Biophysics 230: Nuclear Magnetic Resonance
Haacke Chapter 10

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10.1: Imaging in More Dimensions

The 1D signal $s(k)$ given by

$$ s(k) = \int \rho(z) e^{-i2\pi kz} \, dk $$

is generalized to a signal $s(\vec{r})$ in 3D $\vec{r}$ from a single RF excitation

$$ s(\vec{k}) = \int \rho(\vec{r}) e^{-i2\pi \vec{k} \cdot \vec{r}} \, d^3r $$

or in terms of Cartesian coordinates

$$ s(k_x, k_y, k_z) = \int \int \int \rho(x, y, z) e^{-i2\pi (k_xx + k_yy + k_zz)} \, dx \, dy \, dz $$

$$ = \mathcal{F}[\rho(x, y, z)] $$

this is the 3D imaging equation.
10.1: Imaging in More Dimensions

The three time $t$ and gradient $G(t)$ dependent $k$-space coordinates are

$$k_x(t) = \gamma \int_t^0 G_x(t') \, dt',$$

$$k_y(t) = \gamma \int_t^0 G_y(t') \, dt',$$

$$k_z(t) = \gamma \int_t^0 G_z(t') \, dt'$$

which are the areas between the gradient curves and the primed time axis.
10.1: Imaging in More Dimensions

Need to apply the gradients $G_x, G_y, G_z$ in a particular fashion so that the (frequency and phase) $k$-space can be sufficiently covered and sampled to reconstruct (estimate) the spin density $\hat{\rho}(\vec{r})$

with the use of the inverse (discrete) Fourier transform

$$\hat{\rho}(\vec{r}) = \int s_m(\vec{k}) e^{i2\pi\vec{k}\cdot\vec{r}} \, d^3k$$  \hspace{1cm} (10.3)
10.1: Imaging in More Dimensions

Consider the covering of \((k_x, k_y)\) space in 3D imaging for a particular \(k_z\).

But let’s first take a look at 1D imaging, \(k_y = 0\) line.

The 2D signal with \(k_y = 0\) (and \(k_z = 0\)) will be according to

\[
s(k_x, 0) = \int \rho(\vec{r}) e^{-i2\pi(k_x x + \underbrace{k_y y}_{=0} + \underbrace{k_z z}_{=0})} d^3r
\]  \hspace{1cm} (10.4)
10.1: Imaging in More Dimensions

The inverse FT of the signal data along this line (as in Chapter 9) is*

\[ \hat{\rho}_{1D}(x) = \int s(k_x, 0) e^{i2\pi(k_x x + 0 + 0)} dk_x \]

Insert (10.5)

\[ = \int \left[ \int \rho(r') e^{-i2\pi k_x x'} d^3r' \right] e^{i2\pi k_x x} dk_x \]

Exponential inside

\[ = \int \left[ \int \rho(r') e^{-i2\pi k_x (x-x')} d^3r' \right] dk_x \]

Cartesian

\[ = \int \left[ \int \int \int \rho(r') e^{-i2\pi k_x (x-x')} dx'dy'dz' \right] dk_x \]

Switch order

\[ = \int \int \int \rho(r') \left[ \int e^{-i2\pi k_x (x-x')} dk_x \right] dx'dy'dz' \]

Is delta

\[ = \int \int \int \rho(x', y', z') \delta(x - x') dx'dy'dz' \]

Only for \( x = x' \)

\[ = \int \int \rho(x, y, z) dydz \]

(10.6)

* Note: “’” is not rotating frame
10.1: Imaging in More Dimensions, 1D Coverage

The $\left(\frac{\pi}{2}\right)_x$ RF pulse at $t = 0$ tips the spins in the $y$ direction.

The negative gradient between $t_1$ and $t_2$ moves us to $k_{min}$ (left).

The positive gradient between $t_3$ and $t_4$ moves us to the right.

We measure the signal $s(t)$ between $t_3$ and $t_4$.

Note that $t_2 - t_3 = (t_4 - t_3)/2$. 

![Diagram](image-url)
10.1: Imaging in More Dimensions

The generalization of the signal equation for all \( k_y \) (and a particular \( k_z \)) is

\[
s(k_x, k_y) = \int \int \int \rho(x, y, z) e^{-i2\pi(k_x x + k_y y)} \, dx \, dy \, dz \quad (10.7)
\]

\( \hat{\rho}(x, y) \) can be found in a similar fashion.

\[
\hat{\rho}(x, y) = \int k_y \int k_x s(k_x, k_y) e^{i2\pi(k_x x + k_y y)} \, dk_x dk_y
\]

\[
= \int k_y \int k_x \left[ \int z' \int y' \int x' \rho(x', y', z') e^{-i2\pi(k_x x + k_y y)} \, dx' \, dy' \, dz' \right] e^{i2\pi(k_x x + k_y y)} \, dk_x dk_y
\]

\[
= \int k_y \int k_x \left[ \int z' \int y' \int x' \rho(x', y', z') e^{-i2\pi[k_x(x-x') + k_y(y-y')]} \, dx' \, dy' \, dz' \right] \, dk_x dk_y
\]

\[
= \int z' \int y' \int x' \rho(x', y', z') \left[ \int k_y \int k_x e^{-i2\pi[k_x(x-x') + k_y(y-y')]} \, dk_x \, dk_y \right] \, dx' \, dy' \, dz'
\]

\[
= \int z' \int y' \int x' \rho(x', y', z') \delta(x - x', y - y') \, dx' \, dy' \, dz'
\]

\[
= \int z \rho(x, y, z) \, dz \quad (10.8)
\]
10.1: Imaging in More Dimensions, 2D Coverage

1. The \( (\frac{\pi}{2})_x \) RF pulse at \( t = 0 \) tips the spins in the \( y \) direction.
2. The negative gradient \(-G_y\) moves us to \( k_y, \text{min} \) (bottom).
3. The negative gradient \(-G_x\) moves us to \( k_x, \text{min} \) (left).
4. The \( +G_x \) gradient moves us from \( k_x, \text{min} \) to \( k_x, \text{max} \). Data every \( \Delta t \) or \( \Delta k_x = \gamma \Delta G_x \Delta t \).
5. When we reach $k_{x,\text{max}}$, turn on $+\Delta G_y$ for time $\tau_y$. Up 1 line $\Delta k_y = \gamma \Delta G_y \tau_y$.

6. The $-G_x$ gradient moves us from $k_{x,\text{max}}$ to $k_{x,\text{min}}$. Data at $\Delta t$ or $\Delta k_x = \gamma \Delta G_x \Delta t$.

7. When we reach $k_{x,\text{min}}$, turn on $+\Delta G_y$ on for time $\tau_y$. Up 1 line $\Delta k_y = \gamma \Delta G_y \tau_y$

8. Repeat steps 4.-7. until the $(k_x, k_y)$ space is covered.
10.1: Imaging in More Dimensions, 3D Coverage

3D imaging similar to 2D.

Don’t like Figure 10.4.
Take Figure 10.2
Add negative $G_x$, $G_y$, and $G_z$.
Add positive $G_z$ blip.

1. The $(\frac{\pi}{2})_x$ RF pulse at $t = 0$ tips the spins in the $y$ direction.
2. The negative gradient $-G_z$ moves us to $k_{z,min}$ (bottom $z$).
3. The negative gradient $-G_y$ moves us to $k_{y,min}$ (bottom).
4. The negative gradient $-G_x$ moves us to $k_{x,min}$ (left).
5. The $+G_x$ gradient moves us from $k_{x,min}$ to $k_{x,max}$. Data every $\Delta t$ or $\Delta k_x = \gamma \Delta G_x \Delta t$. 
10.1: Imaging in More Dimensions, 3D Coverage

3D imaging similar to 2D.

Don’t like Figure 10.4.
Take Figure 10.2
Add negative $G_x$, $G_y$, and $G_z$.
Add positive $G_z$ blip.

6. When we reach $k_{x,\text{max}}$, turn on $+\Delta G_y$ for time $\tau_y$. Up 1 line $\Delta k_y = \gamma \Delta G_y \tau_y$.
7. The $-G_x$ gradient moves us from $k_{x,\text{max}}$ to $k_{x,\text{min}}$. Data at $\Delta t$ or $\Delta k_x = \gamma \Delta G_x \Delta t$.
8. When we reach $k_{x,\text{min}}$, turn on $+\Delta G_y$ on for time $\tau_y$. Up 1 line $\Delta k_y = \gamma \Delta G_y \tau_y$.
9. Repeat steps 5.-8. until $(k_x, k_y)$ space covered.
10. Turn $+\Delta G_z$ on for $\tau_z$. Up one slice in $\Delta k_z = -\gamma \Delta G_z \tau_z$. Repeat 1.-9 until $k_z$ covered.
10.1: Imaging in More Dimensions, Multiple RF

**Time Constraints and Collecting Data over Multiple Cycles**

Typically only one slice of $k$ space is collected following each RF excitation.

This is because of signal loss due to decay.

This means multiple excitations at every $T_R$.

When doing this, all $k_x$ are collected from left to right.

Look at Figures 10.4 and 10.5.

Let $N_y$ and $N_z$ denote the number of phase encoding steps. Then it takes

$$T_{acq} = N_y N_z T_R$$  \hspace{1cm} (10.11)

for 3D and

$$T_{acq} = N_y T_R$$  \hspace{1cm} (10.12)

for 2D. These are increased by $N_{acq}$ if we repeat $N_{acq}$ to reduce noise.
10.1: Imaging in More Dimensions, Multiple RF

**Variations in \( k \)-Space Coverage**

The way in which \( k \)-space is covered can be varied as long as each point is covered (according to the book).

Methods were developed so that every every \( k \)-space point is not needed.

Instead of sampling at time \( \Delta t \) while a constant \( G_x = G_0 \) is applied,

May instead change the gradient by the amount \( \Delta G_x \) each time we sample to generate the same \( \Delta k_x \).

These are identical.
10.2: Slice Selection with Boxcar Excitations

Slice Selection
The ‘slice select axis’ is perpendicular to the desired slice.

\( z - axis \) gradient selects transverse  
\( y - axis \) gradient selects coronal  
\( x - axis \) gradient selects sagittal

Let’s take \( z - axis \) or transverse slices.

The frequency along this direction (with a constant linear gradient) is

\[
f(z) = f_0 + \gamma G_z \]

(10.15)

where \( f_0 = \gamma B_0 \) is the Larmor frequency (see figure 10.7).

To select (excite) the slice \( (z_0 - \Delta z/2) \) to \( (z_0 + \Delta z/2) \), the RF pulse must be uniform over \( (\gamma G_z z_0 - \gamma G_z \Delta z/2) \) to \( (\gamma G_z z_0 + \gamma G_z \Delta z/2) \), which is shown in Figure 10.8.
10.2: Slice Selection with Boxcar Excitations

The bandwidth $\Delta f$ is

$$BW_{rf} \equiv \Delta f$$

(10.16)

$$= (\gamma G_z z_0 + \gamma G_z \Delta z/2) - (\gamma G_z z_0 - \gamma G_z \Delta z/2)$$

(10.17)

$$= \gamma G_z \Delta z$$

(10.18)

Denote

$$\Delta z \equiv TH$$

(10.19)

and as a result

$$TH = \frac{BW_{rf}}{\gamma G_z}$$

(10.20)

In order to get a uniform flip, the frequency profile must be

a boxcar function $\text{rect}\left(f \frac{\Delta f}{\Delta f}\right)$ of bandwidth $\Delta f$.

This means the temporal envelope of the rf pulse $B_1(t)$ is a sinc function

$$B_1(t) \propto \text{sinc}(\pi \Delta ft)$$

(10.21)

This is in Chapter 16. Just accept it for now.
10.2: Slice Selection with Boxcar Excitations

The sinc function is extremely important in fMRI.

You can set it equal to zero and find that its first zero crossing is \( t_1 = \frac{1}{\Delta f} \).

In general, the zeros of \( \text{sinc}(x) = \sin(x)/x \) are \( x = n\pi \) for all nonzero integers \( n \).

Realistic sinc pulses are only on for time \( \tau_{rf} \) and thus truncated with

\[
    n_{zc} = \left\lfloor \frac{\tau_{rf}}{t_1} \right\rfloor = \left\lfloor \frac{\Delta f \tau_{rf}}{1} \right\rfloor
    \tag{10.22}
\]

where the brackets denote the largest integer less than its value.

The longer the pulse is on, the more zero crossings and the closer the real approximate pulse is to its theoretical value.
10.2: Slice Selection with Boxcar Excitations

Examples:

\[ G_z = 20 \text{mT/m}. \]

\[ \tau_{rf} = 0.002 \text{s vs } \tau_{rf} = 0.0005 \text{s}. \]

(Blue) \hspace{1cm} (red)

\[ G_z = 20 \text{mT/m vs } G_z = 5 \text{mT/m}. \]

(Blue) \hspace{1cm} (red)

\[ \sin(\pi \times 8516 \times t) / (\pi \times 8516 \times t) \]

\[ \sin(\pi \times 2129 \times t) / (\pi \times 2129 \times t) \]

The longer more precise the sinc, the more ideal square slice profile.
10.2: Slice Selection with Boxcar Excitations

Gradient Rephasing

As in 1D GRE imaging case, get an echo by adding a rephasing gradient.

Look at Figure 10.9

If slice is instantaneously excited at $t = 0$ as in Figure 10.9 with constant gradient strength $G_z = G_{SS}$, then the phase of the transverse magnetization from Equation 9.13 is

$$\phi(z, t) = -\gamma G_{SS} z t$$  \hspace{1cm} (10.23)

The signal $s(t)$ in the slice of width $\Delta z$ at $t$ follows Equation (9.14)

$$s(t) = \int \rho(z) e^{i\phi G(z, t)} \, dz$$  \hspace{1cm} (9.14)

It effectively has a one-dimensional spin density $\rho(z)$ as defined in Equation (9.7) given by

$$\rho(z) \equiv \int \int \rho(\vec{r}) \, dx dy$$  \hspace{1cm} (9.7)
10.2: Slice Selection with Boxcar Excitations

Gradient Rephasing
The spin density can also be assumed to be constant over time (but not $\phi$ which is a function of $z$) resulting in

$$s(t) \simeq \rho(z_0) \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} e^{i\phi(z,t)} \, dz$$  \hspace{1cm} (10.24)$$

As time increases, the spins at different positions $z$ accumulate differing amounts of phase. The signal decreases as a result of the integral

$$\int_{z_0-\Delta z/2}^{z_0+\Delta z/2} e^{i\phi(z,t)} \, dz \quad \rightarrow \quad \text{dephasing 0}$$  \hspace{1cm} (10.25)$$

Arbitrary Slice Orientation
Not Covering.
10.3: 2D Imaging and $k$-space, Gradient Echo Example

Slice Selecting
After the rephase lobe of the slice select gradient, the signal is described by

$$s(\tau_rf) = \int \int \left[ \int \frac{z_0 + \Delta z}{2} \rho(x, y, z) \, dz \right] \, dxdy$$

(10.29)

We now want to explore the spin density within the $z$ slices.
10.3: 2D Imaging and $k$-space, Gradient Echo Example

Phase Encoding

Phase encoding $y$ gradient is applied and while on, the signal described by

$$s(t + \tau_y) = \int \left[ \int e^{-i2\pi\gamma G_y\tau_y y} dy \right] dx$$

$$[\int \rho(x, y, z) dz]$$

(10.30)

which is a $y$ dependent adaptation of Equation (10.23)

and in terms of $k$-space with

$$k_y(G_y) = \mp G_y\tau_y.$$  (10.31)

The $y$ gradient $G_y$ is varied in steps $\Delta G_{PE}$ as in 10.1.4.
10.3: 2D Imaging and $k$-space, Gradient Echo Example

Reading the Data

As in Chapter 9, an $x$ gradient is applied in order to obtain an echo at $T_E$.

In terms of $t' = t - T_E$ with boxcar gradient of Figure 10.13, the signal is

$$s(t', G_y) = \int \int \left[ \int_{z_0 - \frac{\Delta z}{2}}^{z_0 + \frac{\Delta z}{2}} \rho(x, y, z) \, dz \right] e^{-i2\pi \gamma G_y \tau_y y \, dy} e^{-i2\pi \gamma G_x t' x \, dx} \quad -T_s/2 < t' < T_s/2 \quad (10.32)$$

and in terms of $k$ space with

$$k_x(t') = \gamma G_x t' \quad (10.33)$$

and Equation (10.31) leads to

$$s(k_x, k_y) = \int \int \left[ \int_{z_0 - \frac{\Delta z}{2}}^{z_0 + \frac{\Delta z}{2}} \rho(x, y, z) \, dz \right] e^{-i2\pi \gamma (k_x x + k_y y) \, dx dy} \quad (10.34)$$
10.3: 2D Imaging and $k$-space, Gradient Echo Example

Assumption:
The average spin density over the z slice is often approximated by

the spin density at the center of the slice $z_0$

$$s(k_x, k_y) = \int \int \rho(x, y, z_0) e^{-i2\pi\gamma(k_xx+k_yy)} \, dx \, dy$$  \hspace{1cm} (10.35)

By repeating the gradients in Figure 10.13 with the $y$ gradient stepped up each time gives additional $k_y$ lines in $(k_x, k_y)$ space for the selected $z$ slice.

Repeat for each $z$ slice.
10.3: 2D Imaging and $k$-space, Gradient Echo Example

Superposition of Phase Effects

Look at Figure 10.14

The phase encoding and the dephasing gradient of the read lobe along with the rephasing lobe of the slice select gradient can be simultaneously applied as in Figure 10.14.

The results are the same.

This does not apply during the application of the rf pulse or data acquisition.
10.3: 2D Imaging and $k$-space, Gradient Echo Example

$k$-Space coverage
The gradient echo sequences given in Figures 10.13 and 10.14 cover $k$-space as in Figure 10.15.

Refer to Figure 10.14

Read rest of Chapter 10.3 on own.

Spin Echo Example
Read Chapter 10.4 on own.

Homework
Do 10.1, 10.2, 10.3 (Use $H^1$ and $\gamma = 42.58$Mhz/T and watch your units.)
Read the rest of the Chapter, 10.4 on.