An Exploration of Food Webs using Graph Theory

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Abstract

A food web can be modeled by a digraph,D, where there is an arc from vertex x to vertex y if species x preys on species y. The (1,2)-step competition graph of this food web, $C_{1,2}(D)$ introduced in 2001 by Factor and Merz, has the same vertex set as D and has an edge (x, y) if and only if there exists a vertex $z \neq x, y$ such that either $d_{D-y}(x, z) = 1$ and $d_{D-x}(y, z) \leq 2$ or $d_{D-x}(y, z) = 1$ and $d_{D-y}(x, z) \leq 2$. In this paper we characterize the digraphs which generate (1,2)-step competition graphs which are complete on all non basal vertices.

1 Introduction

The aim of this paper is to apply graph theory to the study of food webs. Food webs are towers of organisms in which each organism depends, for energy, on one or many other organisms in the ecosystem represented in the web [1]. Mathematically, these webs can be modeled by acyclic digraphs where the set of vertices $V(D)$ represents the organisms and the set of directed arcs $A(D)$ represents relationships between predators and prey. An cycle in a digraph is a path in $D(V, A)$ $u_1, a_1, u_2, a_2, ..., u_t, a_t, u_{t+1}$ where each $u_i 1 \leq i \leq t$ is distinct and $u_1 = u_{t+1}$ [4]. An acyclic digraph is a digraph which has no cycles. From these acyclic digraphs we are then able to study more closely interactions and energy transfer in an ecosystem. One specific area of interest is the study of competition within an ecosystem. Competition graphs of the digraphs representing food webs highlight relationships between predators which share prey and are therefore, to an extent, codependent. By understanding competition relationships, ecologist are able to study the resiliency and adaptability of an ecosystem. A competition graph of $D(V, A)$, a digraph which models a food web, is a graph $G(V, E)$ on the same vertex set V with edge $\{(v_i, v_j)\} \in E(G)$ if and only if there is a vertex $v_k \in V(D)$ such that (v_i, v_k) and (v_j, v_k) are directed arcs in $E(D)$ [2]. This original competition graph, presented by Cohen, shows the relationships between species which compete directly. A variation, the 1,2-step competition graph, developed by Factor and Merz, allows for the study of relationships between predators who may share a food sources more indirectly. Factor and Merz define the (1,2)-step competition graph of digraph D as a graph $C_{1,2}(D)$ on the same vertex set V with edge $\{x, y\} \in E(C_{1,2}(D))$ if there exists some $x, y \neq z \in V$ such that one of the following hold:

1) $d_{D-y}(x, z) \leq 1$ and $d_{D-x}(y, z) \leq 2$
2) $d_{D-x}(y, z) \leq 1$ and $d_{D-y}(x, z) \leq 2$

Let $D - v$ for $v \in V$ denote the digraph D less vertex v and all arcs incident with v, and let $d_D(x, y)$ represent the distance from x to y in D [3]. The (1,2)-step competition graph then allows us to represent the food web by edges which show relationships between species which directly and indirectly compete for food. We intend to use properties of the (1,2)-step competition graph to inform the study of acyclic digraphs which generate them, and conversely, to use the digraph representations of food webs to inform the study of the properties of (1,2)-step competition graphs.

2 Methods

There are two ways from which we are able to approach the problem of determining the properties of a digraph with a (1,2)-step competition graph which is complete on all non-basal vertices. First we may look at forbidden subgraphs of the digraphs. Forbidden subgraphs would be relationships in the digraph which fail to generate connections in the (1,2)-step competition graph. We then use these forbidden subgraphs to illuminate properties of digraphs which do produce complete components of (1,2)-step competition graphs. The second approach to this question is to look at all the possible digraphs which give rise to (1,2)-step competition graphs which are complete on all non-basal vertices. The goal with the second approach is to use these digraphs to observe patterns which are constant for all digraphs which have compete components in their (1,2)-step competition graph. These patterns then in turn may reveal properties which are necessary for a digraph to generate a (1,2)-step competition graph which is complete on all non-basal vertices.

3 Definitions

Because we are looking at applying graph theory to ecology, we must first clarify the language which will be used throughout the following results. Let us define a basal vertex, or basal species, as a vertex which has no outgoing arc. Conversely, we will define a top vertex, or a top predator, as a vertex which has no incoming arcs. For our purposes, we will also define a herbivore as a species which only has basal vertices in its outset.

4 Results

Much of this work arose from somewhat linear thinking. As (1,2)-step competition concerns itself with path lengths, it logically follows that if we wish our (1,2)-step competition graph to be complete on all non-basal species that the overall path lengths in these digraphs, or food webs, must somehow be similarly limited. The first two conditions below arose from this path of inquiry.

Lemma 1. *for all acyclic digraphs, there exists atleast one vertex which contains only basal vertices in its outset*

Proof. Suppose not. Let D be an acyclic digraph which has no vertices with an outset which contains only basal vertices. By Theorem 11.13 in Roberts [4], D is acyclic if and only if it has a topological sorting, or a labeling of its vertices such that $(v_i, v_j) \in A(D)$ implies $i < j$. Let D have n vertices, suppose that $k < n$ of those vertices are sink vertices. Label these vertices $v_{n-k}, v_{n-(k-1)}, ..., v_n$. Consider all remaining vertices which contain one or more of the sink vertices in their outset. Without loss of generality we can let $v_{n-(k+1)}$ be one such vertex. By our choice of D however, the outset of $v_{n-(k+1)}$ must contain at minimum one vertex which is not a sink vertex. Therefore there exists $(v_{n-(k+1)}, v_{n-(k+h)}) \in A(D)$ where $h \geq 2$. We then have $i > j$ which implies there is no topological sorting for D which in turn implies that D has a cycle which is a contradiction. □

Lemma 1 basically informs us that every food web has atleast one herbivore. As herbivores behave differently from other intermediary vertices, it is important to note their existence in the acyclic digraphs with which we are concerned. Lemma 1 also directly provides for Theorem 1 which, by the definition of (1,2)-step competition and the existence of herbivores enables us to limit the height of our food webs.

Theorem 1. *If the (1,2)-step competition graph of an acyclic digraph is complete on all non-basal vertices, then each vertex which is not a basal vertex is connected to some basal vertex by a path of length no greater than two.*

Proof. Consider D an acyclic digraph. Assume there exists some $y \in V(D)$ with a minimum path length of 3 to any basal vertex. By Lemma 1, there is atleast one vertex in D such that the only vertices in its outset are basal vertices. Let us denote this vertex as x. By the definition of a (1,2)-step competition graph, x only (1,2)-competes with vertices that are in the inset of one of the vertices in its outset, and with the vertices in the insets of those vertices. All such vertices have a path of length of 1 or 2 to the basal vertices, which precludes y from (1,2)-competing with x. Therefore the (1,2)-step competition graph of D is not complete. □

The conditions above are clearly necessary as Lemma 1 provides for Theorem 1, which in turn describes a forbidden subgraph. The condition however which places a maximum-minimum path length in the digraph from top vertex to basal vertex is not sufficient to imply completeness of the (1,2)-step competition graph on all non-basal vertices. The remaining conditions arose from counterexamples, graphs which showed that Theorem 1 is not sufficient to imply completeness.

Lemma 2. *If the (1-2)-step competition graph of an acyclic digraph is complete on all non-basal species, all herbivores must directly compete with all other herbivores.*

Proof. Suppose not. Let D be an acyclic digraph which has vertices x, y such that $O^+(x) \bigcap O^+(y) = \emptyset$. By definition of an herbivore, the greatest path length from any herbivore to any basal species is a maximum length of one, therefore, there is no possible way for x and y to (1,2)-compete, and by our definition of x and y, they do not directly compete. Therefore x and y are not connected in the (1,2)-step competition graph of D and as such the (1,2)-step competition graph is not complete on all non-basal vertices, which is a contradiction. □

Lemma 2 is self evident to the extreme. It however states directly the necessity of distinguishing herbivores from other intermediary species. This becomes relevant when looking at the various ways in which top species and omnivores interact both with each other and with herbivores and basal vertices.

Lemma 3. *If there is a vertex with a minimum path length greater than one, which has an outset of size one, in an acyclic digraph, then the (1,2)-step competition graph of the acyclic digraph is not complete on all non-basal vertices.*

Proof. Consider D, an acyclic digraph. Let y be a vertex such that the minimum path from y to any basal species is 2 and $O^+(y) = \{x\}$ thus $|O^+(y)| = 1$. Clearly, $O^+(y) \bigcap O^+(x) = \emptyset$ so x and y do not directly compete. By our definition of y, any path from y to an element in $O^+(x)$ must, by necessity, pass through x and the path therefore does not exist is $D - x$ and therefore by the definition of (1,2)-step competition, x and y do not (1,2)-compete. Thus x and y are not connected in the (1,2)-step competition graph of D and as such, it is not complete on all non-basal species. □

Theorem 2. *If the (1,2)-step competition graph of an acyclic digraph is complete on all non-basal vertices, then all vertices with a minimum path length of two to a basal species has a second path of maximum length two to the same basal species.*

Proof. Suppose not. Let D be an acyclic digraph with a complete (1,2)-step competition graph. By Theorem 1, all acyclic digraphs with complete (1,2)-step competition graphs must have a maximum-minimum path length of two between each vertex and a basal species. Consider x, a vertex with a minimum path length of two to any basal species. By Lemma 3 $|O^+(x)| \geq 2$. Suppose therefore $|O^+(x)| = 2$. Let x, w, z be the original path of length two to basal species z. First, define x, u, v, z a path of length three to z. There is no path of length two in $D - w$ to z therefore x and w do not (1,2)-compete and the (1,2)-step competition graph of D is not complete which is a contradiction. Now consider a path x, u, y, a path of length two to a different basal species y. There is in this case also no path in $D - w$ from x to z of length two which implies that x and w do not (1,2)-compete and the (1,2)-step competition graph of D is not complete which is a contradiction. □

5 Future Work

Although Theorem 2 is also a condition which is necessary for a digraph to generate a (1,2)-step competition which is complete on all non-basal species, it is not, nor is Theorem 1, nor Theorems 1 and 2, sufficient to imply completeness on all non-basal species in a (1,2)-step competition graph. It appears that studying the ratio of top vertices to intermediary vertices and manipulating the size of the insets and outsets of these vertices may enable us to coerce a digraph with a (1,2)-step competition graph which is complete on all non-basal vertices. This may or may not be relevant however because we may easily coerce a complete on all non-basal vertices (1,2) step competition graph by forcing all vertices to compete for a single basal vertex. This however in rather uninteresting and not a pracical model for a real food web. I am therefore working with the following theorem in conjunction with the previous results.

Theorem 3. *Let n be the number of intermediary vertices in an acyclic digraph D and $v_i 1 \leq i \leq k$ be top vertices in D, If $|O^+(v_i)| \geq n \forall v_i \in V(D)$ then $C_{1,2}(D)$ is complete on all non-basal vertices.*

This Theorem has yet to be proven or disproven.

References

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