# The $(i_1, \ldots, i_m)$-Step Competition Hypergraph

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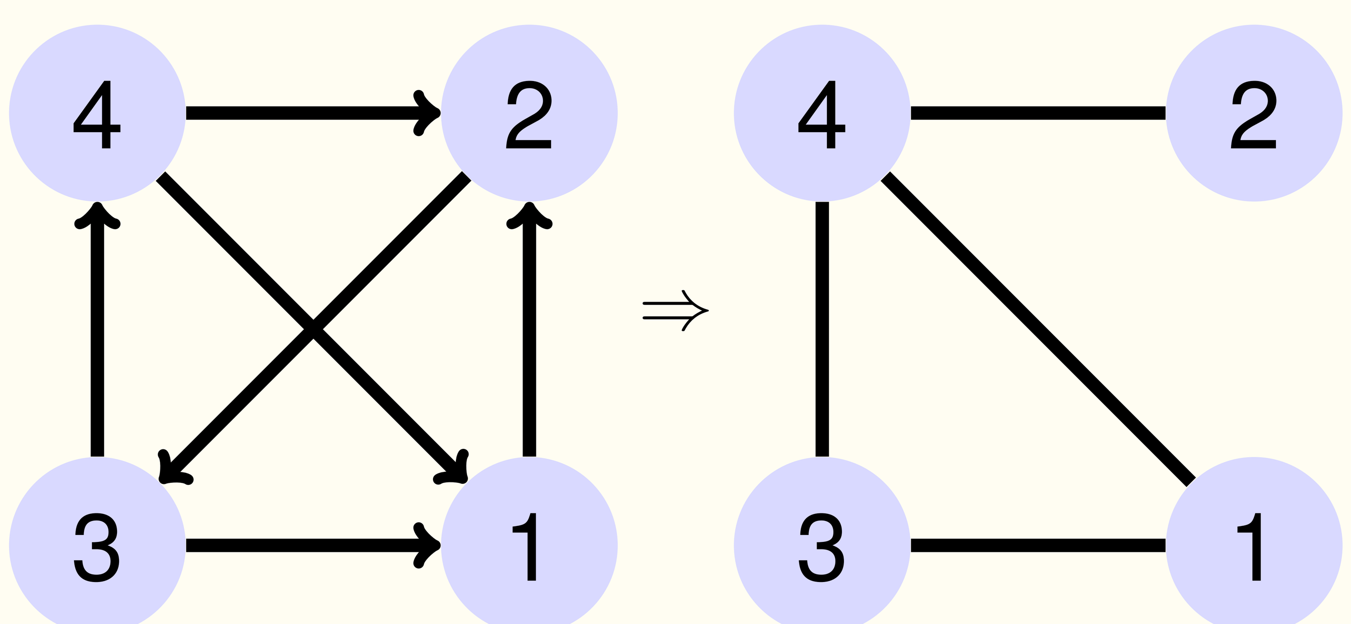
## Background

- In 2010, Drs. Factor and Merz introduced the $(i,j)$-step competition graph, a generalization of the $(1,2)$-step competition graph.
- We define the $(i,j)$-step competition graph as follows: if for some $z \in V(D) - \{x,y\}$, $d_{D-y}(x,z) \le i$ and $d_{D-x}(y,z) \le j$ or $d_{D-x}(y,z) \le i$ and $d_{D-y}(x,z) \le j$.




$(1,2)$-step competition graph of a digraph

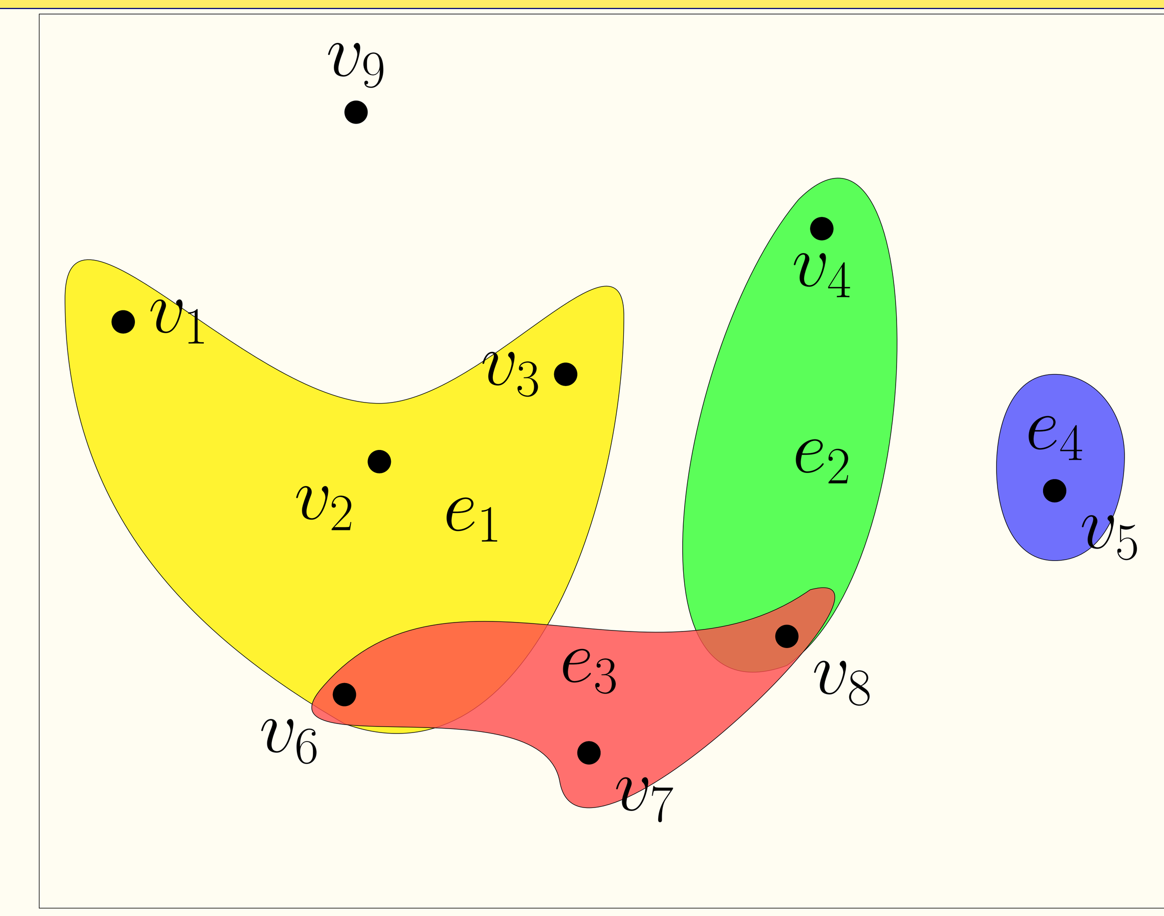
- They derived a surprising result: if the digraph $D$ is a tournament, then the $(i,j)$-step competition graph is equivalent to the $(1,2)$-step competition graph (for $i \ge 1$ and $j \ge 2$).

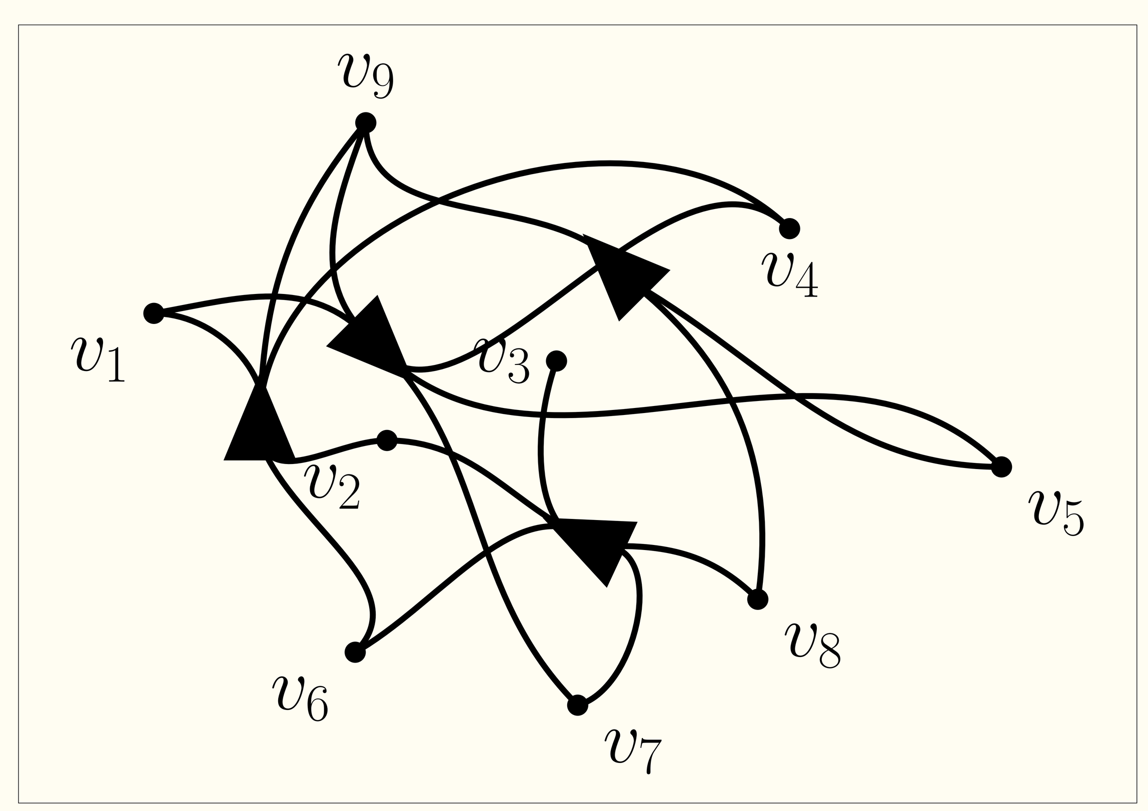
## Objectives

Our main objective is extending the $(i,j)$-step competition graph of a digraph to an object that can represent a relationship between two or more vertices.

## Definitions

Hypergraph: a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ is the set of vertices and $\mathcal{E} = \{E_1, E_2, \cdots, E_m\}$, with $E_i \subseteq \mathcal{V}$ for $i = 1, \cdots, m$, is the set of hyperedges.




The *directed hypergraph* $\mathcal{D}$,(*hyperdigraph*), is the pair $(\mathcal{V}, \mathcal{A})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the vertex set and $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, where every $A_i$ consists of the ordered pair $(T_i, H_i)$, is the set of *directed hyperedges*, or *hyperarcs*. We call $T_i$ and $H_i$ of $A_i$ the *tail* of $A_i$ and *head* of $A_i$, respectively.

**Definition:** The $(i_1, i_2, \ldots, i_m)$-step competition hypergraph of a hyperdigraph $\mathcal{D}$, denoted $C_{i_m}(\mathcal{D})$, is the m-hypergraph on $\mathcal{V}(\mathcal{D})$ where a set of m vertices from $\mathcal{V}(\mathcal{D})$, $\{x_1, x_2, \ldots, x_m\}$, is a hyperedge on $C_{i_m}(\mathcal{D})$ if and only if there exists a vertex $z \ne x_1, x_2, \ldots, x_m$, such that $d_{\mathcal{D}-x_j}(x_k, z) \le i_q$ and is a unique combination of the positive integers $j,\ k,\ q$, where $1 \le j,\ k,\ q \le m$ and $j \ne k$.

## Results

- **Lemma:** Let $\mathcal{T}$ be a strongly-connected tournament with $1 \le i_1, \ldots, i_m$. The hyperedge $\{x_1, \ldots, x_m\} \in \mathcal{E}(C_{i_m}(\mathcal{D}))$ if and only if the outset of some $x_j$ is equal to any number of the other $x_{m-1}$ vertices in the potential hyperedge.
- **Lemma:** Let $\mathcal{T}$ be an $n$-tournament with strong decomposition $\mathcal{T}_1, \ldots, \mathcal{T}_k$. If $\{x_1, \ldots, x_m\} \in \mathcal{E}(C_{i_m}(\mathcal{T}))$, then $x_1, \ldots, x_m \in \mathcal{V}(\mathcal{T}_k)$ or $|\mathcal{V}(\mathcal{T}_k)| = 1$ and $C_{i_m}(\mathcal{T}) = \mathcal{K}_{n-1} \cup \mathcal{K}_1$.
- **Theorem:** If $\mathcal{T}$ is an $n$-tournament, $i_1 \ge 1$ and $i_2, \ldots, i_m \ge 2$, then $C_{(i_1,\ldots,i_m)}(\mathcal{T}) = C_{(1,2,\ldots,2)}(\mathcal{T})$.

## References


[1] G. Gallo, G. Longo, S. Pallottino, and S. Nguyen, "Directed hypergraphs and applications.," *Discrete Applied Mathematics.*, vol. 42, no. 2-3, p. 177, 1993.

[2] K. A. S. Factor and S. K. Merz, "The $(1,2)$-step competition graph of a tournament.," *Discrete Applied Mathematics.*, vol. 159, no. 2-3, p. 100, 2011.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