Signal and Noise in Complex-Valued SENSE MR Image Reconstruction

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OUTLINE

1. Motivation
2. Background
3. Methods
4. Results
5. Discussion
Motivation
In MRI it is not voxel values that are measured.

The actual measurements are spatial frequencies ($k$-space).

The $k$-space measurements are not acquired instantaneously.

In parallel imaging, $k$-space is subsampled and measured in parallel then combined to form a single image.

Image and volume measurement time is decreased at the expense of increased image reconstruction difficulty and time.

One popular parallel imaging method is SENSE.
Background

Image inverse Fourier Reconstruction for single coil.

\[(\Omega_{yR} + i\Omega_{yI}) \ast (F_R + iF_I) \ast (\Omega_{xR} + i\Omega_{xI})^T = (V_R + iV_I)\]
Background

In parallel imaging there is more than one receive coil.

Each coil measures a $k$-space array where every $A^{th}$ line is skipped.

Full $k$-space.  
Skipped $k$-space.
Background
The $k$-space arrays where every $A^{th}$ line is skipped are reconstructed into an aliased image to be combined to form a single image.

![Skipped $k$-space.](image1.png) ![Aliased images.](image2.png) ![Combined image.](image3.png)
Background
The combination of aliased images to form a single image utilizes coil sensitivities.

\[ a_C \quad \text{Aliased images.} \quad S_C \quad \text{Coil sensitivities.} \quad v_C \quad \text{Combined image.} \]
Methods

The SENSE model for aliased voxel values from $n$ coils is

$$a_C = S_C v_C + e_C, \quad e_C \sim CN(0, \Psi_C)$$

where for each voxel

- $a_C$ is a vector of the $n$ complex-valued aliased voxel values
- $v_C$ is a vector of the $A$ unaliased voxel value
- $S_C$ is an $n \times A$ matrix of complex-valued coil sensitivities
- $e_C$ is a vector of the $n$ complex-valued error values

$$a_C = a_R + i a_I$$
$$v_C = v_R + i v_I$$
$$S_C = S_R + i S_I$$
$$e_C = e_R + i e_I$$

Bruce, Karaman, and Rowe: In Submission, 2011.
\[ \alpha_C = S_C + v_C + \epsilon_C \]

Bruce, Karaman, and Rowe: In Submission, 2011.
Methods

The SENSE process

\[
a_C = S_C v_C + \varepsilon_C
\]

uses the complex-valued normal distribution

\[
f(\varepsilon_C) = (2\pi)^{-n} |\Psi_C|^{-1} e^{-1/2 \varepsilon_C^H \Psi_C^{-1} \varepsilon_C}
\]

and for \( n \) coil measurements

\[
f(a_C) = (2\pi)^{-n} |\Psi_C|^{-1} e^{-1/2 (a_C - S_C v_C)^H \Psi_C^{-1} (a_C - S_C v_C)}
\]


Bruce, Karaman, and Rowe: In Submission, 2011.
**Methods**

From the distribution for the \( n \) coil measurements

\[
f(a_C) = (2\pi)^{-n} \left| \Psi_C \right|^{-1} e^{-\frac{1}{2}(a_C - S_C \nu_C)^H \Psi_C^{-1} (a_C - S_C \nu_C)}
\]

the voxel values can be estimated as

\[
u_C = (S_C^H \Psi_C^{-1} S_C)^{-1} S_C^H \Psi_C^{-1} a_C
\]

with knowledge of \( S_C \) and \( \Psi_C \).
Methods

Instead of writing the model with complex numbers as

\[
\begin{align*}
\alpha_C & = S_C \nu_C + \varepsilon_C, \\
\alpha_C & = a_R + ia_I, \quad S_C = S_R + iS_I, \quad \nu_C = v_R + iv_I, \quad \varepsilon_C = \varepsilon_R + i\varepsilon_I
\end{align*}
\]

we can write the model using an isomorphism as

\[
\begin{align*}
\alpha & = S \nu + \varepsilon \\
\alpha & = \begin{pmatrix} \alpha_R \\ \alpha_I \end{pmatrix}, \quad S = \begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}, \quad \nu = \begin{pmatrix} v_R \\ v_I \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_R \\ \varepsilon_I \end{pmatrix}.
\end{align*}
\]

Bruce, Karaman, and Rowe: In Submission, 2011.
Methods

Then the distribution for $n$ coil measurements is

$$f(a) = (2\pi)^{-n} |\Psi|^{-1/2} e^{-1/2(a-S\nu)'\Psi^{-1}(a-S\nu)},$$

with

$$a = \begin{pmatrix} a_R \\ a_I \end{pmatrix}, \quad S = \begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}, \quad \nu = \begin{pmatrix} \nu_R \\ \nu_I \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_R \\ \varepsilon_I \end{pmatrix},$$

and the imposed skew-symmetric covariance structure

$$\Psi = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}.$$
Methods

The SENSE voxel values can be estimated by

\[ v_C = (S_C^H \Psi_C^{-1} S_C)^{-1} S_C^H \Psi_C^{-1} a_C \]

or in terms of an isomorphism

\[
\begin{pmatrix}
    v_R \\
    v_I
\end{pmatrix}
= \begin{pmatrix}
    S_R & -S_I \\
    S_I & S_R
\end{pmatrix}^T
\begin{pmatrix}
    \Psi_R & -\Psi_I \\
    \Psi_I & \Psi_R
\end{pmatrix}^{-1}
\begin{pmatrix}
    S_R & -S_I \\
    S_I & S_R
\end{pmatrix}^{-1}
\begin{pmatrix}
    S_R & -S_I \\
    S_I & S_R
\end{pmatrix}^T
\begin{pmatrix}
    \Psi_R & -\Psi_I \\
    \Psi_I & \Psi_R
\end{pmatrix}^{-1}
\begin{pmatrix}
    a_R \\
    a_I
\end{pmatrix}
\]

\[
\begin{pmatrix}
    v_R \\
    v_I
\end{pmatrix}
= U \ast \begin{pmatrix}
    a_R \\
    a_I
\end{pmatrix}
\]

\[ n=4 \text{ real / } n=4 \text{ imaginary true aliased voxel values} \]

\[ \text{Acceleration } A = 3 \text{ real / } A = 3 \text{ imaginary un-aliased fold values} \]
Methods

SENSE unfolding

\[
\begin{pmatrix}
  v_R \\
  v_I \\
\end{pmatrix}
= U \ast
\begin{pmatrix}
  a_R \\
  a_I \\
\end{pmatrix}
\]
Methods
Real-valued isomorphism

Methods

\[ \nu = I_n \otimes \Omega \ast f \]

- Image 1 aliased
- Image n aliased
- Coil 1 k-space
- Coil n k-space

Bruce, Karaman, and Rowe: In Submission, 2011.
Methods

\[ y = P_u \]

\[ U \]

\[ U_1 \]

\[ U_p \]

\[ \Psi \]

\[ P_s P_c \]

here is \( a \) for each voxel

permute to by folded voxel

Bruce, Karaman, and Rowe: In Submission, 2011.
Methods

Here is a for each voxel

processing on each coil k-space vector

processing on each coil image vector

$k$-space vector of $n$ images

permuting to by folded voxel

reconstruct $n=4$ images

$y = P_u$

$U$

$U_p$

$P_s P_c$

$(I_n \otimes \Omega)$

$U_1$

Bruce, Karaman, and Rowe: In Submission, 2011.
Methods

\[ y = \left[ O_I \quad P_u \quad U \quad PS \quad PC \right] \underbrace{\left( I_n \otimes \Omega_a O_k \right)}_{O} f \]

where

- \( f = (f_1, \ldots, f_n)' \) are coil k-space
- \( O_k \) is k-space preprocessing
- \( \Omega_a \) is adj. inverse Fourier matrix
- \( P_u, P_s, P_c \) are permutation matrices
- \( U \) is SENSE unfolding matrix
- \( O_I \) is image space preprocessing

- \( f = PC\mathcal{R}\mathcal{C}\mathcal{F} \)
- \( O_k = A \mathcal{Z} \mathcal{H} P^{-1}_R \Omega^{-1}_{row} \Phi \Omega_{row} P_R \)
- \( \Omega_a = \Omega \) adjusted for \( \Delta B \) and for \( T_2^* \)
- \( O_I = S_m \) Image smoothing

Bruce, Karaman, and Rowe: In Submission, 2011.
Methods
Statistical Expectation and Covariance.

If $E(f) = f_0$, then for $Mf$, $E(Mf) = Mf_0$.

If $\text{cov}(f) = \Gamma$, then for $Mf$, $\text{cov}(Mf) = M\Gamma M'$.

This means that with $y = Of$,

$$E(y) = Of_0 \quad \text{and} \quad \text{cov}(y) = O\Gamma O' = \Sigma$$

$$\Rightarrow \text{cor}(\nu) = D_{\Sigma}^{-1/2} \Sigma D_{\Sigma}^{-1/2}$$

So even if $\Gamma = \sigma^2 I$, it is not necessarily true that $\Sigma = \sigma^2 I$ !

This has $H_0$ fMRI noise and fcMRI connectivity implications!


Results

Since
\[ y = Of, \]
we inverted and made the \( n \) coil spatial frequencies from
\[ (O^T O)^{-1} O^T v = f \]
where \( O \) and \( v \) are known

\( v \) is true/noiseless Shepp-Logan phantom (scaled by 50)

\[ O = S_m P_U U P_S P_C (I_n \otimes \Omega) \]

The number of coils, \( n \), and the reduction factor, \( A \), are specified in the dimensions of operators, \( O \).
Results

Noiseless data $f = (O^T O)^{-1} O^T v$ generated for $N_X = N_Y = 96$, $n=4$, $A=3$

$O = S_m P_U U P_S P_C (I_n \otimes \Omega)$

$O$ had diagonal blocks $U_j = (S_j^T \Psi^{-1} S_j)^{-1} S_j^T \Psi^{-1}$

Sensitivities, $S$

Markovian coil covariance, $\Psi$

$\Psi = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix}$

$\Psi_R = \begin{pmatrix} 1 & .33 & .11 & .33 \\ .33 & 1 & .33 & .11 \\ .11 & .33 & 1 & .33 \\ .33 & .11 & .33 & 1 \end{pmatrix}$

Not skew-symmetric

$\Psi_I = \Psi_R$

$\Psi_{RI} = \begin{pmatrix} 0 & -.11 & -.07 & -.11 \\ .26 & 0 & -.11 & -.07 \\ .42 & .26 & 0 & -.11 \\ .26 & .42 & .26 & 0 \end{pmatrix}$

Bruce, Karaman, and Rowe: In Submission, 2011.
Results

Gaussian Smoothing applied in image-space
- FWHM = 3 voxels,
- Normalized to leave variance unaffected (Scales mean by 4.516)

By definition, smoothing induces a covariance and correlation between voxels and their neighbors.

This effect is in turn transferred to the correlated voxels from each fold in SENSE.

Gaussian smoothing kernel, $S_m$, was applied in image-space to reconstructed images.

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Results

Ghosting because symmetric coil cov \( \Psi \) used \( \rightarrow \)

Alternatice symmetric coil cov \( \Psi \) proposed.

Phase is important in complex-valued fMRI!

\[
\Psi = \begin{pmatrix}
\Psi_R & -\Psi_I \\
\Psi_I & \Psi_R
\end{pmatrix}
\]

\[
\Psi' = \begin{pmatrix}
\Psi_R & \Psi_{RI} \\
\Psi_{RI} & \Psi_I
\end{pmatrix}
\]
Results

5x5 image

\[ \text{cor} = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\ 0.6 & 0.7 & 0.8 & 0.9 & 1.0 \\ 1.1 & 1.2 & 1.3 & 1.4 & 1.5 \\ 1.6 & 1.7 & 1.8 & 1.9 & 2.0 \\ 2.1 & 2.2 & 2.3 & 2.4 & 2.5 \end{pmatrix} \]

25x25 correlation matrix

5x5 correlation image
Correlations induced about the center voxel.

**Results**

- $N_x = 96$
- $N_y = 96$
- $n = 4$
- $A = 3$

Functional connectivity implications

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Extrapolate to human, mistakenly conclude regions correlated!
Discussion
The SENSE image reconstruction method was described.

Wrote SENSE reconstruction with an isomorphism

\[ y = O_I P_U U P_S P_C (I_n \otimes \Omega O_k) f = Of. \]

The new mean \( E(y) = Of_0 \) and covariance \( \Sigma = O \Gamma O' \)
of complex-valued SENSE described.

Theoretical results of SENSE reconstruction presented.

Ghosting present in SENSE magnitude and phase images.

Induced correlation between folds of no biological origin.
Thank You

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Sweet 16!

Go Marquette!