Class 9

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Agenda:
Recap Chapter 5.1 - 5.3
Lecture 5.3 continued
Review Chapters 3 and 4
Recap Chapter 5.1 - 5.3
Random Variables: … assumes a unique … value for each of the outcomes in the sample space … .

Probability Function: A rule \( P(x) \) that assigns probabilities to the values of the random variable \( x \).

Example:
Let \( x = \# \) of heads when we flip a coin twice.
\( x = \{0,1,2\} \)

\[
P(x) = \frac{2!}{x!(2-x)!} \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{2-x}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>
5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

An experiment with two outcomes does not have to be $H$ and $T$.

More general than $H$ and $T$, call one *Success* and other *Failure*.

Generally call the one we’re interested in the *Success*.

Now that we’ve established that the formula works.

We can determine the theoretical mean number of heads, $\mu$, and the theoretical variance for the number of heads, $\sigma^2$. 
We can determine the theoretical mean number of heads, $\mu$, and the theoretical variance for the number of heads, $\sigma^2$.

So if we flip a coin twice and record the number of heads, repeat this an infinite number of times, then compute the sample mean $\bar{x}$ and variance $s^2$, we would get theoretical mean $\mu$ and variance $\sigma^2$.

$$\bar{x} \to \mu \quad \text{as number of repeats} \to \infty.$$
5: Probability Distributions (Discrete Variables)

5.3 Mean and Variance of a Discrete Random Variable

\[ \mu = \sum_{i=1}^{n} [x_i P(x_i)] = x_1 P(x_1) + x_2 P(x_2) + \ldots + x_n P(x_n) \]

For the # of \( H \) when we flip a coin twice discrete distribution:

\[ \mu = (x_1) \cdot P(x_1) + (x_2) \cdot P(x_2) + (x_3) \cdot P(x_3) \]

\[ \mu = (0) \cdot P(0) + (1) \cdot P(1) + (2) \cdot P(2) \]

\[ \mu = (0) \cdot (1/4) + (1) \cdot (1/2) + (2) \cdot (1/4) \]

\[ \mu = 0 + 1/2 + 1/2 \]

\[ \mu = 1 \]
5: Probability Distributions (Discrete Variables)

5.3 Mean and Variance of a Discrete Random Variable

\[ \sigma^2 = \sum_{i=1}^{n} [(x_i - \mu)^2 P(x_i)] = (x_1 - \mu)^2 P(x_1) + (x_2 - \mu)^2 P(x_2) + \ldots + (x_n - \mu)^2 P(x_n) \]

For the # of \( H \) when we flip a coin twice discrete distribution:

\[ \sigma^2 = (x_1-\mu)^2 \cdot P(x_1) + (x_2-\mu)^2 \cdot P(x_2) + (x_3-\mu)^2 \cdot P(x_3) \]

\[ \mu = 1 \]

\[ \begin{array}{c|c}
 x & P(x) \\
 0 & \frac{1}{4} \\
 1 & \frac{1}{2} \\
 2 & \frac{1}{4} \\
 \end{array} \]

\[ \sigma^2 = (0-1)^2 \cdot P(0) + (1-1)^2 \cdot P(1) + (2-1)^2 \cdot P(2) \]

\[ \sigma^2 = (-1)^2 \cdot (1/4) + (0)^2 \cdot (1/2) + (1)^2 \cdot (1/4) \]

\[ \sigma^2 = 1/4 + 0 + 1/4 \]

\[ \sigma^2 = 1/2 \]
5: Probability Distributions (Discrete Variables)
5.3 Mean and Variance of a Discrete Random Variable

\[ \sigma^2 = \sum_{i=1}^{n} [x_i^2 P(x_i)] - \mu^2 = [x_1^2 P(x_1) + x_2^2 P(x_2) + \ldots + x_n P(x_n)] - \mu^2 \]

Alternate Formula

For the # of H when we flip a coin twice discrete distribution:

\[ \sigma^2 = x_1^2 \cdot P(x_1) + x_2^2 \cdot P(x_2) + x_3^2 \cdot P(x_3) - \mu^2 \]

\[ \sigma^2 = [0^2 \cdot P(0) + 1^2 \cdot P(1) + 2^2 \cdot P(2)] - 1^2 \]

\[ \sigma^2 = [0 \cdot (1/4) + 1^2 \cdot (1/2) + 2^2 \cdot (1/4)] - 1 \]

\[ \sigma^2 = 0 + 1/2 + 4/4 - 1 \]

\[ \sigma^2 = 1/2 \]
5: Probability Distributions (Discrete Variables)
5.3 Mean and Variance of a Discrete Random Variable

\[ \sigma^2 = \sum_{i=1}^{n} [(x_i - \mu)^2 P(x_i)] \quad \text{or} \quad \sigma^2 = \sum_{i=1}^{n} [x_i^2 P(x_i)] - \mu^2 \]

For the \# of \( H \) when we flip a coin twice discrete distribution:

\[ \sigma = \sqrt{\sigma^2} \]

\[ \sigma^2 = 1/2 \]

\[ \sigma = 1/\sqrt{2} \]

\[ \sigma \approx 0.7071 \]
5: Probability Distributions (Discrete Variables)
5.3 The Binomial Probability Distribution

An experiment with only two outcomes is called a Binomial exp. Call one outcome *Success* and the other *Failure*. Each performance of expt. is called a trial and are independent.

\[ P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \]

\[ x = 0, \ldots, n \]

\[ \binom{n}{x} = \frac{n!}{x!(n-x)!} \]

\[ n = \text{number of trials or times we repeat the experiment.} \]
\[ x = \text{the number of successes out of } n \text{ trials.} \]
\[ p = \text{the probability of success on an individual trial.} \]

Bi means two like bicycle
5: Probability Distributions (Discrete Variables)
5.5 The Binomial Probability Distribution

Flip coin ten times. $x = \# \text{ of Heads} \quad n(x) = \text{ways to get } x \text{ Heads}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>$n(x)$</td>
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<td>120</td>
<td>210</td>
<td>252</td>
<td>210</td>
<td>120</td>
<td>45</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>$P(x)$</td>
<td>$\frac{1}{1024}$</td>
<td>$\frac{10}{1024}$</td>
<td>$\frac{45}{1024}$</td>
<td>$\frac{120}{1024}$</td>
<td>$\frac{210}{1024}$</td>
<td>$\frac{252}{1024}$</td>
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<td>$\frac{120}{1024}$</td>
<td>$\frac{45}{1024}$</td>
<td>$\frac{10}{1024}$</td>
<td>$\frac{1}{1024}$</td>
</tr>
</tbody>
</table>

$n=10$
$x=0,\ldots,10$
$p=1/2$

$$n(x) = \frac{n!}{x!(n-x)!}$$

$$p^x (1-p)^{n-x} = 1/1024$$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Note:
1. $0 \leq P(x) \leq 1$
2. $\Sigma P(x)=1$
5: Probability Distributions (Discrete Variables)

5.5 The Binomial Probability Distribution

Flip coin ten times.

\[ P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \]

\( n=10, \ x=7, \ p=0.5 \)

\[ P(7) = \frac{10!}{7!(10-7)!} \left( \frac{1}{2} \right)^7 \left( 1 - \frac{1}{2} \right)^{10-7} \]

\[ P(7) = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!3!} \left( \frac{1}{2} \right)^7 \left( \frac{1}{2} \right)^{10-7} \]

\[ P(7) = \frac{10 \cdot 3 \cdot 2 \cdot 4}{3 \cdot 2} \left( \frac{1}{2} \right)^{10} \]

\[ P(7) = \frac{120}{1024} \]

\[ x = \# \ of \ Heads \]

\[ n(x) = \text{ways to get } x \text{ Heads} \]
Lecture Chapter 5.3 continued
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: \( n = 10, \ p = 0.5 \)

\[
P(x) = \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x}
\]

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These Binomial probabilities can also be found in the back of the book. Table 2 in Appendix B

Please turn to page 713
### Table 2: Binomial Probabilities

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<td>0.125</td>
<td>0.064</td>
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<td>0.961</td>
</tr>
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</table>

\[
\binom{n}{x} = \frac{n!}{x!(n-x)!}
\]

Figure from Johnson & Kuby, 2012.
5: Probability Distributions (Discrete Variables)
5.3 The Binomial Probability Distribution

\[ P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \]

<table>
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<tr>
<th>( n \times x )</th>
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</table>

\( n=2, p=1/2 \)

\[ P(0) = \frac{2!}{0!(2-0)!} (1/2)^0 (1-1/2)^{2-0} \]
\[ P(1) = \frac{2!}{1!(2-1)!} (1/2)^1 (1-1/2)^{2-1} \]
\[ P(2) = \frac{2!}{2!(2-2)!} (1/2)^2 (1-1/2)^{2-2} \]

Figure from Johnson & Kuby, 2012.
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

The binomial probability distribution is given by

\[
P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}
\]

where \( n \) is the number of trials, \( p \) is the probability of success on each trial, and \( x \) is the number of successes.

**Table 2** Binomial Probabilities \( \binom{n}{x} \cdot p^x \cdot q^{n-x} \) (continued)

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<td>.121</td>
<td>.268</td>
<td>.387</td>
</tr>
<tr>
<td>10</td>
<td>0+</td>
<td>0+</td>
<td>0+</td>
<td>0+</td>
<td>0+</td>
<td>0+</td>
<td>0+</td>
<td>.001</td>
<td>.006</td>
<td>.028</td>
<td>.107</td>
</tr>
</tbody>
</table>

**Page 713**

Figure from Johnson & Kuby, 2012.
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

TABLE 2

Binomial Probabilities $\binom{n}{x} \cdot p^x \cdot q^{n-x}$

<table>
<thead>
<tr>
<th>n</th>
<th>p = 2/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.020</td>
</tr>
<tr>
<td>2</td>
<td>0.002</td>
</tr>
</tbody>
</table>

$P(x) = \frac{n!}{x!(n-x)!} \cdot p^x \cdot (1-p)^{n-x}$

<table>
<thead>
<tr>
<th>x</th>
<th>P(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/9</td>
</tr>
<tr>
<td>1</td>
<td>4/9</td>
</tr>
<tr>
<td>2</td>
<td>4/9</td>
</tr>
</tbody>
</table>

$P(x)$ for $n=2$, $p=2/3$

Not every $p$ is on the table.

Figure from Johnson & Kuby, 2012.
Example: \( n=10, \ p=1/2 \)
What is the probability of getting 4, 5, or 6 heads?

\[
P(4 \leq x \leq 6) = P(4) + P(5) + P(6)
\]

\[
P(4 \leq x \leq 6) = P(4) + P(5) + P(6)
\]

\[
P(4 \leq x \leq 6) = P(4) + P(5) + P(6)
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(x) )</td>
<td>( \frac{1}{1024} )</td>
<td>( \frac{10}{1024} )</td>
<td>( \frac{45}{1024} )</td>
<td>( \frac{120}{1024} )</td>
<td>( \frac{210}{1024} )</td>
<td>( \frac{252}{1024} )</td>
<td>( \frac{210}{1024} )</td>
<td>( \frac{120}{1024} )</td>
<td>( \frac{45}{1024} )</td>
<td>( \frac{10}{1024} )</td>
<td>( \frac{1}{1024} )</td>
</tr>
</tbody>
</table>
5: Probability Distributions (Discrete Variables)
5.3 The Binomial Probability Distribution

Example: \( n = 10, \ p = 1/2 \)
What is the probability of getting 7 or fewer heads?

\[
P(x \leq 7) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7)
\]

\[
\begin{array}{cccccccccc}
x & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
P(x) & \frac{1}{1024} & \frac{10}{1024} & \frac{45}{1024} & \frac{120}{1024} & \frac{210}{1024} & \frac{252}{1024} & \frac{210}{1024} & \frac{120}{1024} & \frac{45}{1024} & \frac{10}{1024} & \frac{1}{1024} \\
\end{array}
\]
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: \( n=10, \ p=1/2 \)

What is the probability of getting 7 or fewer heads?

\[
P(x \leq 7) = 1 - P(x \geq 8)
\]

\[
P(x \leq 7) = \frac{1}{1024} \quad \frac{10}{1024} \quad \frac{45}{1024} \quad \frac{120}{1024} \quad \frac{210}{1024} \quad \frac{252}{1024} \quad \frac{210}{1024} \quad \frac{120}{1024} \quad \frac{45}{1024} \quad \frac{10}{1024} \quad \frac{1}{1024}
\]
5: Probability Distributions (Discrete Variables)
5.3 Mean and Standard Deviation of the Binomial Distribution

An experiment with two outcomes does not have to be $H$ and $T$.

More general than $H$ and $T$, call one *Success* and other *Failure*.

Generally call the one we’re interested in the *Success*.

Now that we’ve established that the formula works.

We can determine the theoretical mean number of heads, $\mu$, and the theoretical variance for the number of heads, $\sigma^2$. 
5: Probability Distributions (Discrete Variables)
5.3 Mean and Standard Deviation of the Binomial Distribution

The general formula for determining the theoretical mean $\mu$ of a discrete distribution is:

$$\mu = \sum_{i=1}^{n} [x_i P(x_i)]$$

And upon insertion of the Binomial distribution

$$\mu = \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x}$$

$$= np$$

(5.7)
5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

The general formula for determining the theoretical variance $\sigma^2$ of a discrete distribution is:

$$\sigma^2 = \sum_x (x - \mu)^2 P(x)$$

And upon insertion of the Binomial distribution

$$\sigma^2 = \sum_{x=0}^{n} (x - \mu)^2 \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x}$$

$$= np(1 - p)$$

$$\rightarrow \sigma = \sqrt{np(1 - p)} \tag{5.8}$$
5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

Example:
Before, using \( \mu = \sum_{x=0}^{n} [xP(x)] \) , we found \( \mu = 1 \).

Now using \( \mu = np \) , we get \( \mu = (2) \cdot (1/2) = 1 \).

Before, using \( \sigma^2 = \sum_{x=0}^{n} [(x - \mu)^2 P(x)] \) , we found \( \sigma^2 = 1/2 \).

Now using \( \sigma^2 = np(1 - p) \),
we get \( \sigma^2 = (2) \cdot (1/2) \cdot (1/2) = 1/2 \).

<table>
<thead>
<tr>
<th>x</th>
<th>P(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>
Example: $n=10$, $p=1/2$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>$\frac{1}{1024}$</td>
<td>$\frac{10}{1024}$</td>
<td>$\frac{45}{1024}$</td>
<td>$\frac{120}{1024}$</td>
<td>$\frac{210}{1024}$</td>
<td>$\frac{252}{1024}$</td>
<td>$\frac{210}{1024}$</td>
<td>$\frac{120}{1024}$</td>
<td>$\frac{45}{1024}$</td>
<td>$\frac{10}{1024}$</td>
<td>$\frac{1}{1024}$</td>
</tr>
</tbody>
</table>

What are $\mu$ and $\sigma^2$?

$\mu = np = $

$\sigma^2 = np(1 - p) =$
Review Chapters 3 and 4 (Exam 2 Chapters)

Just the highlights!
3: Descriptive Analysis and Bivariate Data
3.1 Bivariate Data: two qualitative

Cross-tabulation tables or contingency tables

Example:
Construct a $2 \times 3$ table.

<table>
<thead>
<tr>
<th>Name</th>
<th>Gender</th>
<th>Major</th>
<th>Name</th>
<th>Gender</th>
<th>Major</th>
<th>Name</th>
<th>Gender</th>
<th>Major</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adams</td>
<td>M</td>
<td>LA</td>
<td>Feeney</td>
<td>M</td>
<td>T</td>
<td>McGowan</td>
<td>M</td>
<td>BA</td>
</tr>
<tr>
<td>Argento</td>
<td>F</td>
<td>BA</td>
<td>Flanigan</td>
<td>M</td>
<td>LA</td>
<td>Mowers</td>
<td>F</td>
<td>BA</td>
</tr>
<tr>
<td>Baker</td>
<td>M</td>
<td>LA</td>
<td>Hodge</td>
<td>F</td>
<td>LA</td>
<td>Ornt</td>
<td>M</td>
<td>T</td>
</tr>
<tr>
<td>Bennett</td>
<td>F</td>
<td>LA</td>
<td>Holmes</td>
<td>M</td>
<td>T</td>
<td>Palmer</td>
<td>F</td>
<td>LA</td>
</tr>
<tr>
<td>Brand</td>
<td>M</td>
<td>T</td>
<td>Jopson</td>
<td>F</td>
<td>T</td>
<td>Pullen</td>
<td>M</td>
<td>T</td>
</tr>
<tr>
<td>Brock</td>
<td>M</td>
<td>BA</td>
<td>Kee</td>
<td>M</td>
<td>BA</td>
<td>Rattan</td>
<td>M</td>
<td>BA</td>
</tr>
<tr>
<td>Chun</td>
<td>F</td>
<td>LA</td>
<td>Kleeberg</td>
<td>M</td>
<td>LA</td>
<td>Sherman</td>
<td>F</td>
<td>LA</td>
</tr>
<tr>
<td>Crain</td>
<td>M</td>
<td>T</td>
<td>Light</td>
<td>M</td>
<td>BA</td>
<td>Small</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>Cross</td>
<td>F</td>
<td>BA</td>
<td>Linton</td>
<td>F</td>
<td>LA</td>
<td>Tate</td>
<td>M</td>
<td>BA</td>
</tr>
<tr>
<td>Ellis</td>
<td>F</td>
<td>BA</td>
<td>Lopez</td>
<td>M</td>
<td>T</td>
<td>Yamamoto</td>
<td>M</td>
<td>LA</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Gender</th>
<th>LA</th>
<th>BA</th>
<th>T</th>
<th>Row Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>18</td>
</tr>
<tr>
<td>F</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>Col. Total</td>
<td>11</td>
<td>10</td>
<td>9</td>
<td>30</td>
</tr>
</tbody>
</table>

M = male
F = female
LA = liberal arts
BA = business admin
T = technology

Figures from Johnson & Kuby, 2012.
3: Descriptive Analysis and Bivariate Data

3.1 Bivariate Data: one qualitative and one quantitative

Example:

<table>
<thead>
<tr>
<th>Design A (n = 6)</th>
<th>Design B (n = 6)</th>
<th>Design C (n = 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>37, 36, 38, 34, 40</td>
<td>33, 35, 38, 34, 40</td>
<td>40, 39, 40, 41, 43</td>
</tr>
</tbody>
</table>

Figure from Johnson & Kuby, 2012.

Vertical box-and-whiskers
3: Descriptive Analysis and Bivariate Data
3.1 Bivariate Data: two quantitative, Scatter Diagram

**Example:** Push-ups

<table>
<thead>
<tr>
<th>Student</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Push-ups, ( x )</td>
<td>27</td>
<td>22</td>
<td>15</td>
<td>35</td>
<td>30</td>
<td>52</td>
<td>35</td>
<td>55</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>Sit-ups, ( y )</td>
<td>30</td>
<td>26</td>
<td>25</td>
<td>42</td>
<td>38</td>
<td>40</td>
<td>32</td>
<td>54</td>
<td>50</td>
<td>43</td>
</tr>
</tbody>
</table>

**Input variable:** independent variable, \( x \).
**Output variable:** dependent variable, \( y \).

**Scatter Diagram:** A plot of all the ordered pairs of bivariate data on a coordinate axis system.

\((x, y)\) ordered pairs.

Figures from Johnson & Kuby, 2012.

Rowe, D.B.
3: Descriptive Analysis and Bivariate Data
3.2 Linear Correlation

Linear Correlation, $r$, is a measure of the strength of a linear relationship between two variables $x$ and $y$. 

$$-1 \leq r \leq 1$$

Will discuss its computation in a minute.

$r \approx 0$  $r \approx 0.5$  $r \approx 0.8$  $r \approx -0.5$  $r \approx -0.8$

Figure from Johnson & Kuby, 2012.
3: Descriptive Analysis and Bivariate Data

3.2 Linear Correlation

Example:

<table>
<thead>
<tr>
<th>Student</th>
<th>Push-ups, x</th>
<th>x²</th>
<th>Sit-ups, y</th>
<th>y²</th>
<th>xy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>27</td>
<td>729</td>
<td>30</td>
<td>900</td>
<td>810</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td>484</td>
<td>26</td>
<td>676</td>
<td>572</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>225</td>
<td>25</td>
<td>625</td>
<td>375</td>
</tr>
<tr>
<td>4</td>
<td>35</td>
<td>1,225</td>
<td>42</td>
<td>1,764</td>
<td>1,470</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>900</td>
<td>38</td>
<td>1,444</td>
<td>1,140</td>
</tr>
<tr>
<td>6</td>
<td>52</td>
<td>2,704</td>
<td>40</td>
<td>1,600</td>
<td>2,080</td>
</tr>
<tr>
<td>7</td>
<td>35</td>
<td>1,225</td>
<td>32</td>
<td>1,024</td>
<td>1,120</td>
</tr>
<tr>
<td>8</td>
<td>55</td>
<td>3,025</td>
<td>54</td>
<td>2,916</td>
<td>2,970</td>
</tr>
<tr>
<td>9</td>
<td>40</td>
<td>1,600</td>
<td>50</td>
<td>2,500</td>
<td>2,000</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>1,600</td>
<td>43</td>
<td>1,849</td>
<td>1,720</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\Sigma x &= 351 \\
\Sigma x^2 &= 13,717 \\
\Sigma y &= 380 \\
\Sigma y^2 &= 15,298 \\
\Sigma xy &= 14,257 \\
\end{align*}
\]

Figures from Johnson & Kuby, 2012.
3: Descriptive Analysis and Bivariate Data

3.2 Linear Correlation

Example:

<table>
<thead>
<tr>
<th>Push-ups, x</th>
<th>27</th>
<th>22</th>
<th>15</th>
<th>35</th>
<th>30</th>
<th>52</th>
<th>35</th>
<th>55</th>
<th>40</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sit-ups, y</td>
<td>30</td>
<td>26</td>
<td>25</td>
<td>42</td>
<td>38</td>
<td>40</td>
<td>32</td>
<td>54</td>
<td>50</td>
<td>43</td>
</tr>
</tbody>
</table>

\[
SS(x) = \sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left( \sum_{i=1}^{n} x_i \right)^2 = 13717 - \frac{(351)^2}{10} = 1396.9
\]

\[
SS(y) = \sum_{i=1}^{n} y_i^2 - \frac{1}{n} \left( \sum_{i=1}^{n} y_i \right)^2 = 15298 - \frac{(380)^2}{10} = 858.0
\]

\[
SS(xy) = \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \left( \sum_{i=1}^{n} x_i \right) \left( \sum_{i=1}^{n} y_i \right) = 14257 - \frac{(351)(380)}{10} = 919.0
\]

\[
r = \frac{SS(xy)}{\sqrt{SS(x)SS(y)}} = \frac{919.0}{\sqrt{(1396.9)(858.0)}} = 0.84
\]

\[
\sum_{i=1}^{n} x_i = 351
\]

\[
\sum_{i=1}^{n} x_i^2 = 13717
\]

\[
\sum_{i=1}^{n} y_i = 380
\]

\[
\sum_{i=1}^{n} y_i^2 = 15298
\]

\[
\sum_{i=1}^{n} x_i y_i = 14257
\]

Figure from Johnson & Kuby, 2012.
3: Descriptive Analysis and Bivariate Data

3.4 Linear Regression

We try different lines until we find the “best” one, \( \hat{y} = b_0 + b_1 x \)

Move line until sum of the squared residuals is a minimum.

\[ \sum_{i=1}^{n} e_i^2 \]

\( b_0 \) is estimated y-intercept
\( b_1 \) is estimated slope.
3: Descriptive Analysis and Bivariate Data

3.4 Linear Regression

\((x,y)\) pairs: \((1,1),(3,2),(2,3),(4,4)\)

Plotted points.

The line goes through \((\bar{x}, \bar{y})\).

The slope is \(b_1=0.8\).

The \(y\) - intercept \(b_0=0.5\).

Two points \((2.5, 2.5)\) and \((0, 5)\).
3: Descriptive Analysis and Bivariate Data

3.3 Linear Regression

Example: Our data!

Height vs. Weight

\[
b_1 = \frac{SS(xy)}{SS(x)} = \frac{2456.7}{472.4} = 5.2
\]

units of lbs/in

\[
b_0 = (143.2) - (5.2)(66.7) = -203.7
\]

point-slope formula

Rowe, D.B.
3: Descriptive Analysis and Bivariate Data

Questions?

Homework: Chapter 3 # 33, 44, 53, 59, 75
Read 4.1 and 4.2
4: Probability
4.1 Probability of Events

An **experiment** is a process by which a measurement is taken or observations is made. i.e. flip coin or roll die

An **outcome** is the result of an experiment. i.e. Heads, or 3

**Sample space** is a listing of possible outcomes. i.e. \( S=\{H,T\} \)

An **event** is an outcome or a combination of outcomes. i.e. even number when rolling a die

Rowe, D.B.
4: Probability
4.1 Probability of Events

Approaches to probability.
(1) Empirical (AKA experimental)

\[
\text{empirical probability of } A = \frac{\text{number of times } A \text{ occurred}}{\text{number of trials}}
\]

(2) Theoretical (AKA classical or equally likely)

\[
\text{theoretical probability of } A = \frac{\text{number of times } A \text{ occurs in sample space}}{\text{number of elements in the sample space}}
\]

Property 1: \(0 \leq P(A_i) \leq 1\)

Property 2: \(\sum_{i=1}^{n} P(O_i) = 1\)

\(A_i\) are events

\(O_i\) are outcomes
4: Probability - Empirical
4.1 Probability of Events – Law of large numbers

Had computer flip a single coin 1000 times.

Flip # on $x$ axis

$P'(H)$ on $y$ axis.

This shows convergence to true value of 1/2.

$$P'(H) = \frac{\text{# of heads}}{\text{# coin flips}}$$

Rowe, D.B.
4: Probability
4.1 Probability of Events
So let's flip a coin three times.

Can flip three times.

Sample space: listing of outcomes for 3 flips

\[ S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \]

\[ P(HHH) = \frac{\text{# times } HHH \text{ occurs in } S}{\text{# elements in } S} \]
4: Probability
4.3 Rules of Probability

**Example:** Pick Card, \( A=\text{Heart}, \ B=\text{Ace} \)

\[
P(\bar{A}) = 1 - P(A)
\]

\[
P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)
\]

\[
P(A \text{ and } B) = P(A) \cdot P(B \mid A)
\]

\[
P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}
\]

Figure from Johnson & Kuby, 2012.
4: Probability
4.4 Mutually Exclusive Events

Mutually exclusive events:
Events that share no common elements

In algebra: \( P(A \text{ and } B) = 0 \)

In words:
1. If one event has occurred, the other cannot.
2. None of the elements in one is in other.
3. In Venn diagrams, no intersection.
4. Intersection of events has a probability of zero.
4: Probability
4.5 Independent Events

**Independent events:** … the occurrence or nonoccurrence of one gives no information about … occurrence for the other.

\[ P(A) = P(A \mid B) = P(A \mid \text{not } B) \]

**Dependent events:** … occurrence of one event does have an effect on the probability of occurrence of the other event.

\[ P(A) \neq P(A \mid B) \]

**Special multiplication rule:**
Let \( A \) and \( B \) … independent events … in a sample space \( S \).

\[ P(A \text{ and } B) = P(A) \cdot P(B) \]
4: Probability

Questions?

Homework: Chapter 4 # 59, 63, 65, 69, 85, 89, 91, 97, 105, 107, 113