Bivariate Transformation of Variables

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Outline

• Bivariate Continuous Distributions
  Joint PDF, Conditional PDF, Marginal PDF

• Bivariate Transformation of Variables
  Sum of RVs
Bivariate Continuous Distributions

A bivariate (2D) PDF \( f(x_1, x_2 \mid \theta) \) of two continuous random variables \((x_1, x_2)\) depending upon parameters \(\theta\) satisfies

1) \( 0 \leq f(x_1, x_2 \mid \theta) \leq 1, \ \forall (x_1, x_2) \)

2) \( \int_{x_1} \int_{x_2} f(x_1, x_2 \mid \theta) dx_1 dx_2 = 1 \).
Bivariate Continuous Distributions

Marginal Distributions

\[ f(x_1 \mid \theta) = \int_{x_2} f(x_1, x_2 \mid \theta) \, dx_2 \]

\[ f(x_2 \mid \theta) = \int_{x_1} f(x_1, x_2 \mid \theta) \, dx_1 \]

Marginal Expectations

\[ E(g(X_1) \mid \theta) = \int_{x_1} g(x_1) f(x_1 \mid \theta) \, dx_1 \]

\[ E(g(X_2) \mid \theta) = \int_{x_2} g(x_2) f(x_2 \mid \theta) \, dx_2 \]

Provided integral exists

Marginals Normal
Bivariate Continuous Distributions

Marginal Means

\[ E(X_1 \mid \theta) = \int_{x_1} x_1 f(x_1 \mid \theta) \, dx_1 = \mu_1 \]

\[ E(X_2 \mid \theta) = \int_{x_2} x_2 f(x_2 \mid \theta) \, dx_2 = \mu_2 \]

Marginal Variances

\[ E([X_1 - E(X_1 \mid \theta)]^2 \mid \theta) = \int_{x_1} [x_1 - E(X_1 \mid \theta)]^2 f(x_1 \mid \theta) \, dx_1 = \sigma_1^2 \]

\[ E([X_2 - E(X_2 \mid \theta)]^2 \mid \theta) = \int_{x_2} [x_2 - E(X_2 \mid \theta)]^2 f(x_2 \mid \theta) \, dx_2 = \sigma_2^2 \]
Bivariate Continuous Distributions

Conditional Distributions

\[
f(x_1 \mid x_2, \theta) = \frac{f(x_1, x_2 \mid \theta)}{f(x_2 \mid \theta)}
\]

\[
f(x_2 \mid x_1, \theta) = \frac{f(x_1, x_2 \mid \theta)}{f(x_1 \mid \theta)}
\]

Conditional Expectations

\[
E(g(X_1) \mid X_2, \theta) = \int_{x_1} g(x_1) f(x_1 \mid x_2, \theta) dx_1
\]

\[
E(g(X_2) \mid X_1, \theta) = \int_{x_2} g(x_2) f(x_2 \mid x_1, \theta) dx_2
\]

Conditionals Normal

\[
f(x_1 \mid x_2, \theta)
\]

has this shape

Provided integral exists
Bivariate Continuous Distributions

**Conditional Means**

\[ E(X_1 \mid X_2, \theta) = \int_{x_1} x_1 f(x_1 \mid x_2, \theta) dx_1 \]

\[ E(X_2 \mid X_1, \theta) = \int_{x_2} x_2 f(x_2 \mid x_1, \theta) dx_2 \]

**Conditional Variances**

\[ E([X_1 - E(X_1 \mid X_2, \theta)]^2 \mid X_2, \theta) = \int_{x_1} [x_1 - E(X_1 \mid X_2, \theta)]^2 f(x_1 \mid x_2, \theta) dx_1 \]

\[ E([X_2 - E(X_2 \mid X_1, \theta)]^2 \mid X_2, \theta) = \int_{x_2} [x_2 - E(X_2 \mid X_1, \theta)]^2 f(x_2 \mid x_1, \theta) dx_2 \]
Bivariate Continuous Distributions

Marginal Distributions

\[ f(x_2 \mid \theta) = \int_{x_1} f(x_1, x_2 \mid \theta) \, dx_1 \]

\[ f(x_2 \mid \theta) = \frac{f(x_1, x_2 \mid \theta)}{f(x_1 \mid x_2, \theta)} \]
Bivariate Continuous Distributions

Covariance

\[ \text{cov}(X_1, X_2 \mid \theta) = \iint_{x_1 x_2} \left[ x_1 - E(X_1 \mid \theta) \right] \left[ x_2 - E(X_2 \mid \theta) \right] f(x_1, x_2 \mid \theta) dx_1 dx_2 \]

\[ = \sigma_{12} \]

Correlation

\[ \text{corr}(X_1, X_2 \mid \theta) = \frac{\text{cov}(X_1, X_2 \mid \theta)}{\sigma_1 \sigma_2} \]
Bivariate Continuous Distributions

Statistical Independence

Two random variables $x_1$ and $x_2$ are independent if and only if

$$f(x_1, x_2 \mid \theta) = f(x_1 \mid \theta_1)f(x_2 \mid \theta_2).$$

If two random variables $x_1$ and $x_2$ are uncorrelated,

$$\text{corr}(X_1, X_2 \mid \theta) = 0$$

then they are not necessarily independent.
Given two continuous random variables, \((x_1, x_2)\) with joint probability distribution function \(f_{X_1,X_2}(x_1, x_2 \mid \theta)\).

Let \(
\begin{pmatrix}
y_1(x_1, x_2) \\
y_2(x_1, x_2)
\end{pmatrix}
\) be a transformation from \((x_1, x_2)\) to \((y_1, y_2)\) with inverse transformation \(
\begin{pmatrix}
x_1(y_1, y_2) \\
x_2(y_1, y_2)
\end{pmatrix}
\)
Bivariate Change of Variable

Then, the joint probability distribution function \( f_{Y_1, Y_2}(y_1, y_2 \mid \theta) \) of \((y_1, y_2)\) can be found via

\[
f_{Y_1, Y_2}(y_1, y_2 \mid \theta) = f_{X_1, X_2}(x_1(y_1, y_2), x_2(y_1, y_2) \mid \theta) \times | J(x_1, x_2 \rightarrow y_1, y_2) |
\]

where \( J(x_1, x_2 \rightarrow y_1, y_2) = \begin{vmatrix} dx_1(y_1, y_2) & dx_1(y_1, y_2) \\ dy_1 & dy_2 \\ dx_2(y_1, y_2) & dx_2(y_1, y_2) \\ dy_1 & dy_2 \end{vmatrix} \).

\[
| J(x_1, x_2 \rightarrow y_1, y_2) | = 1/ | J(y_1, y_2 \rightarrow x_1, x_2) |
\]
Bivariate Change of Variable - Sum

Let $x_1$ have PDF $f_{x_1}(x_1 \mid \theta_1)$ and $x_2$ have PDF $f_{x_2}(x_2 \mid \theta_2)$, then, the PDF of $y_1 = x_1 + x_2$ can be found via the bivariate change of variable technique

$$f_{Y_1,Y_2}(y_1, y_2 \mid \theta) = f_{X_1,X_2}(x_1(y_1, y_2), x_2(y_1, y_2) \mid \theta) \times \left| J(x_1, x_2 \rightarrow y_1, y_2) \right|$$

with marginalization $f_{Y_1}(y_1 \mid \theta) = \int f_{Y_1,Y_2}(y_1, y_2 \mid \theta) dy_2$. 
The joint PDF of $\mathbf{(x_1, x_2)}$ is

$$f_{x_1, x_2}(x_1, x_2 \mid \theta) = f_{x_1}(x_1 \mid \theta_1) f_{x_2}(x_2 \mid \theta_2).$$

(need not be independent)

Let $y_1 = x_1 + x_2$ and $y_2 = x_2$, then $x_1 = y_1 - y_2$ and $x_2 = y_2$.

$$J(x_1, x_2 \rightarrow y_1, y_2) = \begin{vmatrix} dx_1(y_1, y_2) & dx_1(y_1, y_2) \\ dy_1 & dy_2 \\ dx_2(y_1, y_2) & dx_2(y_1, y_2) \\ dy_1 & dy_2 \end{vmatrix} = 1$$
Bivariate Change of Variable - Sum

Example: Normal
Let \( x_1 \sim \text{normal}(\mu_1, \sigma_1^2) \) and \( x_2 \sim \text{normal}(\mu_2, \sigma_2^2) \), \( x_1 \& x_2 \) independent. The joint PDF of \((x_1, x_2)\) is

\[
f(x_1, x_2 \mid \mu_1, \sigma_1^2, \mu_2, \sigma_2^2) = \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2} \cdot \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2}.
\]

With \( x_1 = y_1 - y_2 \) \( x_2 = y_2 \) \( J(x_1, x_2 \rightarrow y_1, y_2) = 1 \)

\[
f_{Y_1, Y_2}(y_1, y_2 \mid \theta) = \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{y_1 - y_2 - \mu_1}{\sigma_1} \right)^2} \cdot \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{y_2 - \mu_2}{\sigma_2} \right)^2} \times 1
\]
Bivariate Change of Variable - Sum

Rearranging leads to

\[ f_{Y_1,Y_2}(y_1, y_2 \mid \theta) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left( \frac{(y_1-y_2-\mu_1)^2 + (y_2-\mu_2)^2}{\sigma_1^2 + \sigma_2^2} \right)} \]

Complete square in exponent to get

\[ f_{Y_1,Y_2}(y_1, y_2 \mid \theta) = \frac{1}{\tau \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{(y_2-\delta)^2}{\tau^2} \right)} \frac{\tau}{\sigma_1\sigma_2 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \gamma - \tau^{-2} \delta^2 \right)} \]

\( \delta \) does not depend on \( y_2 \)

\[ \delta = \frac{\sigma_2^2 (y_1 - \mu_1) + \mu_1 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \]

\[ \tau^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \]

\[ \gamma = \frac{(y_1 - \mu_1)^2 \sigma_2^2 + \mu_2^2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2} \]
Bivariate Change of Variable - Sum

Marginalizing leads to

\[ f_{Y_1}(y_1 \mid \theta_1) = \int f_{Y_1, Y_2}(y_1, y_2 \mid \theta) \, dy_2 \]

algebra leads to

\[ f_{Y_1}(y_1 \mid \theta_1) = \frac{\tau}{\sigma_1 \sigma_2 \sqrt{2\pi}} e^{-\frac{1}{2}(\gamma - \tau \delta)^2} \int_{y_2} \frac{1}{\tau \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y_2 - \delta}{\tau})^2} \, dy_2 = 1 \]

\[ \tau^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \]

\[ \delta = \frac{\sigma_2^2 (y_1 - \mu_1) + \mu_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2} \]

\[ \gamma = \frac{(y_1 - \mu_1)^2 \sigma_2^2 + \mu_2^2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2} \]

\[ y_1 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \]
Bivariate Change of Variable - Sum

This change of variable technique can be repeated.

If $x_3 \sim \text{normal}(\mu_3, \sigma_3^2)$, then if we let $y_3 = x_3 + y_1$
(don’t forget $y_1 = x_1 + x_2$),

then we can find that $y_3 \sim N(\mu_1 + \mu_2 + \mu_3, \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$

we can repeat the procedure to get $y_n \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$.

We can also find that $y = \frac{1}{n} \sum_{i=1}^{n} x_i \sim N\left(\frac{1}{n} \sum_{i=1}^{n} \mu_i, \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2\right)$. 
Homework 8:

1) Let $x_1 \sim \text{normal}(\mu_1, \sigma_1^2)$, $x_2 \sim \text{normal}(\mu_2, \sigma_2^2)$, $x_1$ and $x_2$ independent.
   a) Derive the distribution of $y = x_1 + x_2$.
   b) Derive the distribution of $y = x_1 - x_2$.
   c) Derive the distribution of $y = x_1 x_2$.
   d) Derive the distribution of $y = x_1 / x_2$.

In any of a)-d) you may need to constrain $\mu$'s and/or $\sigma^2$'s.
(But try not to.)
2) Let $x_1 \sim \text{normal}(5,4)$, $x_2 \sim \text{normal}(10,1)$, $x_1$ and $x_2$ independent. Generate $10^6 x_1$’s and $10^6 x_2$’s.

- a) Let $10^6$ new random variates be $y = x_1 + x_2$.
- b) Let $10^6$ new random variates be $y = x_1 - x_2$.
- c) Let $10^6$ new random variates be $y = x_1 x_2$.
- d) Let $10^6$ new random variates be $y = x_1 / x_2$.
- e) For a)-d) generate histogram, means, variances 50th and 99th percentiles.

In any of a)-d) reconsider with any constraints you put on $\mu$’s and/or $\sigma^2$’s.