Bivariate Transformation of Variables (continued)

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Outline

- Distributions
  Uniforms to Normals
  Normals to Chi-Square
  Normal and Chi-Square to t
  Chi-Squares to F
Bivariate Change of Variable

Given two continuous random variables, \((x_1, x_2)\)

with joint probability distribution function \(f_{x_1,x_2}(x_1, x_2 \mid \theta)\).

Let \(\begin{pmatrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \end{pmatrix}\) be a transformation from \((x_1, x_2)\) to \((y_1, y_2)\)

with inverse transformation \(\begin{pmatrix} x_1(y_1, y_2) \\ x_2(y_1, y_2) \end{pmatrix}\).
Bivariate Change of Variable

Then, the joint probability distribution function \( f_{Y_1,Y_2}(y_1, y_2 \mid \theta) \) of \((y_1, y_2)\) can be found via

\[
f_{Y_1,Y_2}(y_1, y_2 \mid \theta) = f_{X_1,X_2}(x_1(y_1, y_2), x_2(y_1, y_2) \mid \theta) \times |J(x_1, x_2 \to y_1, y_2)|
\]

where \( J(x_1, x_2 \to y_1, y_2) = \begin{vmatrix} dx_1(y_1, y_2) & dx_1(y_1, y_2) \\ dy_1 & dy_2 \\ dx_2(y_1, y_2) & dx_2(y_1, y_2) \\ dy_1 & dy_2 \end{vmatrix} \).
Bivariate Change of Variable - Normals

Let \( u_1 \sim \text{uniform}(0,1) \) and \( u_2 \sim \text{uniform}(0,1) \). The joint PDF of \((u_1, u_2)\) is

\[
 f(u_1, u_2) = \begin{cases} 
 1 & \text{if } u_1 \in [0,1] \text{ and } u_2 \in [0,1] \\
 0 & \text{if } u_1 \notin [0,1] \text{ or } u_2 \notin [0,1] 
\end{cases}.
\]

If \( z_1 = z_1(u_1, u_2) \), \( z_2 = z_2(u_1, u_2) \), the joint distribution of \((z_1, z_2)\) is

\[
 f_{Z_1,Z_2}(z_1, z_2 \mid \theta) = f_{U_1,U_2}(u_1(z_1, z_2), u_2(z_1, z_2) \mid \theta) \times |J(u_1, u_2 \rightarrow z_1, z_2)|
\]

\[
 J(u_1, u_2 \rightarrow z_1, z_2) = \begin{vmatrix} 
 du_1(z_1, z_2) & du_1(z_1, z_2) \\
 dz_1 & dz_2 \\
 du_2(z_1, z_2) & du_2(z_1, z_2) \\
 dz_1 & dz_2 
\end{vmatrix}
\]
Bivariate Change of Variable - Normals

Let \( z_1 = \sqrt{-2 \ln(u_1)} \cos(2\pi u_2) \) and \( z_2 = \sqrt{-2 \ln(u_1)} \sin(2\pi u_2) \), then

\[
\begin{align*}
 u_1(z_1, z_2) &= e^{-\frac{1}{2}(z_1^2 + z_2^2)} \\
u_2(z_1, z_2) &= \frac{1}{2\pi} \arctan \left( \frac{z_2}{z_1} \right).
\end{align*}
\]

\[
J(u_1, u_2 \rightarrow z_1, z_2) = \left| \begin{array}{cc}
\frac{du_1(z_1, z_2)}{dz_1} & \frac{du_1(z_1, z_2)}{dz_2} \\
\frac{du_2(z_1, z_2)}{dz_1} & \frac{du_2(z_1, z_2)}{dz_2}
\end{array} \right| = -\frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2 + z_2^2)}
\]
Bivariate Change of Variable - Normals

Therefore,

\[ f_{z_1, z_2} (z_1, z_2 \mid \theta) = f_{u_1, u_2} (u_1(z_1, z_2), u_2(z_1, z_2) \mid \theta) \times \left| J(u_1, u_2 \rightarrow z_1, z_2) \right| \]

which upon insertion yields

\[ f_{z_1, z_2} (z_1, z_2 \mid \theta) = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2 + z_2^2)} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_2^2} . \]

This means \( z_1 \sim N(0, 1) \), \( z_2 \sim N(0, 1) \), \( z_1 \) and \( z_2 \) are independent.
Bivariate Change of Variable - Normals

Generate $10^6$ independent uniform(0,1)’s.

The first half of the $10^6$ standard uniform random variates were used as $u_1$’s and the second half used as $u_2$’s.

Take each $(u_1, u_2)$ pair to produce a $(z_1, z_2)$ pair.

$$z_1 = \sqrt{-2\ln(u_1)} \cos(2\pi u_2) \quad z_2 = \sqrt{-2\ln(u_1)} \sin(2\pi u_2)$$

$(z_1, z_2)$ are independent normally distributed.
Bivariate Change of Variable - Normals

\[ n = 10^6; \]
\[ u_1 = \text{rand}(n/2,1); \]
\[ u_2 = \text{rand}(n/2,1); \]
\[ z_1 = \sqrt{-2 \cdot \log(u_1)} \cdot \cos(2 \pi u_2); \]
\[ z_2 = \sqrt{-2 \cdot \log(u_1)} \cdot \sin(2 \pi u_2); \]
\[ \text{figure(1)} \]
\[ \text{hist([u1;u2],50)} \]
\[ \text{figue(2)} \]
\[ \text{hist([z1;z2],50)} \]

\[ \text{[mean(u1),var(u1)]} \]
\[ \text{[mean(u2),var(u2)]} \]
\[ \text{[mean(z1),var(z1)]} \]
\[ \text{[mean(z2),var(z2)]} \]
\[ \text{[corr(u1,u2),corr(z1,z2)]} \]

Uncorrelated and since normal are independent

0.5000 0.0832
0.5006 0.0833
0.0000 1.0011
-0.0016 0.9970
0.0025 0.0013

Uncorrelated and since normal are independent
We discussed how we can obtain a random variable $x$ that has a general normal distribution with mean $\mu$ and variance $\sigma^2$ via the transformation $x = \sigma z + \mu$.

The PDF of $x$ can be obtained by the change of variable

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}$$

where, $x, \mu \in \mathbb{R}, 0 < \sigma$. That is, $x \sim \text{normal}(\mu, \sigma^2)$. 

Bivariate Change of Variable - Chi-Square

We also discussed how the change of variable technique can be repeated. If \( x_i \sim \text{normal}(\mu, \sigma^2) \) for \( i=1,\ldots,n \), and \( x_i \)'s are independent, then

\[
y = \frac{1}{n} \sum_{i=1}^{n} x_i \sim \mathcal{N} \left( \mu, \frac{\sigma^2}{n} \right).
\]
Bivariate Change of Variable - Chi-Square

We also discussed how the change of variable technique can be applied to $y_1 = \left( \frac{x_1 - \mu}{\sigma} \right)^2$. If $x_1 \sim \text{normal}(\mu, \sigma^2)$, then the distribution $y_1$ is $\chi^2(1)$. This process can be duplicated so that if $x_2 \sim \text{normal}(\mu, \sigma^2)$, then the distribution of $y_2 = \left( \frac{x_2 - \mu}{\sigma} \right)^2$ is $\chi^2(1)$.

Now what is the distribution of $y_1 + y_2$?
Bivariate Change of Variable - Chi-Square

Let \( y_1 \) and \( y_2 \) have independent chi-square PDFs

\[
\begin{align*}
f(y_i) &= \frac{y_i^{1/2-1} e^{-y_i/2}}{\Gamma(1/2)2^{1/2}}, \quad y_i > 0, \quad i = 1, 2.
\end{align*}
\]

We can find the distribution of \( w_1 = y_1 + y_2 \) (and \( w_2 = y_2 \)) via the bivariate change of variable technique

\[
\begin{align*}
f_{W_1,W_2}(w_1, w_2 | \theta) &= f_{Y_1,Y_2}(y_1(w_1, w_2), y_2(w_1, w_2) | \theta) \times | J(y_1, y_2 \rightarrow w_1, w_2) | \\
\end{align*}
\]

with marginalization

\[
\begin{align*}
f_{W_1}(w_1 | \theta) &= \int_{w_2} f_{W_1,W_2}(w_1, w_2 | \theta) dw_2.
\end{align*}
\]
It turns out that if $y_1 \sim \chi^2(1)$, $y_2 \sim \chi^2(1)$, and independent, then

$$w_1 = y_1 + y_2 \sim \chi^2(2).$$

Or more generally, if $y_1 \sim \chi^2(v_1)$, $y_2 \sim \chi^2(v_2)$, and independent, then $w_1 = y_1 + y_2 \sim \chi^2(v_1 + v_2)$.

So what this means is that

$$y = \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)!$$

Homework problem.
Bivariate Change of Variable - Chi-Square

If \( y_i = \left( \frac{x_i - \mu}{\sigma} \right)^2 \sim \chi^2(1) \), then \( y = \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 \sim \chi^2(n) \).

n=10; mu=5; sigma=2;
x=sigma*randn(10^6,n)+mu;
y=sum(((x-mu)/sigma).^2,2);
figure(1)
hist(y,(0:.1:40)')
axis([0 40 0 11000])
mean(y)
var(y)

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Bivariate Change of Variable - Chi-Square

If the mean $\mu$ is unknown, then we can estimate it by $\overline{x}$ and lose one degree of freedom!

$$\sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} \left( \frac{x_i - \overline{x}}{\sigma} \right)^2 + \left( \frac{\overline{x} - \mu}{\sigma / \sqrt{n}} \right)^2$$

We just showed

add and subtract $\overline{x}$ in the numerator

Because $df$ add, or by transformation!

Since

$$\overline{x} \sim N(\mu, \sigma^2 / n)$$

$$\frac{\overline{x} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$$

$$\left( \frac{\overline{x} - \mu}{\sigma / \sqrt{n}} \right)^2 \sim \chi^2(1)$$
Bivariate Change of Variable - Chi-Square

\[ y_2 = \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\sigma} \right)^2 \sim \chi^2(n-1) \]

Already have \( x \)'s.

\[ x_{\text{bar}} = \text{mean}(x,2); \]
\[ y_2 = \text{sum}(((x-x_{\text{bar}})*\text{ones}(1,n)).../\sigma).^2,2); \]
\[ \text{figure}(2) \]
\[ \text{hist}(y2,(0:.1:40)') \]
\[ \text{axis}([0 40 0 11000]) \]
\[ \text{mean}(y2) \]
\[ \text{var}(y2) \]

\[ n = 10 \]
\[ \bar{y} = 9.0000 \]
\[ s_y^2 = 18.0031 \]
Bivariate Change of Variable - Chi-Square

\[ y_1 = \left( \frac{x - \mu}{\sigma / \sqrt{n}} \right)^2 \sim \chi^2(1) \]

Already have x-bars’s.

\[ y_1 = ((xbar - mu) / (sigma / sqrt(n))).^2; \]

figure(3)

hist(y1,(0:.1:15))'

axis([0 15 0 10^5])

mean(y1)

var(y1)

\[ n = 10 \]

\[ \bar{y}_1 = 0.9968 \]

\[ s^2_{y_1} = 1.9870 \]
Bivariate Change of Variable - Chi-Square

\[ y = \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 \sim \chi^2(n) \quad \text{and} \quad y_2 = \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\sigma} \right)^2 \sim \chi^2(n-1) \]

\[
\begin{align*}
n & = 10 \\
\bar{y} & = 9.9968 \\
s_y^2 & = 19.9985
\end{align*}
\]

\[
\begin{align*}
n & = 10 \\
\bar{y} & = 9.0000 \\
s_y^2 & = 18.0031
\end{align*}
\]
Bivariate Change of Variable - Chi-Square

\[ \chi^2(10) \]

\[ n = 10 \]
\[ \bar{y} = 9.9968 \]
\[ s_y^2 = 19.9985 \]

\[ \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 \sim \chi^2(n) \]

Toggle with next slide!
Bivariate Change of Variable - Chi-Square

\[ n = 10 \]
\[ \bar{y}_2 = 9.0000 \]
\[ s^2_{y_2} = 18.0031 \]

\[ \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\sigma} \right)^2 \sim \chi^2(n - 1) \]
We showed that if $x_i \sim \text{normal}(\mu, \sigma^2)$ for $i=1,\ldots,n$, then

the distribution of $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ and $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

and that the distribution of $y_2 = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma}\right)^2 \sim \chi^2(n-1)$.

Note that $y_2 = \frac{(n-1)s^2}{\sigma^2}$.

It turns out that $z$ and $\frac{(n-1)s^2}{\sigma^2}$ are statistically independent!
Bivariate Change of Variable - Student-t

So \( z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0,1) \) and \( y_2 = \frac{\nu s^2}{\sigma^2} \sim \chi^2(\nu), \ \nu = n - 1. \)

Let \( t = \frac{z}{\sqrt{y_2 / \nu}} \) and \( s = y_2. \)

Then \( z = \frac{t\sqrt{s}}{\sqrt{\nu}} \) and \( y_2 = s, \) the Jacobian of the transformation is

\[
J(z, y \rightarrow t, s) = \begin{vmatrix}
\frac{dz(t, s)}{dt} & \frac{dz(t, s)}{ds} \\
\frac{dy_2(t, s)}{dt} & \frac{dy_2(t, s)}{ds}
\end{vmatrix} = \frac{\sqrt{s}}{\sqrt{\nu}}
\]
The joint distribution of \((t,s)\) is

\[
f_{T,S}(t, s \mid \theta) = f_{y2,z}(y(240,262),(577,866)(234,262),(574,866), z(t, s) \mid \theta) \times |J(y(234,262),(574,866)(234,262),(574,866), z \rightarrow t, s)|
\]

\[
f_{T,S}(t, s \mid \theta) = \frac{s^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2}(1 + \frac{1}{\nu} t^2)}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\nu/2} \sqrt{2\pi}} \times \left|\frac{\sqrt{s}}{\sqrt{\nu}}\right|
\]

and by integrating out \(s\) the distribution of \(t\) is

\[
f_T(t \mid \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{1}{\nu} t^2\right)^{-\frac{\nu+1}{2}}.
\]

The distribution of \(t = \frac{z}{\sqrt{y_2/(n-1)}} \sim t(n-1)\)!

Here we use the assumption that \(z\) and \(y\) are independent!
Bivariate Change of Variable - F

Recall that
\[
\sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\sigma} \right)^2 + \left( \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right)^2,
\]

It turns out that
\[
y_1 = \left( \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right)^2 \quad \text{and} \quad y_2 = \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\sigma} \right)^2
\]
are statistically independent.

But of interest to us (hypothesis testing) is the distribution of
\[
f = \frac{y_1 / \nu_1}{y_2 / \nu_2}, \quad \text{where} \quad y_1 \sim \chi^2(\nu_1) \quad \text{and} \quad y_2 \sim \chi^2(\nu_2).
\]
Bivariate Change of Variable - F

Let \( y_1 \) and \( y_2 \) have independent \( \chi^2 \) PDFs with \( \nu_1 \) and \( \nu_2 \) df

\[
f(y_i \mid \nu_i) = \frac{y_i^{\nu_i/2-1} e^{-y_i/2}}{\Gamma(\nu_i / 2)2^{\nu_i/2}}, \quad y_i > 0 \, , \, i = 1, 2.
\]

We can find the distribution of \( f = \frac{y_1 / \nu_1}{y_2 / \nu_2} \) (and \( g=y_2 \)) via the bivariate change of variable technique

\[
f_{F,G}(f, g \mid \theta) = f_{Y_1,Y_2}(y_1(f, g), y_2(f, g) \mid \theta) \times |J(y_1, y_2 \to f, g)|
\]

and marginalization \( f_F(f \mid \theta) = \int f_{F,G}(f, g \mid \theta)dg \).
The joint distribution of \((f, g)\) is

\[
f_{F,G}(f, g \mid \theta) = f_{Y_1,Y_2}(y_1(f, g), y_2(f, g) \mid \theta) \times |J(y_1, y_2 \rightarrow f, g)|
\]

the original variables in terms of the new variables are

\[
y_1 = \frac{\nu_1}{\nu_2} g f \quad \text{and} \quad y_2 = g \quad \text{with Jacobian}
\]

\[
J(y_1, y_2 \rightarrow f, g) = \begin{vmatrix}
\frac{dy_1(f, g)}{df} & \frac{dy_1(f, g)}{dg} \\
\frac{dy_2(f, g)}{df} & \frac{dy_2(f, g)}{dg}
\end{vmatrix} = \frac{\nu_1}{\nu_2} g.
\]
Bivariate Change of Variable - F

The joint distribution of \((f, g)\) is

\[
f_{F,G}(f, g \mid \theta) = f_{Y_1,Y_2}(y_1(f, g), y_2(f, g) \mid \theta) \times |J(y_1, y_2 \rightarrow f, g)|
\]

\[
f_{F,G}(f, g \mid \theta) = \left(\frac{f}{g}ight)^{\nu_1/2 - 1} \cdot e^{-\left(\frac{f}{g}\right)^{1/2}} \cdot \frac{g^{\nu_2/2 - 1} e^{-g^{1/2}}}{\Gamma(\nu_2 / 2) 2^{\nu_2/2}} \times \left|\frac{\nu_1}{\nu_2} \frac{g}{f}\right|
\]

\[
f_F(f \mid \theta) = \int_g f_{F,G}(f, g \mid \theta) dg
\]

\[
f_F(f \mid \nu_1, \nu_2) = \frac{\Gamma((\nu_1 + \nu_2) / 2)}{\Gamma(\nu_1 / 2) \Gamma(\nu_2 / 2)} \left(\frac{\nu_1 f}{\nu_1 f + \nu_2}\right)^{\nu_1/2} \left(1 - \frac{\nu_1 f}{\nu_1 f + \nu_2}\right)^{\nu_2/2}
\]

\[
y_1 = \frac{\nu_1}{\nu_2} \cdot g \quad y_2 = g
\]
**Bivariate Change of Variable - F**

The joint distribution of \( f = \frac{y_1 / \nu_1}{y_2 / \nu_2} \) is F distributed with \( \nu_1 \) numerator df and \( \nu_2 \) denominator df.

\[
f_F(f \mid \nu_1, \nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1 f}{\nu_1 f + \nu_2}\right)^{\nu_1/2} \left(1 - \frac{\nu_1 f}{\nu_1 f + \nu_2}\right)^{\nu_2/2}
\]

where \( \nu_1, \nu_2 = 1, 2, ... \)

Therefore,

\[
f = \left[\left(\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}\right)^2\right] / \left[\sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma}\right)^2 / (n-1)\right] \sim F(1, n-1)
\]
Bivariate Change of Variable - F/Student-t

We just showed that

\[ f = \frac{y_1 / \nu_1}{y_2 / \nu_2} \sim F(1, n-1) \]

where \( y_1 = \left( \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right)^2 \) and \( y_2 = \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\sigma} \right)^2 \).

Recall that we showed that

\[ t = \frac{z}{\sqrt{y_2/(n-1)}} \sim t(n-1) \]

where \( z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \) and \( y_2 = \frac{(n-1)s^2}{\sigma^2} \) ?

What this means is, when \( \nu_1 = 1 \), \( f = t^2 \)!

\[ t^2 = f = \left[ \frac{\left( \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right)^2}{1} \right] \sqrt{n} \left[ \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\sigma} \right)^2 \right] / (n-1) \]
Bivariate Change of Variable - normal, $\chi^2$, t, F

Recap: $u_1$ and $u_2 \sim \text{uniform}(0,1)$ and independent

$$z_1 = \sqrt{-2 \ln(u_1) \cos(2\pi u_2)} \quad z_2 = \sqrt{-2 \ln(u_1) \sin(2\pi u_2)}$$

$z_1 \sim N(0,1), z_2 \sim N(0,1), z_1$ and $z_2$ are independent

$$x_i = \sigma z_i + \mu \sim N(\mu, \sigma^2), \quad \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right), \quad z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$$

$$y_1 = \left(\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}\right)^2 \sim \chi^2(1), \quad y_2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1) \quad y_1 \text{ and } y \text{ are independent}$$

$$t = \frac{z}{\sqrt{y_2/(n-1)}} \sim t(n-1), \quad f = \frac{y_1 / 1}{y_2 / (n-1)} \sim F(1, n-1).$$
Homework 10:

1) Show analytically that if \( y_1 \sim \chi^2(\nu_1) \), \( y_2 \sim \chi^2(\nu_2) \), indep. then
\[ \nu_1 = y_1 + y_2 \sim \chi^2(\nu_1 + \nu_2) . \]

2) Generate \( 10^6 \) \( y_1 \sim \chi^2(5) \) and \( 10^6 \) \( y_2 \sim \chi^2(7) \) random variates.
   a) Make a histogram of the \( y_1 \)’s. Compute mean and variance.
   b) Make a histogram of the \( y_2 \)’s. Compute mean and variance.
   c) Add \( y_1 \) to \( y_2 \) to obtain \( y=y_1+y_2 \) random variates.
   d) Make a histogram of the \( y \)’s. Compute mean and variance.
   e) Comments?
Homework 10:

3) Generate 10*10^6 independent N(μ=5, σ^2=4) random variates.
   a) Compute the sample mean and variance for each group of 10.
   b) Make a histogram of the 10^6 z’s.
   c) Compute mean and variance of z’s.
   d) Make a histogram of the 10^6 y’s.
   e) Compute mean and variance of y’s.
   f) Compute the correlation between z’s and y’s.
   g) Form 10^6 t’s. Histogram, mean variance.
   h) Square each of the 10^6 t’s to get f’s. Histogram, mean variance.
   i) Comments.

\[ z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \]

\[ y = \frac{(n - 1)s^2}{\sigma^2} \]

\[ t = \frac{z}{\sqrt{y/(n-1)}} \]