Math Review

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Outline

• Differentiation
  Definition
  Analytic Approach
  Numerical Approach

• Integration
  Definition
  Analytic Approach
  Numerical Approach

• Summary
Differentiation - Definition

A **function** \( f \) is a rule that assigns to each element \( x \) in a set \( A \) exactly one element, called \( f(x) \), in a set \( B \).

The set \( A \) is the domain of \( f \) and the set \( B \) is the range of \( f \).
Differentiation - Definition

$f(x)$

$x$

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Differentiation - Definition

Slope of Line: $h = \Delta x$

$$m_l = \frac{f(a+h) - f(a)}{h}$$
Differentiation - Definition

Slope of Line: \( h = \Delta x \)

\[
m_l = \frac{f(a + h) - f(a)}{h}
\]
Differentiation - Definition

Slope of Line: \( h = \Delta x \)

\[ m_l = \frac{f(a + h) - f(a)}{h} \]
Differentiation - Definition

Slope of Line: \( h = \Delta x \)

\[ m_l = \frac{f(a + h) - f(a)}{h} \]

Slope of Tangent Line:

\[ m_t = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]
Differentiation - Analytic Approach

\[ m_t = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \frac{df(x)}{dx} = f'(x) \]

\[ f(x) = -(x - 1)^2 \]

\[ f'(x) = -2x + 2 \]
Differentiation - Analytic Approach

```matlab
% analytical derivative

f = '-(x-1)^2';
fprime = '-2*x+2';

figure(1)
fplot(f, [-1 3], 'b')
figure(2)
fplot(fprime, [-1 3], 'b')
```

\[ f(x) = -(x - 1)^2 \]

\[ f'(x) = -2x + 2 \]
Differentiation - Analytic Approach

A function $f(x)$ is differentiable at $x=a$ if there exists only one unique tangent line to the graph of $f(x)$ at $x=a$. 

Differentiable

Not Differentiable
Differentiation - Analytic Approach

\[
\frac{d}{dx} c = 0
\]

\[
\frac{d}{dx} x^n = nx^{n-1}
\]

\[
\frac{d}{dx} \sin(x) = \cos(x)
\]

\[
\frac{d}{dx} \cos(x) = -\sin(x)
\]

\[
\frac{d}{dx} e^x = e^x
\]
Differentiation - Analytic Approach

Let $f'(x)$ and $g'(x)$ exist.

Linearity Rule: \[
\frac{d}{dx} \left[ c_1 f(x) + c_2 g(x) \right] = c_1 f'(x) + c_2 g'(x)
\]

Product Rule: \[
\frac{d}{dx} \left[ c f(x) g(x) \right] = c \left[ f'(x) g(x) + f(x) g'(x) \right]
\]

Quotient Rule: \[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = c \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}
\]

Chain Rule: \[
\frac{d}{dx} \left[ c f(g(x)) \right] = c f'(g(x)) g'(x)
\]

$f'(g(x))$ must exist
Differentiation - Analytic Approach

Examples:

Linearity Rule:  \[ \frac{d}{dx} \left[ c_1 x + c_2 x^2 \right] = c_1 1 + c_2 2x \]

Product Rule: \[ \frac{d}{dx} \left[ cx \sin(x) \right] = c \left[ 1 \sin(x) + x \cos(x) \right] \]

Quotient Rule: \[ \frac{d}{dx} \left[ c \frac{\sin(x)}{\cos(x)} \right] = c \frac{\cos(x) \cos(x) - \sin(x) [-\sin(x)]}{[\cos(x)]^2} = c \sec^2(x) \]

Chain Rule: \[ \frac{d}{dx} \left[ c (x^2 + 1)^{1/2} \right] = c \frac{1}{2} (x^2 + 1)^{-1/2} 2x \]
Differentiation - Numerical Approach

Slope of Line: $h$ small

\[ \frac{d}{dx} f(x) \bigg|_{x=a} \approx \frac{f(a + h) - f(a)}{h} \]

$f(x)$

\[ \frac{d}{dx} f(x) \bigg|_{x=a} \approx \frac{f(a + h) - f(a)}{h} \]
Differentiation - Numerical Approach

Slope of Line: $h$ small

$$\left. \frac{d}{dx} f(x) \right|_{x=a+h} \approx \frac{f(a+2h) - f(a+h)}{h}$$
Differentiation - Numerical Approach

Slope of Line: \( h \) small

\[
\frac{d}{dx} f(x) \bigg|_{x=a+2h} \approx \frac{f(a+3h) - f(a+2h)}{h}
\]
Differentiation - Numerical Approach

\[ f(x) = -(x - 1)^2 \]

\[ f'(x) = -2x + 2 \]

\[ \text{polyder([-1 2 -1])} \]
\[ \text{ans} = -2 \quad 2 \]

If \( h=0.01 \), then lines look same!
Differentiation - Numerical Approach

```matlab
1 % numerical derivative
2 3 f = -(x-1)^2
4 fprime = -2*x+2
5 6 xpts = (-1:.1:3)'
7 fpts = -(xpts-1).^2
8 9 figure(1)
10 fplot(f, [-1 3], 'b')
11 hold on
12 plot(xpts, fpts, 'r')
13 14 numder = zeros(length(xpts)-1, 1);
15 for count = 1:length(xpts)-1
16     numder(count, 1) = ...
17         (fpts(count+1, 1) - fpts(count, 1)) / (xpts(count+1, 1) - xpts(count, 1));
18 end
19 20 figure(2)
21 fplot(fprime, [-1 3], 'b')
22 hold on
23 plot(xpts(1:length(xpts)-1, 1), numder, 'r')
```
Maximization - Analytic Approach

Given a function $f(x)$, if it has a maxima in the interval $[a,b]$ at $x_0$, then the slope of $f(x)$ is zero at $x_0$. This means that $f'(x)=0$ at $x=x_0$.

$x_0$ is a global maxima if it is unique.
Maximization - Numerical Approach

Define values of $x$ want to find max of $f(x)$ for, $a$ to $b$. Assume complicated function we can’t analytically differentiate.
Maximization - Numerical Approach

Define values of $x$ want to find max of $f(x)$ for, $a$ to $b$.

Set $\Delta x$ to evaluate $f(x)$ at increments.

$x_0$ is value that makes $f(x)$ largest.

Assume complicated function we can’t analytically differentiate.

better results for $\Delta x$ smaller
Integration - Area Under Curve

\[ f(x) \]
Integration - Area Under Curve

Area under curve between $a$ and $b$. 

$f(x)$

$a$ $b$ $x$
Integration - Area Under Curve

Divide into intervals: $\Delta x$ small

$\Delta x = \frac{(b-a)}{n}$  \hspace{1cm} $\Delta x = x_i - x_{i-1}$

\[ a = x_0 \quad x_1 \quad x_2 \quad \ldots \quad b = x_n \]

$n = 5$
Integration - Area Under Curve

Find midpoints: $\Delta x$ small

$$\Delta x = (b-a)/n \quad \Delta x = x_i - x_{i-1}$$

$$x_i^* = a + \Delta x / 2 + (i-1)\Delta x$$

$$i = 1, \ldots, n$$
Integration - Area Under Curve

Area by rectangles: $\Delta x$ small

$\Delta x = (b-a)/n \quad \Delta x = x_i - x_{i-1}$

$x_i^* = a + \Delta x / 2 + (i-1)\Delta x$

$A = \sum_{i=1}^{n} f(x_i^*) \Delta x \quad i = 1, \ldots, n$

$n = 5$
Integration - Area Under Curve

\[ A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x^*_i)\Delta x \]

\[ = \int_{x=a}^{b} f(x)dx \]

\[ \Delta x = (b - a) / n \]

\[ x^*_i = a + \Delta x / 2 + (i - 1)\Delta x \]
Integration - Analytic Approach

```matlab
% analytical integral
f='4-(x-1)^2'
fint='4*x-(x-1)^3/3'
figure(1)
plot(f,[-1 3],'b')
figure(2)
plot(fint,[-1 3],'b')
```

\[
f(x) = 4 - (x-1)^2
\]

\[
\int f(x)dx = 4x - (x-1)^3 / 3
\]
Integration - Analytic Approach

\[ \int c \, dx = cx + C \]

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \]

\[ \int \sin(x) \, dx = -\cos(x) + C \]

\[ \int \cos(x) \, dx = \sin(x) + C \]

\[ \int e^x \, dx = e^x + C \]
Integration - Analytic Approach

Linearity: \[ \int c_1 f(x) + c_2 g(x) \, dx = c_1 \int f(x) \, dx + c_2 \int g(x) \, dx \]

By Parts: \[ \int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx \]
Assuming \( f'(x) \) and \( g'(x) \) exist.

Other methods:
Trigonometric Substitution
Partial Fractions
Integration - Numerical Approach

\[
\int_{1}^{3} (4x - 1) \, dx = -10.6667
\]

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analytic

\[ f(x) \approx \sum_{i=1}^{n} f(x_i) \cdot \Delta x \]

\[ n = 10, \Delta x = 0.4 \]

\[ n = 50, \Delta x = 0.08 \]
Integration - Numerical Approach

$$
\int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} f(x_i) \Delta x
$$

Analytic:

$$
\int_{a}^{b} f(x) \, dx = \text{Value}
$$

Numerical:

$$
\text{numint} = \sum_{i=1}^{n} f(x_i) \Delta x
$$

For $n=10, \Delta x=0.4$, 
numint = 10.7200

For $n=50, \Delta x=0.08$, 
numint = 10.6688

- Analytic
- Numerical
Integration - Area Under Curve

Trapezoidal Rule:
Area by trapezoids:

\[ \Delta x = \frac{(b-a)}{n} \]
Integration - Area Under Curve

Simpson’s Rule:
Area by parabola arcs:
\[ \Delta x = \frac{(b-a)}{n} \]
Summary

• Differentiation
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• Integration
  Definition
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  Numerical Approach
Homework 1:

\[ f(x) = x^3, x \in \mathbb{R} \]

1) Differentiate analytically and evaluate at \( a=-1 \) and \( b=1 \).

2) Differentiate by hand numerically with \( \Delta x=0.5 \).

3) Write a Matlab program for 2) then let \( \Delta x=1/100 \).

4) Integrate analytically and evaluate from \( a=-1 \) to \( b=1 \).

5) Integrate by hand numerically using \( n=4 \).

\[ \Delta x = 0.5 \quad \left( x_1^*, x_2^*, x_3^*, x_4^* \right) = (-0.75, -0.25, 0.25, 0.75) \]

6) Write a Matlab program for 5) then let \( n=100 \).