Class 7

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Agenda:

Recap Chapter 4.1, 4.2

Lecture Chapter 4.3 - 4.5

Go over Exam 1.
Recap Chapter 4.1, 4.2
4: Probability
4.1 Probability of Events

An **experiment** is a process by which a measurement is taken or observations is made. i.e. *flip coin* or *roll die*

An **outcome** is the result of an experiment. i.e. *Heads*, or 3

**Sample space** is a listing of possible outcomes. i.e. $S=\{H,T\}$

An **event** is an outcome or a combination of outcomes. i.e. $A=$even number when rolling a die=$\{2,4,6\}$
4: Probability

4.1 Probability of Events

Property 1: $0 \leq P(A_i) \leq 1$

Property 2: $\sum_{i=1}^{n} P(O_i) = 1$

Approaches to probability.

(1) Empirical  (AKA experimental)

empirical probability of $A = \frac{\text{number of times } A \text{ occurred}}{\text{number of trials}}$

(2) Theoretical  (AKA classical or equally likely)

theoretical probability of $A = \frac{\text{number of times } A \text{ occurs in sample space}}{\text{number of elements in the sample space}}$
4: Probability - Empirical
4.1 Probability of Events – Law of large numbers

Had computer flip a single coin 1000 times.

Flip # on $x$ axis

$P'(H)$ on $y$ axis.

This shows convergence to true value of 1/2.

$$P'(H) = \frac{\text{# of heads}}{\text{# coin flips}}$$

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4: Probability - Theoretical

4.1 Probability of Events

So let's flip a coin twice.

Can flip three times.

Sample space: listing of outcomes for 2 flips

\[ S = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \} \]

\[ P(HHH) = \frac{\text{# times } HHH \text{ occurs in } S}{\text{# elements in } S} \]
4: Probability
4.2 Conditional Probability

Example: Draw card from deck. Let $A = \text{red card}$, $B = \text{heart}$.

$P(A) = ?$ vs. $P(A|B) = ?$

$P(A) = \frac{1}{2}$ vs. $P(A|B) = 1$

Figure from Johnson & Kuby, 2012.
Chapter 4: Probability continued

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4: Probability
4.3 Rules of Probability - Probability of “Not $A$”

**Complimentary Events:** The *compliment of $A$, $\bar{A}$* is the set of all sample points in the sample space that does not belong to event $A$. i.e. If $A$, is heads, then $\bar{A}$ is tails.

**Compliment Rule:**
In words: probability of $A$ compliment $= \text{one} - \text{probability of } A$

In algebra: $P(\bar{A}) = 1 - P(A)$

From $P(A) + P(\bar{A}) = 1 \quad (4.3)$

i.e. $P(T) = 1 - P(H)$
4: Probability
4.3 Rules of Probability - Probability of “Not $A$”

Compliment: $S = \{A, \bar{A}\}$

Venn Diagram:

$P(\bar{A}) = 1 - P(A)$
4: Probability
4.3 Rules of Probability - Probability of “A or B”

General Addition Rule
Let $A$ and $B$ be two events defined in the sample space, $S$.

In words: probability of $A$ or $B$ = probability of $A$
+ probability of $B$
- probability of $A$ and $B$

In algebra: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ (4.4)
4: Probability

4.3 Rules of Probability - Probability of “A or B”

**Union:** $A$ or $B$

**Venn Diagram:**

\[
P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)
\]

Double count so have to subtract one off.
4: Probability

4.3 Rules of Probability – “A or B”

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

**Example:** Pick Card, \( A = \text{Heart} \), \( B = \text{Ace} \)

\[ P(\text{Heart or Ace}) \]

\[ P(\text{Heart}) = \frac{13}{52} \]

\[ P(\text{Ace}) = \frac{4}{52} \]

\[ P(\text{Heart and Ace}) = \frac{1}{52} \]

\[ P(\text{Heart or Ace}) = P(\text{Heart}) + P(\text{Ace}) - P(\text{Heart and Ace}) \]

\[ P(\text{Heart or Ace}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} \]

Figure from Johnson & Kuby, 2012.
4: Probability
4.3 Rules of Probability - Probability of “A and B”

General Multiplication Rule
Let $A$ and $B$ be two events defined in the sample space, $S$.

In words: probability of $A$ and $B = \text{probability of } A \times \text{probability of } B$, knowing $A$

In algebra: $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

(4.5)
4: Probability

4.3 Rules of Probability - Probability of “A and B”

**Event Intersection:** $A$ and $B$

**Venn Diagram:**

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

$S$
4: Probability

4.3 Rules of Probability – “A and B”

\[ P(A \text{ and } B) = P(B)P(A \mid B) \]

**Example:** Pick Card, \( A = \text{Heart} \), \( B = \text{Ace} \)

\[ P(\text{Heart and Ace}) \]

\[ P(\text{Ace}) = \frac{4}{52} \]

\[ P(\text{Heart} \mid \text{Ace}) = \frac{1}{4} \]

\[ P(\text{Heart and Ace}) = P(\text{Ace})P(\text{Heart} \mid \text{Ace}) \]

\[ P(\text{Heart and Ace}) = \left( \frac{4}{52} \right) \left( \frac{1}{4} \right) = \frac{1}{52} \]

Figure from Johnson & Kuby, 2012.
Conditional Probability: Probability of event $A$ given that event $B$ has occurred is

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$$

the “$|$” is spoken as “given” or “knowing”
4: Probability
4.3 Rules of Probability – “A and B”

\[ P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \]

**Example:** Pick Card, \( A = \text{Heart}, \ B = \text{Ace} \)

\[ P(\text{Heart} \mid \text{Ace}) \]

\[ P(\text{Heart and Ace}) = \frac{1}{52} \]

\[ P(\text{Ace}) = \frac{4}{52} \]

\[ P(\text{Heart} \mid \text{Ace}) = \frac{P(\text{Heart and Ace})}{P(\text{Ace})} = \frac{1/52}{4/52} = \frac{1}{4} \]

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**4: Probability**

**4.3 Rules of Probability**

**Union Example** \((A \text{ or } B)\): Rolling a single die.  
\(A=\text{event } #1,2,3.\ B=\text{event odd number.}\)
4: Probability
4.3 Rules of Probability

**Union Example** \((A \text{ or } B)\): Rolling a single die.
\(A=\text{event #1,2,3.} \quad B=\text{event odd number.}\)
\(A=\{1,2,3\}\)
4: Probability
4.3 Rules of Probability

Union Example ($A$ or $B$): Rolling a single die.
$A$=event #1,2,3. $B$=event odd number.
$B=$\{1,3,5\}

\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\hline
\hline
\hline
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S
\end{array}
4: Probability
4.3 Rules of Probability

Union Example \((A \text{ or } B)\): Rolling a single die.
\(A=\text{event #1,2,3. } B=\text{event odd number.}\)
\((A \text{ or } B)\)

\[
P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)
\]
4: Probability
4.3 Rules of Probability

**Intersection Example** (*A* and *B*): Rolling a single die.
*A*=event #1,2,3.  *B*=event odd number.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</table>

\[ S \]
4: Probability
4.3 Rules of Probability

Intersection Example \((A \text{ and } B)\): Rolling a single die.

\(A\)=event #1,2,3. \(B\)=event odd number.

\(A=\{1,2,3\}\)
4: Probability
4.3 Rules of Probability

Intersection Example \((A \text{ and } B)\): Rolling a single die.

\(A\) = event #1,2,3. \(B\) = event odd number.

\(B\) = \{1,3,5\}

\[ S \]

\begin{tabular}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{tabular}
4: Probability
4.3 Rules of Probability

Intersection Example ($A$ and $B$): Rolling a single die.

$A$=event #1,2,3. $B$=event odd number.

$(A \text{ and } B) = \{1,3\}$

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

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4: Probability
4.4 Mutually Exclusive Events

**Mutually exclusive events:**
Events that share no common elements

In algebra: \( P(A \text{ and } B) = 0 \)

In words:
1. If one event has occurred, the other cannot.
2. None of the elements in one is in other.
3. In Venn diagrams, no intersection.
4. Intersection of events has a probability of zero.
4: Probability
4.4 Mutually Exclusive Events

Mutually Exclusive: \( P(A \text{ and } B) = 0 \)

Venn Diagram:
4: Probability
4.4 Mutually Exclusive Events

Mutually Exclusive Example: Rolling a single die.
$A =$ event #1,2. $B =$ event #5,6.

$A \cap B = \emptyset$

$P(A \text{ and } B) = 0$
4: Probability
4.5 Independent Events

**Independent events:** Two events are independent if the occurrence or nonoccurrence of one gives us no information about the likeliness of occurrence for the other.

In algebra: \[ P(A) = P(A \mid B) = P(A \mid \text{not } B) \]

In words:
1. Prob of \( A \) unaffected by knowledge that \( B \) has occurred, not occurred, or no knowledge.
2. …
3. …
4: Probability
4.5 Independent Events

Two events $A$ and $B$ are independent if the probability of one is not “influenced” by the occurrence or nonoccurrence of the other.

Two Events $A$ and $B$ are independent if:

1. $P(A) = P(A \mid B)$
2. $P(B) = P(B \mid A)$
3. $P(A \text{ and } B) = P(A) \cdot P(B)$

Examples:
Dependent events: Events that are not independent. That is, occurrence of one event does have an effect on the probability of occurrence of the other event.

In algebra: $P(A) \neq P(A \mid B)$
4: Probability
4.5 Independent Events - Special multiplication rule

**Special multiplication rule:**
Let $A$ and $B$ be two independent events defined in a sample space $S$.

In words: The probability of $A$ and $B = \text{probability of } A \times \text{probability of } B$

In algebra: $P(A \text{ and } B) = P(A) \cdot P(B)$

More generally

$P(A \text{ and } B \text{ and } C \text{ and } D \text{ and } E) = P(A) \cdot P(B) \cdot P(C) \cdot P(D) \cdot P(E)$ (4.7)
4: Probability

4.6 Are Mutually Exclusive and Independence Related?

Read this section on your own.
4: Probability

Questions?

Homework: Chapter 4 # 59, 63, 65, 69, 85, 89, 91, 97, 105, 107, 113

Read Chapter 5.1-5.2