The Distribution of Magnitude and Complex Voxel Values in MRI

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OUTLINE

1. Image Reconstruction
2. Statistics-Ricean & Normal
3. Estimation-Ricean & Normal
4. Estimation-Bivariate Normal
5. Discussion
Reconstruction:
Ideally measure complex-valued FT of the object.

\[
S(k_x, k_y) = S_R(k_x, k_y) + i S_I(k_x, k_y)
\]

Complex: 96×96
Real: 96×96
Imaginary: 96×96

Actual data!

\[
p = 9216
\]
\# of voxels
Reconstruction:
By complex-valued inverse FT of the object.

\[(\Omega_{yR} + i\Omega_{yI}) \ast (S_R + iS_I) \ast (\Omega_{xR} + i\Omega_{xI})^T = (Y_R + iY_I)\]
Reconstruction:
Due to imperfect reconstruction (noise, $T_2^*$, $\Delta B$, ...), image is complex-valued, $Y_C(x, y) = Y_R(x, y) + iY_I(x, y)$.
Reconstruction:
Toy Example $8 \times 8$, image is complex-valued,
\[ Y_C(x, y) = Y_R(x, y) + iY_I(x, y). \]
Reconstruction:
By complex-valued forward FT of the object.

\[ \Omega \bar{\Omega} = I \]

\[
\begin{align*}
(\Omega_{yR} + i\Omega_{yI}) & \ast (Y_R + iY_I) \ast (\Omega_{xR} + i\Omega_{xI})^T = (S_R + iS_I) \\
\end{align*}
\]
Reconstruction:

$S_R + i S_I$

On Cartesian grid

stack rows of $S_R$ on rows of $S_I$

Reconstruction:

\[
S = \begin{pmatrix}
S_R \\
S_I
\end{pmatrix}
\]

\[
2p \times 1
\]

\[
\Omega = \begin{bmatrix}
\Omega_R & -\Omega_I \\
\Omega_I & \Omega_R
\end{bmatrix}
\]

\[
\Omega_R = [(\Omega_{yR} \otimes \Omega_{xR}) - (\Omega_{yI} \otimes \Omega_{xI})]
\]

\[
\Omega_I = [(\Omega_{yR} \otimes \Omega_{xI}) + (\Omega_{yI} \otimes \Omega_{xR})]
\]

Reconstruction:
Inverse FT reconstruction can be equivalently described as:

\[ y = \Omega * S \]

\[ y_R = \Omega * S_R \]

\[ y_I = \Omega * S_I \]


Real-valued isomorphism
Reconstruction:

\[ y = \begin{pmatrix} y_R \\ y_I \end{pmatrix} \]

take sections of \( y_R \)

\begin{align*}
Y_R & \quad +i \\
Y_I & \quad +i
\end{align*}

transpose


WISC, Waisman

Rowe, MCW
Reconstruction:
Inverse FT reconstruction can be performed as:

\[ y = \Omega * S \]


Real-valued isomorphism
Statistics: Expectation and Covariance.

If $E(s) = s_0$, then for $y = \Omega s$, $E(y) = E(\Omega s) = \Omega s_0$.

If $\text{cov}(s) = \Gamma$, then for $y = \Omega s$, $\text{cov}(y) = \text{cov}(\Omega s) = \Omega \Gamma \Omega'$.

This means that with $\Gamma = \sigma_k^2 I$,

and because $\Omega \Omega' = \sigma^2 I$ where $\sigma^2 = (\sigma_k^2 / p^2)$

$\text{cov}(y) = \sigma^2 I_{2 \times 2}$.
Statistics: Expectation and Covariance.

When we use normal distribution from thermal noise

\[ s = s_0 + \epsilon, \quad \epsilon \sim N(0, \sigma_k^2 I) \]

\[ s \sim N(s_0, \sigma_0^2 I), \text{ then } y \sim N(\Omega s_0, \sigma^2 I). \]

This means that if we choose a voxel, say \( j \)
Statistics: Expectation and Covariance.

from \( y \sim N(\Omega s_0, \sigma^2 I) \), the distribution of \( y_{Rj} \) and \( y_{Ij} \) is

\[
\begin{pmatrix}
  y_{Rj} \\
  y_{Ij}
\end{pmatrix}
\sim N\left(\begin{pmatrix}
  \mu_{Rj} \\
  \mu_{Ij}
\end{pmatrix}, \begin{pmatrix}
  \sigma^2 & 0 \\
  0 & \sigma^2
\end{pmatrix}\right)
\]

where

\[
\begin{align*}
  \mu_{Rj} &= \omega_{j} s_0 \\
  \mu_{Ij} &= \omega_{p+j} s_0
\end{align*}
\]

\( y_{Cj} = y_{Rj} + iy_{Ij} \)

the pdf is

\[
p(y_{Rj}, y_{Ij}) = \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y_{Rj} - \mu_{Rj})^2 + (y_{Ij} - \mu_{Ij})^2 \right] \right\}
\]

product of two normal pdfs

with phase coupled means

\[
\begin{align*}
  \mu_{Rj} &= \rho_j \cos \theta_j \\
  \mu_{Ij} &= \rho_j \sin \theta_j
\end{align*}
\]
Statistics:

Real Image

Imaginary Image

voxel $j$

$y_{Rj}$

$y_{Ij}$
Statistics:

Magnitude Image

Phase Image

voxel $j$

$$m_j = \sqrt{y_{Rj}^2 + y_{Ij}^2}$$

$$\varphi_j = \tan^{-1}(y_{Ij} / y_{Rj})$$
Statistics:

Magnitude Image

Phase Image

voxel \( j \)

\[ m_j = \sqrt{y_{Rj}^2 + y_{Ij}^2} \]

\[ \varphi_j = \tan^{-1}\left(\frac{y_{Ij}}{y_{Rj}}\right) \]

Phase generally discarded!
Statistics:

$\mathbf{R-I}$

$\mathbf{M-P}$

$\mathbf{M-P}$

$\mathbf{M}$

$y_R$

$m$

$\varphi$

$m$

$\varphi$

$m_j = \sqrt{y_{Rj}^2 + y_{Ij}^2}$
Statistics:

Get \( p(m_j) \) from \( p(y_{Rj}, y_{Ij}) \).

\begin{align*}
\mu_{Rj} &= \rho_j \sin \theta_j \\
\mu_{Ij} &= \rho_j \cos \theta_j
\end{align*}

Convert from \( y_{Rj}, y_{Ij} \) to \( m_j, \varphi_j \).

\begin{align*}
p(y_{Rj}, y_{Ij}) &= \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y_{Rj} - \rho_j \cos \theta_j)^2 + (y_{Ij} - \rho_j \sin \theta_j)^2 \right] \right\} \\
p(m_j, \varphi_j) &= \frac{m_j}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} \left[ m_j^2 + \rho_j^2 - 2m_j\rho_j \cos(\varphi_j - \theta_j) \right] \right\} \\
p(m_j) &= \frac{m_j}{\sigma^2} \exp \left\{ -\frac{m_j^2 + \rho_j^2}{2\sigma^2} \right\} I_0 \left( \frac{\rho_j m_j}{\sigma^2} \right)
\end{align*}

zeroth order modified Bessel function of first kind

\[ \frac{1}{2\pi} \int_{\varphi_j = -\pi}^{\pi} e^{\frac{\rho_j m_j}{\sigma^2} \cos(\varphi_j - \theta_j)} d\varphi_j \]

Statistics:

\[ p(m_j) = \frac{m_j}{\sigma^2} \exp \left\{ - \frac{m_j^2 + \rho_j^2}{2\sigma^2} \right\} I_0 \left( \frac{\rho_j m_j}{\sigma^2} \right) \]

\[ SNR = \frac{\rho_j}{\sigma^2} \]

The magnitude, does not have a normal distribution!

Ricean Distribution!
Statistics:

\[ p(m_j) = \frac{m_j}{\sigma^2} \exp \left\{ - \frac{m_j^2 + \rho_j^2}{2\sigma^2} \right\} I_0 \left( \frac{\rho_j m_j}{\sigma^2} \right) \]

\[ SNR = \frac{\rho_j}{\sigma^2} \]

The magnitude, does not have a normal distribution!

Ricean Distribution!

Ricean \( \rightarrow \) Normal as the SNR \( \uparrow \)
Statistics:

The high SNR normality of \( m_j \) can be seen as

\[
m_j = \left[ (y_{Rj})^2 + (y_{Ij})^2 \right]^{1/2} = \left[ (\rho_j \cos \theta_j + \eta_{Rj})^2 + (\rho_j \sin \theta_j + \eta_{Rj})^2 \right]^{1/2} = \left[ \rho_j^2 + (\eta_{Rj}^2 + \eta_{Ij}^2) + 2\rho_j (\eta_{Rj} \cos \theta_j + \eta_{Rj} \sin \theta_j) \right]^{1/2} = \rho_j \left[ 1 + 2 \frac{(\eta_{Rj} \cos \theta_j + \eta_{Rj} \sin \theta_j)}{\rho_j} + \frac{\eta_{Rj}^2 + \eta_{Ij}^2}{\rho_j^2} \right] \approx \rho_j + \varepsilon_j
\]

where \( \varepsilon_j = \eta_{Rj} \cos \theta_j + \eta_{Rj} \sin \theta_j \)

\( \varepsilon_j \sim N(0, \sigma^2) \)

\[
\sqrt{1+u^2} \approx 1 + u / 2, \quad |u| \ll 1
\]
Statistics:
We take $n$ $k$-space arrays under different signal conditions.

Reconstruct each image.
Statistics:
We get $n$ images under different signal conditions

Cartesian coordinates
Real-Imaginary

$t=1$
Statistics:
We get $n$ images under different signal conditions

Cartesian coordinates
Real-Imaginary

voxel $j$

$y_{Rjt}$ Real
$y_{Ijt}$ Imaginary

$t=1$
Statistics:
We get $n$ images under different signal conditions
Statistics:
We get \( n \) images under different signal conditions

\[ j \]

\[ m_{jt} \]

\[ \varphi_{jt} \]

Phase discarded in a lot of MRI especially fMRI

Polar Coordinates
Magnitude-Phase

voxel \( j \)
Statistics:
We take $n$ images under different signal conditions

Toy Example

Cartesian coordinates
Real-Imaginary

$y_{Rjt}$
$y_{Ijt}$
$t=1$
Statistics: \( y_t = \Omega S_t \)

We take \( n \) images under different signal conditions.
Statistics: Bivariate Normal to Ricean

The distribution of measurement $t$ in voxel $j$ is:

$$p(y_{Rjt}, y_{ljt}) = \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y_{Rjt} - \rho_{jt} \cos \theta_{jt})^2 + (y_{ljt} - \rho_{jt} \sin \theta_{jt})^2 \right] \right\}$$

$$p(m_{jt}) = \frac{m_{jt}}{\sigma^2} \exp \left\{ - \frac{m_{jt}^2 + \rho_{jt}^2}{2\sigma^2} \right\} I_0 \left( \frac{\rho_{jt} m_{jt}}{\sigma^2} \right) \quad t = 1, \ldots, n$$

The goal is to estimate a functional form $\rho_{jt} = f(x_t | \beta_j)$ for the magnitude and possibly $\theta_{jt} = g(u_t | \gamma_j)$ for the phase from the data $y_{Cj1}, \ldots, y_{Cjn}$ or $m_{j1}, \ldots, m_{jn}$ in each voxel.

$x$ is a vector of known “dial” settings $\beta$ is a vector of unknown parameters.

$$\mu_{Rjt} = \omega_j s_{0t} = \rho_{jt} \cos \theta_{jt}$$

$$\mu_{ljt} = \omega_{p+j} s_{0t} = \rho_{jt} \sin \theta_{jt}$$

$\omega_j$ is $j^{th}$ row of $\Omega$

$\omega_{p+j}$ is $(p+j)^{th}$ row of $\Omega$
Estimation:
Types of functions to estimate:

Data $y_{C1}, ..., y_{Cn}$ or $m_1, ..., m_n$ in each voxel. No $j$ subscript.

<table>
<thead>
<tr>
<th>$f(x \mid \beta)$</th>
<th>$x$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>1</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$\rho \exp(-TE/T_2)$</td>
<td>TE</td>
<td>$\rho, T_2$</td>
</tr>
<tr>
<td>$S_0 \exp(-b'r'Dr)$</td>
<td>$b, r$</td>
<td>$S_0, D$</td>
</tr>
<tr>
<td>$\rho(1 - 2\exp(t/T_1))$</td>
<td>$t$</td>
<td>$\rho, T_1$</td>
</tr>
<tr>
<td>$x'\beta$</td>
<td>$x'$</td>
<td>$\beta$</td>
</tr>
</tbody>
</table>
Estimation: Ricean

Estimate parameters of function from magnitude data:

\[ p(m_t) = \frac{m_t}{\sigma^2} \exp \left\{ -\frac{m_t^2 + (f(x_t | \beta))^2}{2\sigma^2} \right\} I_0 \left( \frac{f(x_t | \beta)m_t}{\sigma^2} \right) \]

\[ L = \prod_{t=1}^{n} \frac{m_t}{\sigma^{2n}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t^2 + (f(x_t | \beta))^2 \right] \right\} \prod_{t=1}^{n} I_0 \left( \frac{f(x_t | \beta)m_t}{\sigma^2} \right) \]

\[ LL = -n \log(\sigma^2) + \sum_{t=1}^{n} \log(m_t) \]

\[ -\frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t^2 + (f(x_t | \beta))^2 \right] + \sum_{t=1}^{n} \log \left[ I_0 \left( \frac{f(x_t | \beta)m_t}{\sigma^2} \right) \right] \]

Maximize \( LL \):

\[ \frac{\partial LL}{\partial \beta} = 0 \quad \text{and} \quad \frac{\partial LL}{\partial \sigma^2} = 0 \]

Under \( H_1 \) and \( H_0 \)
Estimation: Large SNR Normal

Ricean

\[ p(m_t) = \frac{m_t}{\sigma^2} \exp\left\{ -\frac{m_t^2 + (f(x_t | \beta))^2}{2\sigma^2} \right\} I_0 \left( \frac{f(x_t | \beta)m_t}{\sigma^2} \right) \]

Normal as SNR ↑

\[ p(m_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} [m_t - f(x_t | \beta)]^2 \right\} \]

Then use usual least squares estimation.

Maximize \( LL \):

\[ \frac{\partial LL}{\partial \beta} = 0 \quad \text{and} \quad \frac{\partial LL}{\partial \sigma^2} = 0 \]

Under \( H_1 \) and \( H_0 \)

\[ LL = -2n \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} [m_t - f(x_t | \beta)]^2 \]
Estimation: Large SNR Normal

\[ LL = -2n \log(\sigma^2) + \frac{1}{2\sigma^2} \sum_{t=1}^{n} [m_t - f(x_t | \beta)]^2 \]

Under \( H_I \):

\[ \frac{\partial LL}{\partial \beta} = -\frac{2}{\sigma^2} \sum_{t=1}^{n} [m_t - f(x_t | \beta)] \frac{\partial f(x_t | \beta)}{\partial \beta} \]

Under \( H_0 \): add Lagrange constraint \( h(\beta, \sigma^2) \) to \( LL \)

\[ \frac{\partial LL}{\partial \beta} = -\frac{2}{\sigma^2} \sum_{t=1}^{n} [m_t - f(x_t | \beta)] \frac{\partial f(x_t | \beta)}{\partial \beta} + \frac{\partial h(\beta, \sigma^2)}{\partial \beta} \]

Under \( H_0 \) and \( H_I \):

\[ \frac{\partial LL}{\partial \sigma^2} = -\frac{2n}{\sigma^2} - \frac{1}{\sigma^4} \sum_{t=1}^{n} [m_t - f(x_t | \beta)]^2 \left( + \frac{\partial h(\beta, \sigma^2)}{\partial \sigma^2} \right) \]

May require numerical maximization depending on \( f(x_t | \beta) \).
**Estimation: Large SNR Normal**

GLM: Does not require numerical maximization. $X$ known

Under $H_1$: $\hat{\beta} = (X'X)^{-1}X'm$ \quad $\hat{\sigma}^2 = (y - X\hat{\beta})'(y_j - X\hat{\beta})/n$

Under $H_0$: $h(\beta, \sigma^2) = 2\psi'C\beta/\sigma^2$

$$\tilde{\beta} = \Psi(X'X)^{-1}X'm \quad \tilde{\sigma}^2 = (y - X\tilde{\beta})'(y - X\tilde{\beta})/n$$

$$\Psi = I - (X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C$$

Insert back into likelihoods and take ratio.

$$\lambda = L(\tilde{\beta}, \tilde{\sigma}^2) / L(\hat{\beta}, \hat{\sigma}^2)$$

This is how we get our usual $t$ and $F$ statistics.
Estimation: Large SNR Normal

DTI: Requires numerical maximization. $b$ and $r_t$ known

$$LL = -2n \log (\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} [m_t - S_0 \exp(-br_tD_t)]^2$$

Under $H_1$:

$$\frac{\partial LL}{\partial S_0} = -\frac{2}{\sigma^2} \sum_{t=1}^{n} [m_t - S_0 \exp(-br_tD_t)] \frac{\partial S_0 \exp(-br_tD_t)}{\partial S_0}$$

$$\hat{S}_0 \mid \hat{D} = \left[ \frac{\sum_{t=1}^{n} m_t \exp(-br_tD_t)}{\sum_{t=1}^{n} \exp(-2br_tD_t)} \right]$$

$$\frac{\partial LL}{\partial D} = 0 \quad \text{Does not yield a closed form solution.}$$

Need numerical maximization with say Newton-Raphson or Levenberg-Marquardt.
**Estimation: Large SNR Normal**

Numerical maximization.

\[
LL = -\frac{n}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t - \sqrt{f(x_t | \beta)^2 + \sigma^2} \right]^2
\]

\[\beta^{(0)} : \sum_{t=1}^{n} \left[ m_t - f(x_t | \beta) \right]^2 \quad \text{Minimized by Levenberg-Marquardt}\]

\[(\sigma^2)^{(0)} = \sum_{t=1}^{n} \left[ m_t - f(x_t | \beta^{(0)}) \right]^2 / n\]

\[\beta^{(r+1)} : \sum_{t=1}^{n} \left[ m_t - \sqrt{f(x_t | \beta)^2 + (\sigma^2)^{(r)}} \right]^2 \quad \text{Minimize by Levenberg-Marquardt}\]

\[(\sigma^2)^{(0)} : \quad LL = -\frac{n}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t - \sqrt{f(x_t | \beta)^2 + \sigma^2} \right]^2\]

Minimized by Newton-Raphson

\[\beta^{(0)}, (\sigma^2)^{(0)}, \beta^{(1)}, (\sigma^2)^{(1)}, ..., \beta^{(r+1)}, (\sigma^2)^{(r+1)} \quad \text{sequence converges to MLE!}\]
**Estimation: Small SNR Ricean**

\[
LL = -n \log(\sigma^2) + \sum_{t=1}^{n} \log(m_t) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t^2 + (f(x_t | \beta))^2 \right] \\
+ \sum_{t=1}^{n} \log \left[ I_0 \left( f(x_t | \beta)m_t / \sigma^2 \right) \right]
\]

Under \( H_1 \):
\[
A_t = f(x_t | \beta)m_t / \sigma^2
\]

\[
\frac{\partial LL}{\partial \beta} = \frac{1}{\sigma^2} \sum_{t=1}^{n} \left[ m_t I_1(A_t) / I_0(A_t) - f(x_t | \beta) \right] \frac{\partial f(x_t | \beta)}{\partial \beta}
\]

Under \( H_0 \):
\[
\frac{\partial LL}{\partial \beta} = \frac{1}{\sigma^2} \sum_{t=1}^{n} \left[ m_t I_1(A_t) / I_0(A_t) - f(x_t | \beta) \right] \frac{\partial f(x_t | \beta)}{\partial \beta} + \frac{\partial h(\beta, \sigma^2)}{\partial \beta}
\]

Under \( H_1 \) and \( H_0 \)
\[
\frac{\partial LL}{\partial \sigma^2} = \frac{1}{2\sigma^4} \left[ m_t^2 + (f(x_t | \beta))^2 - 2m_tA_tf(x_t | \beta) - 2n\sigma^2 \right] \left( + \frac{\partial h(\beta, \sigma^2)}{\partial \sigma^2} \right)
\]

No closed form solution. Requires numerical maximization!

Take magnitude variates \( m_1, ..., m_n \) that are Ricean distributed

\[
p(m_t) = \frac{m_t}{\sigma^2} \exp \left\{ -\frac{m_t^2 + f(x_t | \beta)^2}{2\sigma^2} \right\} I_0 \left( \frac{f(x_t | \beta)m_t}{\sigma^2} \right)
\]

Introduce latent phase variables \( \phi_1, ..., \phi_n \) such that

\[
p(m_t, \phi_t) = \frac{m_t}{2\pi\sigma^2} \exp \left\{ -(m_t^2 + f(x_t | \beta)^2 \right. \\
\left. \quad \quad \quad \quad \quad -2m_t f(x_t | \beta) \cos \phi_t \right\} / 2\sigma^2 \]

and

\[
LL = -n \log(2\pi\sigma^2) + \sum_{t=1}^{n} \log(m_t) \\
\quad - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t^2 + f(x_t | \beta)^2 - 2m_t f(x_t | \beta) \cos(\phi_t) \right]
\]

**Estimation:** Small SNR

EM Algorithm. Iterative.

\[
LL = -n \log(2\pi\sigma^2) + \sum_{t=1}^{n} \log m_t - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t^2 + f(x_t | \beta)^2 - 2m_t f(x_t | \beta) \cos \phi_t \right]
\]

**E Step:** Let \( Y_m = (m_1, \ldots, m_n), Y_\phi = (\phi_1, \ldots, \phi_n), Y_x = (x_1, \ldots, x_n) \)

given \( \beta^{(r)}, (\sigma^2)^{(r)} \): Initial values from normal GLM

\[
E[L_c(\beta, \sigma^2 | Y_m, Y_\phi, Y_x) | Y_m, Y_x, \beta^{(r)}, (\sigma^2)^{(r)}] = \\
- n \log(\sigma^2)^{(r)} - \frac{1}{2(\sigma^2)^{(r)}} \sum_{t=1}^{n} \left[ m_t^2 + f(x_t | \beta^{(r)})^2 - 2m_t f(x_t | \beta^{(r)}) A_t^{(r)} \right] \\
A_t^{(r)} = f(x_t | \beta^{(r)}) m_t / (\sigma^2)^{(r)}
\]

with respect to \( p(Y_\phi | Y_m, Y_x, \beta^{(r)}, (\sigma^2)^{(r)}) = \prod_{t=1}^{n} p(\phi_t | m_t, \beta^{(r)}, (\sigma^2)^{(r)}) \)


Estimation: Small SNR
EM Algorithm. Iterative.

M Step:

given $\beta^{(r)}$, $(\sigma^2)^{(r)}$:

$$(\sigma^2)^{(r+1)} = \frac{1}{2n} \sum_{t=1}^{n} \left[ m_t^2 + f(x_t | \beta^{(r)})^2 - 2m_t f(x_t | \beta^{(r)}) A_t^{(r)} \right]$$

$$A_t^{(r)} = f(x_t | \beta^{(r)}) m_t / (\sigma^2)^{(r)}$$

$$\beta^{(r+1)} : \text{minimize} \sum_{t=1}^{n} \left[ f(x_t | \beta)^2 - m_t A_t^{(r)} \right]^2 \quad \text{given} \quad (\sigma^2)^{(r+1)}$$

$\beta^{(0)}, (\sigma^2)^{(0)}, \beta^{(1)}, (\sigma^2)^{(1)}, \ldots, \beta^{(r+1)}, (\sigma^2)^{(r+1)}$ sequence converges to MLE!

Estimation: Small SNR EM Algorithm.

\[ f(S_0, D | r, b) = S_0 \exp(-br'Dr) \]

Fractional Anisotropy, FA

Signal-to-Noise Ratio, \( S_0/\sigma^2 \)

Estimation: Bivariate Normal

Magnitude Image

Phase Image

voxel $j$

\[ m_j = \sqrt{y_{Rj}^2 + y_{ij}^2} \]

\[ \varphi_j = \tan^{-1}\left(\frac{y_{ij}}{y_{Rj}}\right) \]
Statistics:
We get $n$ images under different signal conditions.

Polar Coordinates
Magnitude-Phase

voxel $j$

$M_{jt}$

$\Phi_{jt}$

$t=1$
**Estimation: All SNRs Bivariate Normal**

\[
\begin{pmatrix}
 y_{Rt} \\
y_{It}
\end{pmatrix} = \begin{pmatrix}
 \rho_t \cos \theta_t \\
 \rho_t \sin \theta_t
\end{pmatrix} + \begin{pmatrix}
 \eta_{Rt} \\
 \eta_{It}
\end{pmatrix}, \quad \begin{pmatrix}
 \eta_{Rt} \\
 \eta_{It}
\end{pmatrix} \sim N(0, \Sigma)
\]

\[
p(y_{Rt}, y_{It}) = \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y_{Rt} - \rho_t \cos \theta_t)^2 + (y_{It} - \rho_t \sin \theta_t)^2 \right] \right\}
\]

\[
p(m_t, \varphi_t) = \frac{m_t}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} \left[ m_t^2 + \rho_t^2 - 2m_t \rho_t \cos(\varphi_t - \theta_t) \right] \right\}
\]

\[
\rho_t = f(x_t \mid \beta) \quad \text{and} \quad \theta_t = g(u_t \mid \gamma)
\]

\[
LL = -n \log(2\pi\sigma^2) + \sum_{t=1}^{n} \log(m_t)
\]

\[
- \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t^2 + f(x_t \mid \beta)^2 - 2m_t f(x_t \mid \beta) \cos(\varphi_t - g(u_t \mid \gamma)) \right]
\]
Estimation: All SNR Bivariate Normal

\[ LL = -n \log(2\pi\sigma^2) + \sum_{t=1}^{n} \log(m_t) \]

\[- \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[ m_t^2 + f(x_t | \beta)^2 - 2m_t f(x_t | \beta) \cos(\varphi_t - g(u_t | \gamma)) \right] \]

\[ \frac{\partial LL}{\partial \beta} = \frac{1}{\sigma^2} \sum_{t=1}^{n} \left[ m_t \cos(\varphi_t - g(u_t | \gamma)) - f(x_t | \beta) \right] \frac{\partial f(x_t | \beta)}{\partial \beta} \]

\[ \frac{\partial LL}{\partial \gamma} = \frac{1}{\sigma^2} \sum_{t=1}^{n} \left[ m_t f(x_t | \beta) \sin(\varphi_t - g(u_t | \gamma)) \right] \frac{\partial g(u_t | \gamma)}{\partial \gamma} \]

\[ \frac{\partial LL}{\partial \sigma^2} = \frac{n}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{t=1}^{n} \left[ m_t^2 + f(x_t | \beta)^2 - 2m_t f(x_t | \beta) \cos(\varphi_t - g(u_t | \gamma)) \right] \]

\( \sigma^2 \) can be uniquely solved for given \( \beta, \gamma \)
Estimation:
Time series are complex, bivariate with phase coupled means.

The $y_R$ and $y_I$ time courses have related info! From actual human data!
Estimation:
Time series are complex, bivariate with phase coupled means.

An FT of this ts would show a peak at task freq.

Magnitude: Task related magnitude changes!

Phase: Task related phase changes!

Periodicity!

Estimation:
Magnitude and or phase change.

\[ \rho_t = x'_t \beta \text{ and } \theta_t = u'_t \gamma \]


Estimation:
Magnitude and or phase change.

\[ \rho_t = x_t' \beta \text{ and } \theta_t = u_t' \gamma \]


Estimation:
Magnitude and or phase change.

\[ \rho_t = x'_t \beta \text{ and } \theta_t = u'_t \gamma \]


**Estimation: All SNR**

**GLM:**

\[ \rho_t = x_t' \beta \quad \text{and} \quad \theta_t = u_t' \gamma \]

\[ LL = -n \log(2\pi\sigma^2) + \sum_{t=1}^{n} \log(m_t) \]

\[ H_0 : C \beta = 0 \quad \text{vs.} \quad H_1 : C \beta \neq 0 \]

\[ D \gamma = 0 \quad D \gamma \neq 0 \]

Maximize \( LL \): under \( H_1 \) and \( H_0 \)

\[ \beta^{(0)} : \text{initial value} \]

\[ \hat{\gamma}^{(r)} = \left( \hat{Z}'(r) \hat{Z}(r) \right)^{-1} \hat{Z}'(r) \phi^{(r)} \]

\[ \hat{\beta}^{(r+1)} = (X'X)^{-1} X'm^{(r)}_* \]

\[ (\hat{\sigma}^2)^{(r+1)} = \frac{1}{2n} \sum_{t=1}^{n} \left[ (m - X \hat{\beta}^{(r+1)})' (m - X \hat{\beta}^{(r+1)}) \right] + 2(m - \hat{m}^{(r+1)}_*)' X \hat{\beta}^{(r+1)} \]

\( \varphi^{(r)}_* \) has elements \( \varphi_t \sqrt{m_t x_t' \hat{\beta}^{(r)}(r)} \)

\( \hat{Z}'(r) \) has rows \( u_t' \sqrt{m_t x_t' \hat{\beta}^{(r)}(r)} \)

\( m^{(r)}_* \) has elements \( m_t \cos(\varphi_t - u_t' \hat{\gamma}^{(r)}(r)) \)

Estimation:  GLM:

20s off + 16 × (8 s on 8 s off), 276 TRs
12 axial slices, 96 × 96, FOV = 24 cm
TH = 2.5 mm, TR = 1 s, TE = 34.6 ms
FA = 45°, BW = 125 kHz, ES = .708 ms

20s off + 16 × (8 s on 8 s off), 276 TRs
10 axial slices, 96 × 96, FOV = 24 cm
TH = 2.5 mm, TR = 1 s, TE = 42.8 ms
FA = 45°, BW = 125 kHz, ES = .768 ms

20s off + 10 × (8 s on 8 s off), 180 TRs
9 axial slices, 64 × 64, FOV = 24 cm
TH = 3.8 mm, TR = 1 s, TE = 26.0 ms
FA = 45°, BW = 125 kHz, ES = .680 ms


Hahn, Nencka, Rowe: In progress.
Discussion:

• Not clear how much improvement from Ricean distribution.

• Improvements will show below SNR=5. High $b$-values.

• Other factors hinder it.
  Dynamic field changes
  Image Warping
  Motion
  Image Processing

• Should also use phase for complete data model.

• More biological info extracted with use of phase.
Discussion:

1. Image Reconstruction
2. Statistics - Ricean & Normal
3. Estimation - Ricean & Normal
4. Estimation - Bivariate Normal
5. Discussion

Further research is needed ....
Thank You

Questions?