

Fix $H$. Which $n$-vertex tree $T$ maximizes $\text{hom}(T, H)$?
Fix $H$. Which $n$-vertex tree $T$ maximizes $\text{hom}(T, H)$?

**Theorem (E., Galvin 2014)**

Fix $H$. For $n$ large and any $n$-vertex tree $T$,

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$

However.....(New to me, May 2014)
**Question**

*Fix* $H$. Which $n$-vertex tree $T$ maximizes $\text{hom}(T, H)$?

**Theorem (E., Galvin 2014)**

*Fix* $H$. For $n$ large and any $n$-vertex tree $T$,

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$

However.....(New to me, May 2014)

**Theorem (Sidorenko 1994)**

*Fix* $H$. For any $n$-vertex tree $T$,

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$
What next?

The star $K_{1,n-1}$ maximizes # of $H$-colorings in trees. What minimizes?

Conjecture

Fix $H$. For any $n$-vertex tree $T$,

$$\text{hom}(P_n, H) \leq \text{hom}(T, H).$$
What next?
The star $K_{1,n-1}$ maximizes # of $H$-colorings in trees. What minimizes?

Conjecture

*Fix $H$. For any $n$-vertex tree $T$,*

\[ \text{hom}(P_n, H) \leq \text{hom}(T, H). \]

FALSE! (Even for $n = 7$)
What next?
The star $K_{1,n-1}$ maximizes # of $H$-colorings in trees. What minimizes?

Conjecture

Fix $H$. For any $n$-vertex tree $T$,

$$\text{hom}(P_n, H) \leq \text{hom}(T, H).$$

FALSE! (Even for $n = 7$)

Theorem (E., Galvin 2014)

For a certain class of $H$, for any $n$-vertex tree $T$ we have

$$\text{hom}(P_n, H) \leq \text{hom}(T, H).$$

(This class includes the Widom-Rowlinson graph $H_{WR}$ and the independent set graph $H_{\text{ind}}$.)
What next?
The star $K_{1,n-1}$ maximizes # of $H$-colorings in trees. What minimizes?

Conjecture

Fix $H$. For any $n$-vertex tree $T$,

$$\text{hom}(P_n, H) \leq \text{hom}(T, H).$$

FALSE! (Even for $n = 7$) ← Open question — what $H$ is it true for?

Theorem (E., Galvin 2014)

For a certain class of $H$, for any $n$-vertex tree $T$ we have

$$\text{hom}(P_n, H) \leq \text{hom}(T, H).$$

(This class includes the Widom-Rowlinson graph $H_{WR}$ and the independent set graph $H_{ind}$.)
What next?

Theorem (E., Galvin 2014)

*Fix* $H$. *For $n$ large and any $n$-vertex tree* $T$,

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$

See what else your proof can do!
What next?

**Theorem (E., Galvin 2014)**

*Fix* $H$. For $n$ large and any $n$-vertex tree $T$,

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$

See what else your proof can do!

**Theorem (E., Galvin 2014)**

*Fix a non-regular* $H$. For $n$ large and any 2-connected graph $G$,

$$\text{hom}(G, H) \leq \text{hom}(K_{2,n-2}, H).$$
What next?

Theorem (E., Galvin 2014)

Fix $H$. For $n$ large and any $n$-vertex tree $T$,

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$

See what else your proof can do!

Theorem (E., Galvin 2014)

Fix a non-regular $H$. For $n$ large and any 2-connected graph $G$,

$$\text{hom}(G, H) \leq \text{hom}(K_{2,n-2}, H).$$

$K_{2,n-2}$:

```
  o--o--o--o--o
     |    |
```

**Question:** Does $K_{k,n-k}$ maximize the $H$-colorings among all $k$-connected graphs (for most $H$)?
Idea of proof

Theorem (E., Galvin 2014)

Fix $H$. For $n$ large and any $n$-vertex tree $T$,

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$

Idea: Stability
Idea of proof

**Theorem (E., Galvin 2014)**

Fix $H$. For $n$ large and any $n$-vertex tree $T$, 

$$\text{hom}(T, H) \leq \text{hom}(K_{1,n-1}, H).$$

---

**Idea:** Stability

**Step 0.** Note $\text{hom}(K_{1,n-1}, H_{\text{WR}}) \geq 3^{n-1}$

**Step 1.** Extremal tree must be structurally close to $K_{1,n-1}$

**Step 2.** Small blemishes added to star can’t be extremal
Idea of proof

Step 0. $\text{hom}(K_{1,n-1}, H) \geq 3^{n-1}$
Idea of proof

Step 0. $\text{hom}(K_{1,n-1}, H) \geq 3^{n-1}$

Step 1. Show that an extremal tree can’t contain a long path.
Idea of proof

**Step 0.** $\text{hom}(K_{1,n-1}, H) \geq 3^{n-1}$

**Step 1.** Show that an extremal tree can’t contain a long path.

\[
A(H) = A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]
Idea of proof

Step 0. \( \text{hom}(K_{1,n-1}, H) \geq 3^{n-1} \)

Step 1. Show that an extremal tree can’t contain a long path.

\[ A(H) = A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

\( (A^k)_{ij} \) = \# colorings of \( P_{k+1} \) with endpoints colored \( i,j \)
Idea of proof

Step 0. \( \text{hom}(K_{1,n-1}, H) \geq 3^{n-1} \)

Step 1. Show that an extremal tree can’t contain a long path.

\[
A(H) = A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

- \( (A^k)_{ij} \) = # colorings of \( P_{k+1} \) with endpoints colored \( i,j \)
- Perron-Frobenius: largest eigenvalue is \( \lambda < 3 \) (\( H \) not regular)
Idea of proof

Step 0. \( \text{hom}(K_{1,n-1}, H) \geq 3^{n-1} \)

Step 1. Show that an extremal tree can’t contain a long path.

\[
A(H) = A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\]

- \((A^k)_{ij} = \#\text{ colorings of } P_{k+1} \text{ with endpoints colored } i,j\)
- Perron-Frobenius: largest eigenvalue is \( \lambda < 3 \) (H not regular)
- \( \text{hom}(T, H) \leq c\lambda^k 3^{n-k} < 3^{n-1} \) (for constant \( k \), uses \( n > c_H \))
Idea of proof

Step 2: Any non-star with no long path has fewer $H$-colorings
Idea of proof

Step 2: Any non-star with no long path has fewer $H$-colorings.
No $k$-path implies there is a vertex with at least $\log n$ neighbors.
If not a star:

\[
\begin{align*}
\{ \circ, \ldots, \circ \} & \geq \log n \\
\end{align*}
\]
Idea of proof

**Step 2:** Any non-star with no long path has fewer \( H \)-colorings.

No \( k \)-path implies there is a vertex with at least \( \log n \) neighbors.

If not a star:

\[
\begin{align*}
\{ & v, \ldots, \circ, \circ \} \geq \log n \\
\end{align*}
\]

Number of colorings where:

- \( v \) has color \( w \); \( d(w) < 3 \) \( \implies \) \( < c2^{\log n 3^n - \log n - 1} \leq cn \frac{-1}{3} 3^n = o(1)3^n \)
Idea of proof

Step 2: Any non-star with no long path has fewer $H$-colorings

No $k$-path implies there is a vertex with at least $\log n$ neighbors.

If not a star:

$$\{ \ldots \} \geq \log n$$

Number of colorings where:

- $v$ has color $w$; $d(w) < 3 \implies < c2^{\log n}3^{n-\log n-1} \leq cn^{-1}3^n = o(1)3^n$
- $v$ has color $w$; $d(w) = 3$: constant in leading term dampened if not $K_{1,n-1}$
Thank you!

Slides available on my homepage:
http://www.mscs.mu.edu/~engbers/