Class 13

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Agenda:

Recap Chapter 6.1 - 6.3

Lecture Chapter 6.4 - 6.6

TA Go Over Exam 3.
Recap Chapter 6.1 - 6.3
The mathematical formula for the normal distribution is (p 315):

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \]

where
\[ e = 2.718281828459046… \]
\[ \pi = 3.141592653589793… \]
\[ \mu = \text{population mean} \]
\[ \sigma = \text{population std. deviation} \]

We will not use this formula.
Normal distribution with population mean $\mu$ and variance $\sigma^2$.

We want to know the (red) area under the normal distribution between $x_1$ and $x_2$.

Similar to discrete probabilities adding to 1.

The total area under the normal distribution is 1.
Example: Here is a normal distribution with $\mu = 5$ and $\sigma^2 = 4$.

Let’s say we want to know the red area under the normal distribution between $x_1 = 3$ and $x_2 = 7$.

What is the area under the normal distribution between these two values?
6: Normal Probability Distributions
6.3 The Standard Normal Probability Distributions

Normal distribution with $\mu=5$ and $\sigma^2=4$.

We drew a rectangle?

$$A = (x_2 - x_1) \times \text{(height)}$$

$$x_2 - x_1 = 7 - 3 = 4$$

height $\approx 0.2$

Area $\approx 0.8$
6: Normal Probability Distributions

6.3 The Standard Normal Probability Distributions

Normal distribution with \( \mu = 5 \) and \( \sigma^2 = 4 \).

But the normal distribution is not a rectangle.

Someone had the idea to convert normal distribution to the standard normal.

Subtract \( \mu \) and divide this by \( \sigma \) for every value of \( x \).

\[
z = \frac{x - \mu}{\sigma}.
\]

Area between \( x_1 \) and \( x_2 \) is the same as area between \( z_1 \) and \( z_2 \).
6: Normal Probability Distributions
6.3 The Standard Normal Probability Distributions

Normal distribution $\mu=5$ & $\sigma^2=4$ and standard normal distribution.

Area between $x_1$ and $x_2$ is same as the area between $z_1$ and $z_2$.

If $x_1 = 3$ and $x_2 = 7$ then $z_1 = (x_1 - \mu)/\sigma = -1$ and $z_2 = (x_2 - \mu)/\sigma = 1$?

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6: Normal Probability Distributions
6.3 The Standard Normal Probability Distributions

Now we can simply look up the $z$ areas in a table.

Appendix B Table 3 page 662.
6: Normal Probability Distributions

Appendix B

Table 3

Page 662

Areas of the Standard Normal Distribution

The entries in this table are the probabilities that a random variable, with a standard normal distribution, assumes a value between 0 and \( z \); the probability is represented by the shaded area under the curve in the accompanying figure. Areas for negative values of \( z \) are obtained by symmetry.

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<th>0.00</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
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The entries in this table are the probabilities that a random variable, with a standard normal distribution, assumes a value between 0 and \( z \); the probability is represented by the shaded area under the curve in the accompanying figure. Areas for negative values of \( z \) are obtained by symmetry.

- Total area under curve is 1.
- The area less than 0 is 0.5.
- The area greater than 0 is 0.5.
- The area from 0 to \( z \) is the same as the area from \(-z\) to 0.
- The area greater than \( z \) is 0.5 minus the area between 0 and \( z \).
- The area less than \( z \) is 0.5 plus the area between 0 and \( z \).
6.3 The Standard Normal Probability Distributions

Now we can simply look up the \( z \) areas in a table.

Appendix B Table 3 page 662.

Area between 0 and 1 is 0.3413.

Area between -1 and 1 is 0.6826
Normal curve with $\mu=5$ and $\sigma^2=4$.

With rectangle the area is $A = (x_2 - x_1) \times \text{(height)}$

Area $\approx 0.8$

From the table it is $A = 0.6826$

We over estimated area and can refine by subtracting triangles.
With rectangle the area is
\[ A = (x_2 - x_1) \times \text{(height)} \]
Area \( \approx 0.8 \)

From the table it is
\[ A = 2(0.3413) = 0.6826 \]

We over estimated, refine by subtracting triangles.
\[ A \approx (x_2 - x_1) \times \text{(height)} - 2 \times 0.5 \times \frac{(x_2 - x_1)}{2} \times (0.2 - 0.125) \]
\[ A \approx 0.65 \]
not bad approx
6: Normal Probability Distributions

Questions?

Homework: Chapter 6 # 7, 9, 13, 17, 29, 31, 33, 41, 43, 57, 59, 71, 89
Chapter 6: Normal Probability Distributions continued

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Department of Mathematics, Statistics, and Computer Science
We already discussed converting from a general normal distribution ($x$ axis) to the standard normal distribution ($z$ axis).

**Standard Score**

In words:

$$z = \frac{x - \text{(mean of } x)}{\text{standard deviation of } x}$$

In algebra:

$$z = \frac{x - \mu}{\sigma}$$

(6.3)
Assume that IQ scores are normally distributed with a mean $\mu$ of 100 and a standard deviation $\sigma$ of 16.

If a person is picked at random, what is the probability that his or her IQ is between 100 and 115?

i.e. $P(100 < x < 115)$ ?

Figures from Johnson & Kuby, 2008.
6: Normal Probability Distributions
6.4 Applications of Normal Distributions

IQ scores normally distributed
\( \mu = 100 \) and \( \sigma = 16 \).

\[ P(100 < x < 115) \]

\[ z = \frac{x - \mu}{\sigma} \]

\[ z_1 = \frac{x_1 - \mu}{\sigma} = \]

\[ z_2 = \frac{x_2 - \mu}{\sigma} = \]

Figures from Johnson & Kuby, 2008.
6: Normal Probability Distributions
6.4 Applications of Normal Distributions

IQ scores normally distributed 
$\mu=100$ and $\sigma=16$.

$P(100 < x < 115)$

Now we can use the table.

$P(100 < x < 115) = P(0 < z < 0.94)$

Figures from Johnson & Kuby, 2008.
We can use the table in reverse.

Before we had a $z$ value then looked up the probability (area) between 0 and $z$.

Now we will have a probability (area), call it $\alpha$, and want to know the $z$ value, call it $z(\alpha)$, that has a probability (area) of $\alpha$ larger than it.

\[ \alpha = P(z > z(\alpha)) \]

Figure from Johnson & Kuby, 2008.
Example:
Let $\alpha=0.05$. Find $P(z>z(0.05))=0.05$.

Figures from Johnson & Kuby, 2008.
In Chapter 5 we discussed the binomial distribution

\[ P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \]

\( x = \# \) of heads when we flip a coin \( n \) times

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

\( n=2 \)
\( p=1/2 \)
If we flip the coin a large number of times

\[ P(x) = \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x} \]

\[ x = 0, \ldots, n \]

\( x \) = \# of heads when we flip a coin \( n \) times

\( n = 14 \)
\( p = 1/2 \)

It gets tedious to find the \( n = 14 \) probabilities!

Figure from Johnson & Kuby, 2008.
It gets tedious to find the $n=14$ probabilities!

So what we can do is use a histogram representation,

Figures from Johnson & Kuby, 2008.
6: Normal Probability Distributions
6.6 Normal Approximation of the Binomial Distribution

So what we can do is use a histogram representation,

\[ n=14 \]
\[ p=1/2 \]

Then approximate binomial probabilities with normal areas.

Figures from Johnson & Kuby, 2008.

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Approximate binomial probabilities with normal areas. Use a normal with $\mu = np$, $\sigma^2 = np(1 - p)$

$$
\mu = (14)(.5) = 7 \\
\sigma^2 = (14)(.5)(1 - .5) = 3.5
$$

Figures from Johnson & Kuby, 2008.
6: Normal Probability Distributions
6.6 Normal Approximation of the Binomial Distribution

We then approximate binomial probabilities with normal areas.

\[ P(x = 4) \text{ from the binomial formula} \]

is approximately \[ P(3.5 < x < 4.5) \]

from the normal with \( \mu = 7, \sigma^2 = 3.5 \)

\( n = 14, p = 1/2 \)

Figures from Johnson & Kuby, 2008.
6: Normal Probability Distributions

6.6 Normal Approximation of the Binomial Distribution

From the binomial formula

\[ P(4) = \frac{14!}{4!(14-4)!} \cdot (.5)^4 (1-.5)^{14-4} \]

\[ P(x = 4) = 0.061 \]

\[ P(1.34 < z < 1.87) = 0.0594 \]

From the Normal Distribution

\[ P(3.5 < x < 4.5) \quad \mu = 7, \quad \sigma^2 = 3.5 \]

\[ z_1 = \frac{x_1 - \mu}{\sigma} = \]

\[ z_2 = \frac{x_2 - \mu}{\sigma} = \]

\( n=14, p=1/2 \)
6: Normal Probability Distributions

Questions?

Homework: Chapter 6 # 7, 9, 13, 17, 29, 31, 33, 41, 43, 57, 59, 71, 89
TA Go Over Exam 3.