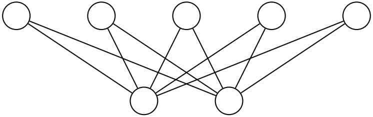


# Counting independent sets in graphs with a given minimal degree

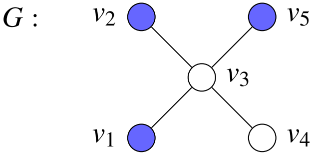
John Engbers*     David Galvin

University of Notre Dame
Department of Mathematics

April 2012

# An extremal question

**Independent set (of vertices):** A set of vertices which are pairwise non-adjacent.



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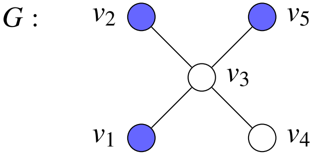


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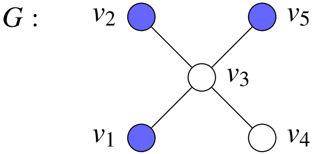


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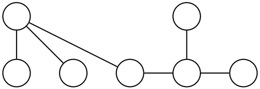
**Question**

*Given a family of graphs $\mathcal{G}$, what is the maximum value of $i(G)$ and $i_t(G)$ as $G$ ranges over $\mathcal{G}$?*

# Fixed order, trees

$\mathcal{G}(n)$: trees on $n$ vertices



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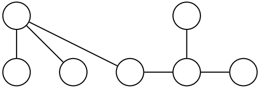

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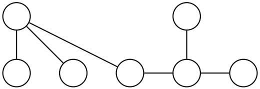

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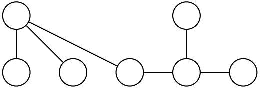

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**Foreshadowing:** $G$ a tree $\implies \delta(G) = 1$

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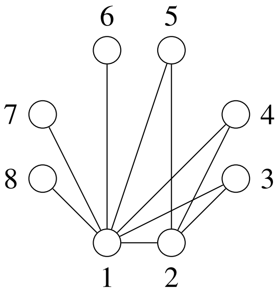


Lex$(8,11)$

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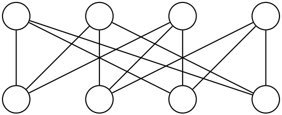
$\mathcal{G}(n, d)$: graphs with $n$ vertices, $d$-regular



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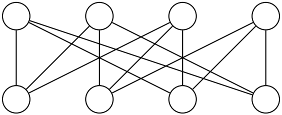

**Theorem (Kahn 2001; Zhao 2011)**

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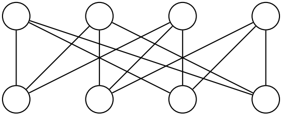

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- Asymptotic evidence for conjecture given by Carroll, G., Tetali (2009)

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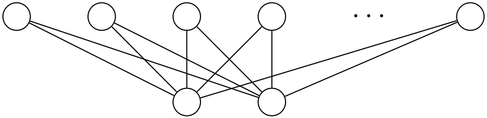

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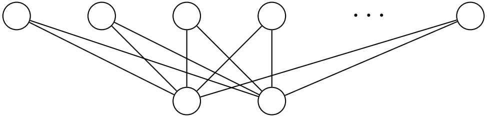

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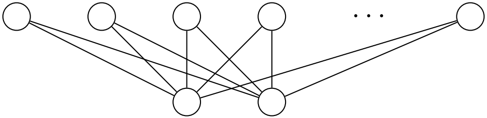


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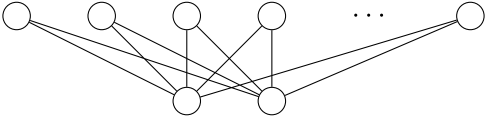

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**Note:** no cut-edges $\implies \delta(G) \geq 2$.

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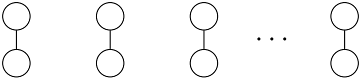
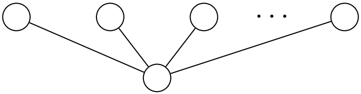


$i(nK_2) = 3^{n/2}$



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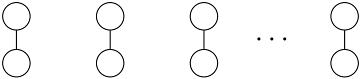

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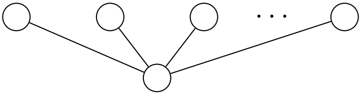

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**New intuition:** Maximize $\alpha(G)$

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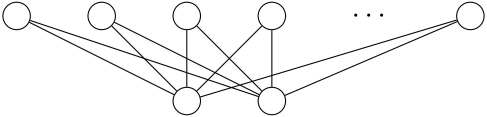

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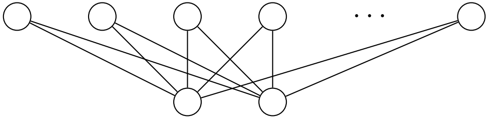

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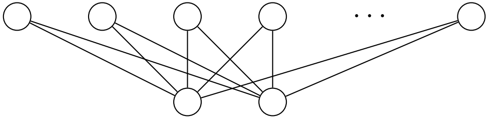

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$i_2(G) = \binom{n}{2} - |E(G)| \implies$ a regular $G$ (*not* $K_{\delta,n-\delta}$!) is maximizer

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*Conjecture true for*

- $\delta = 1, 2, 3$
- $\delta \geq 4$ *and* $t \geq 2\delta + 1$

# Proof for $t \geq 2\delta + 1$

## Observation

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**Proof:** Suppose that for some $t > \delta$,

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Then

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Ordered $(t+1)$-set: choose first $t$, which rules out $t + \delta$ vertices, so:

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$$\#_{>\delta} \;\leq\; k(n - (\delta + 2))(n - (\delta + 3)) \cdots (n - (\delta + t))$$

where $k$ is the number of vertices with degree $> \delta$.

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Ordered independent $t$-sets starting with vertex of degree $= \delta$:

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**Missing term?**

- Worst case situation: each new choice shares same $\delta$ neighbors

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- Can't happen $\delta + 1$ times (or we're in case 1.)
- $(\delta + 1)$st choice (at worst) removes a new vertex

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Have

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$$\#_{=\delta} \leq (n-k)(n-(\delta+1))(n-(\delta+2))\cdots \widehat{(n-(2\delta+1))}(n-(2\delta+2))\cdots(n-(\delta+t)).$$

Worst case is $k=0$:

$$\begin{aligned} i_t(G) &\leq \frac{1}{t!}n(n-(\delta+1))(n-(\delta+2))\cdots \widehat{(n-(2\delta+1))}(n-(2\delta+2))\cdots(n-(\delta+t)) \\ &\leq \frac{1}{t!}(n-\delta)(n-(\delta+1))\cdots (n-(2\delta+1))\cdots(n-(\delta+(t-1)))\text{[uses } t=2\delta+1\text{]} \\ &= \binom{n-\delta}{t}. \end{aligned}$$

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Future improvements?

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- Condition on the degrees of neighbors [linear programming]

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Thank you!