Middle and High School Students' Interpretations of the Equal Sign and Equality: Analysis of Written Task Solutions

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Abstract:

Students develop their experiences with equality and the equal sign in elementary grades, but research shows that most elementary school students have a weak understanding of the equal sign. In this paper we explore how *middle* and *high school* students view and understand the concept equality and the equal sign. We found that overall middle and high school students lack the language to explain the meaning of the equal sign and equality. Most also interpret the equal sign operationally. Those middle and high school students who interpret the equality as sameness do it computationally. Our findings provide the teachers with directions about how to cater to students' needs and help their students better understand this concept.

Background Information

In mathematics, equivalence relations are a type of relation among the elements of a given set that provide a way for those elements to be identified with other elements of that set according to a certain criterion. The power of an equivalence relation relates to its ability to partition a set of elements into the disjoint union of subsets-typically referred to as equivalence classes. Because of its power to partition a set, equivalence relations are one of the most useful tools in mathematics. Equivalence relations have three properties: reflexive, symmetric and transitive. These properties can be defined as follows:

The Definition of Equivalence Relation

Let X be a set and let x, y, and z be elements of X. An equivalence relation, ~, on X is a relation on X such that:

(1) Reflexive Property: x is equivalent to x for all x in X.

(2) Symmetric Property: if x is equivalent to y, then y is equivalent to x.

(3) Transitive Property: if x is equivalent to y and y is equivalent to z, then x is equivalent to z.

In the K-12 mathematics curriculum students develop knowledge of equivalence relations as they discuss equality, study parallel lines, or discuss congruence of geometric figures. Equivalence and the equal sign provide the focus for this paper.

Mathematics Educators often wonder and explore why their students do not do as well in algebra as they might wish. Herscovics and Linchevski may have found a reason. In their 2010 study they noted that students had problems in "the acceptance of the equal symbol." Research has shown that an understanding of equality as an equivalence relation helps students learn algebraic concepts and is an essential aspect of algebraic thinking (Kieran, 1981; McNeil & Alibali, 2005; Knuth, Stephens, McNeil, & Alibali, 2006).

Students develop their experiences with equality and the equal sign in elementary grades. But research shows that most elementary school students have a weak understanding of the equal sign (e.g., Behr, Erlwanger, Nichols, 1980). They typically view the equal sign as an operator symbol, the prompt for an answer (e.g., Kieran, 1981). Less is known how middle and high school students deal with equality and the equal symbol. A few studies suggest that older students might also have a limited understanding of equality and the equal sign (McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur, & Krill, 2006; Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2008). Knuth, Stephens, McNeil, & Alibali (2006) observed that the concept of equality and the equal sign is explicitly taught in the early elementary grades with little attention paid to these ideas in the later grades.

In this paper we explore how middle and high school students view and understand the concept of equality and the equal sign. This information can help teachers to address their students' needs and provide the teachers with directions about how to cater to their students' needs helping them to better understand this concept.

Conceptual Foundations

Matthews, Rittle-Johnson, McEldoon and Taylor (2012) developed a map, presented in Table 1, which describes students' continuum of mathematical thinking about equality and the equal sign. We used this map as a foundation for our work.

Levels of Knowledge	Explanation of Knowledge
Level 4: Comparative Relational	Students interpret the equal sign as "sameness" and understand equality relationally. They use relational strategies to get solutions to equations and are able to recognize that transformations maintain equality.
Level 3: Comparative Computational	Students interpret the equal sign as "sameness" but use computational strategies to determine "sameness." They solve equations by following the rules rather than thinking about the relationship between the quantities on both sides of the equal sign.
Level 2: Flexible Operational	Students interpret the equals sign operationally. They accept the validity of statements that follow an answer equals operations format (e.g. x=x, x= w + y). They also recognize some properties of equality.
Level 1: Rigid Operational	Students interpret the equal sign operationally. They do not accept the validity of statements that follow an answer equals operation format (e.g., 3 + 5 =). They only compute and are not able to recognize that transformations maintain equality

Table 1: Map of students' knowledge of equality and the equal sign

Methodology

Participants

This study is part of a larger study being done by Dr. Marta Magiera and Dr. Leigh van den Kieboom at Marquette University in which they studied how teachers help students learn concepts in algebra. In my summer research I analyzed work of 607 students from 25 different classrooms grades 7-12. The distribution of students by grade level was as followed: 117 seventh graders, 131 eighth graders, 120 ninth graders, 112 tenth graders, 90 eleventh graders, and 37 twelfth graders. The students were asked to respond in writing to 8 problems that were used to facilitate students' thinking about equality and the equal sign.

Data Analysis

Using the Matthews' et al. (2012) map (Table 1) we designed a task specific rubric for each task. This rubric operated on a 5 point scale from 0-4. Students' answers to each question were analyzed and coded according to the rubric. To be sure of consistency of grading the rubric was revised and clarified several times using specific examples of student solutions and recoding was done until everything was deemed 100% reliable. Examples of coding rubrics and illustrations of student responses are included in the results section.

After coding, the data were analyzed quantitatively. Frequencies of different categories of student responses were examined and z-scores were calculated to determine whether the differences in frequencies of student responses aligned with different response category were statistically significant.

Results

Students Interpretation of the Equal Sign

There were two problems (Problem 1 and Problem 2) used to examine how students define and interpret the equal sign.

Problem 1 gave students the following question: *23 + 11 = 34. What is the name of the symbol the arrow is pointing at* (with an arrow pointing towards the equal sign)*? What does that symbol mean?* Students were then asked again, *could this symbol mean anything else?* To further test their understanding, students were then asked the following problem: *Mary was asked to solve 123 + 58. She did 123 + 58 = 123 + 7 = 130 + 51 = 181. Did she show her work correctly? Yes or no? Explain.*
Responses were coded as follows:

Question 1

(0) *unclear;* explanation was not clear enough to determine the student's manner of thinking (e.g. "it means equal")
(1) *rigid operational;* the equal sign was interpreted explicitly as a prompt to give the answer.
(2) *flexible operational;* students gave both a rigid operational response and a comparative computational response.
(3) *Comparative computational;* the equal sign was interpreted as sameness and the determination of sameness resulted from calculations preformed on both sides of the equation
(4) *Comparative relational;* the equal sign was interpreted as sameness and the determination of sameness resulted from examination of the quantities on both sides of the equation

Question 2

(0) unclear; yes or no with an unspecific explanation that is either irrelevant or not clear enough to determine the student's manner of thinking
(1) rigid operational; students focused upon the initial result only (e.g. $123 + 58 \neq 123$)
(2) flexible operational; students focused upon the correctness of the answer (e.g. 123 + 58 = 181)
(3) Comparative computational; students focused upon the computational aspect of sameness (e.g. "$181 \neq 130 \neq 181$")
(4) Comparative relational; students focused upon the difference in quantities in each of the equations (e.g. "$58 \neq 7$") without comparison between sums and without preforming the calculations

Both middle and high school students exhibited difficulty when it came to explicitly defining the equal sign. Around 60% of middle and high school students gave unclear answers when asked to give definition. About a third of students in both groups demonstrated an operational view of equality when explaining its meaning and only a very small percent of students explained the meaning of the equal sign as sameness (4.9%, 6.44% respectively).

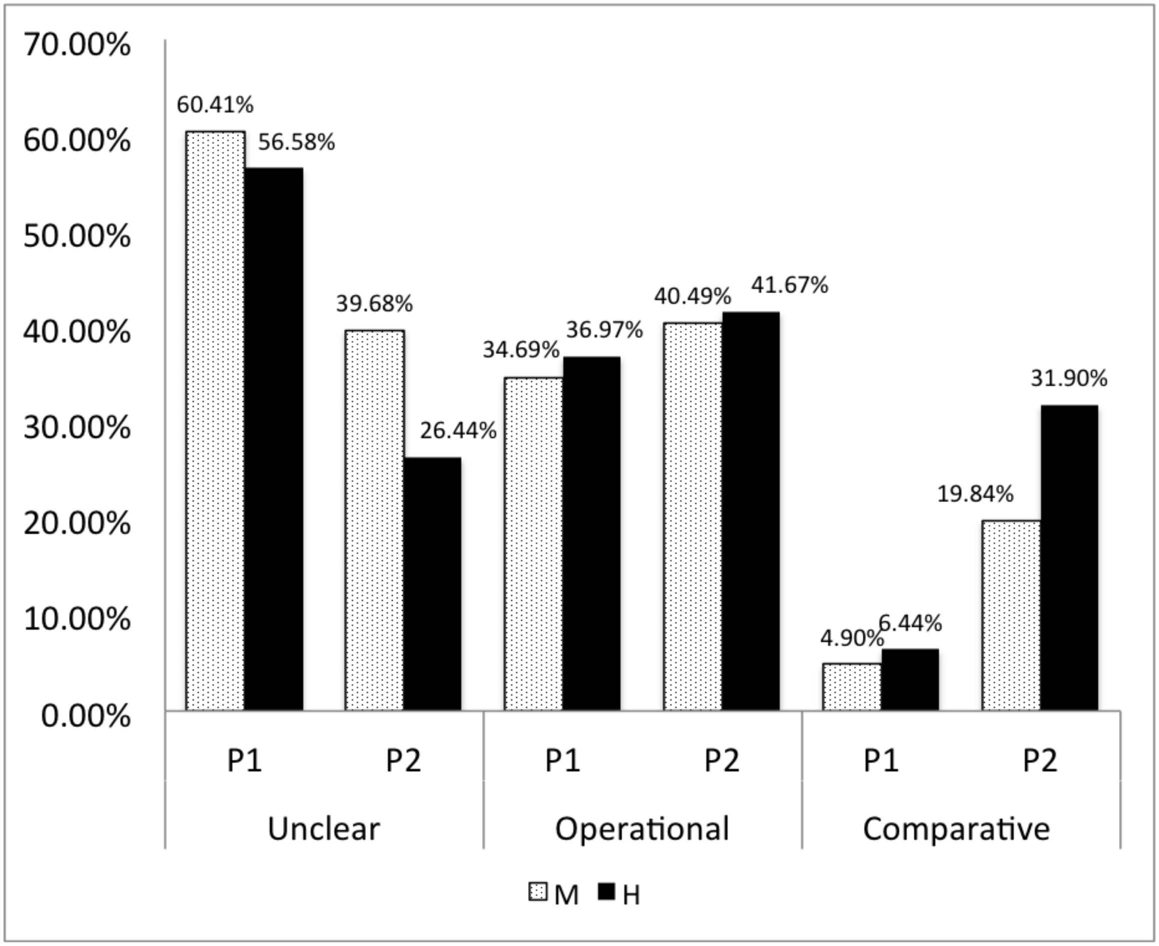

Figure 1: The Interpretation of the Equal Sign

While the students had difficulty to define the equal sign they performed better when asked to interpret the equal sign in context. Looking at Problem 2, we see that 40% of middle school students didn't look at the equal sign at all. 26.4% of high school students also didn't pay attention to the use of the equal sign at all but instead they mostly addressed that they felt the way the work was recorded was confusing.

About 40% of both middle and high school students gave an operational explanation but significantly more high school students who were able to notice incorrect usage of the equal sign answered comparatively than middle school students ($z=3.38$, $p<0.05$).

The analysis of students' responses to these two problems shows that the vast majority of middle and high school students had trouble to clearly define or interpret the equal sign in general. But in the specific arithmetic context (Problem 2) many more middle school and high school students were able to interpret the equal sign comparatively.

Recognizing Properties of Equality

Two problems were used to examine how students recognize the properties of equality to support their work with equations. In Problem 3 students were asked if both equations: $2x + 13 = 35$ *and* $2x + 13 - 8 = 35 - 8$, had the same solution. They then were asked to give an explanation as to *why yes or no.* In Problem 5 students were given the following situation: *$x = 5$ is a solution to the equation $2x + 10 = 20$. Is $x = 5$ a solution to $2x + 10 - 3 = 20 - 3$ as well? Explain why yes or no.*

The responses to both questions were coded according to the rubric:

Questions 3 & 5

(0) *unclear;* explanation was not clear enough to determine the student's manner of thinking

(1) *rigid operational;* students do not recognize that transformations maintain equality and focused solely upon computing answers (e.g. "one equation has more numbers than the other" & " the one on the right is equal to 27 not 35)

(2) *flexible operational;* students recognized the structure of the second equation but were unable to connect the two equations

(3) *Comparative computational;* students determined their response by solving each of the equations and comparing their solutions

(4) *Comparative relational;* students recognized the equivalency of both equations without computing by thinking about the properties of equality (e.g. "-8's on both sides keep it the same")

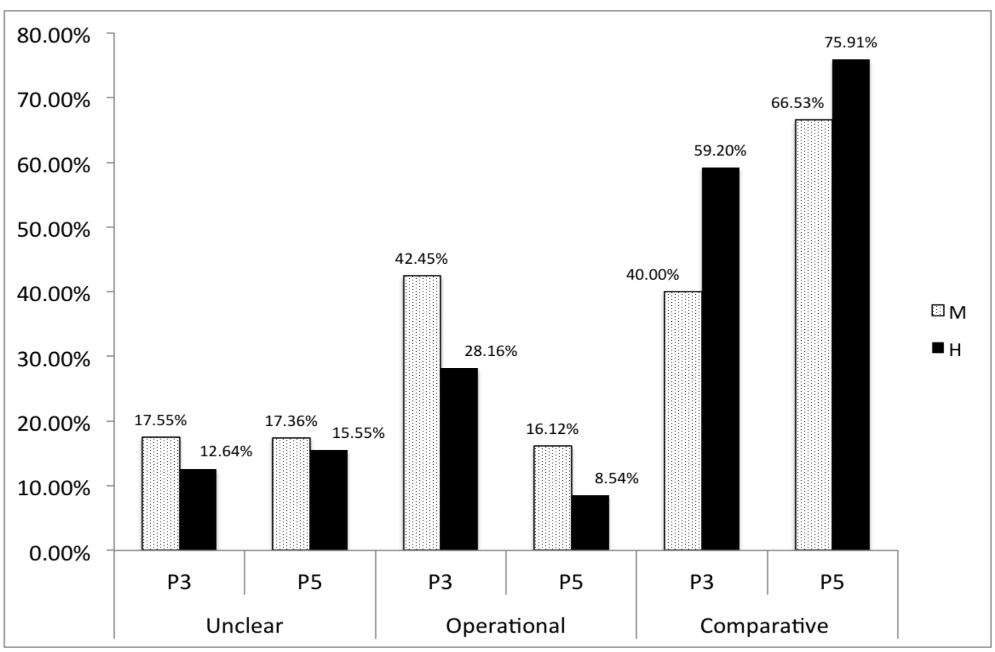

Figure 2: Recognizing Properties of Equality

Figure 2 shows that the middle school students answered Problem 3 in the operational or comparative way with about the same frequency. For Problem 5 the number of middle school students who answered in a comparative way significantly increased ($z=6.07$, $p<0.05$). For high school students there was also significant increase in comparative responses between Problem 3 and Problem 5 ($z=4.71$, $p<0.05$).

Looking at the summary of students' responses in Figure 2 we can see that there is a significant increase in comparative responses among both middle (26%) and high school (17%) students in Problem 5 compared to Problem 3. This begs the question, why most students answered comparatively Problem 5 but not Problem 3 when both problems had similar structures. Breaking down the comparative responses into comparative relational and comparative computational (as shown in Figure 3) we see that significantly more students gave a computational answer to Problem 5 than in Problem 3 for both middle and high school students ($z=6.51$, $p<0.05$; $z=5.26$, $p<0.05$ respectively). The higher number of computational responses to Problem 5 might be because that problem could be solved using more computational strategies.

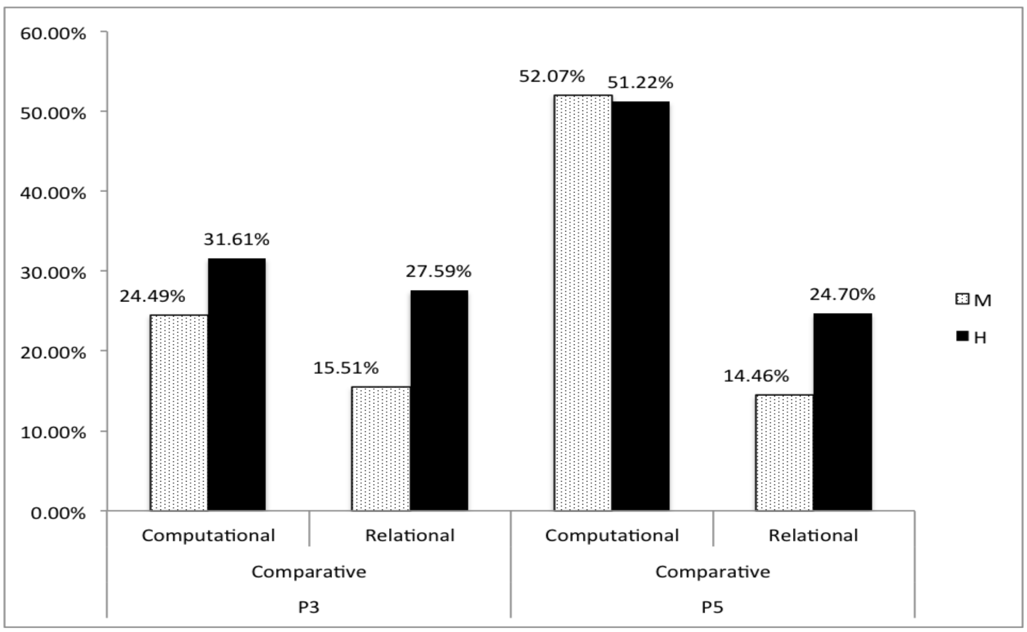

Figure 3: Recognizing Properties of Equality with a comparative view

Comparing Quantities

We analyzed students' responses to three problems (Problem 4, 6, and 7) to determine how middle and high school students think about the equal sign and the properties of numbers. Problem 4 asked students to determine *what value of x made the equation* $44 + 30 = 28 + 44 + x$ *true?* Problem 6 gave the following information: You know that $2x + 15 = 35$. Is this helpful to you to figure out what $2x + 16$ could be? Problem 7 asked the students *whether it is helpful to them to know that* $58 + 67 = 125$ *if they are asked to figure out* $59+60$*?* *All three* problems asked the students to explain their answers. Responses were coded as follows:

Question 4

(0) *unclear;* yes or no with no explanation or unclear explanation (e.g. "it would be x")

(1) *rigid operational;* students were unable to make connections between the two sides of the equation (e.g. "74 because 44+30=74)

(2) *flexible operational;* students were able to recognize some properties of equality and were able to recognize and accept operations = operations number sentences.

(3) *Comparative Computational;* students were able to see that each side of the equation needed to equal the same and computed to get *x* (e.g. "74=72+*x*").

(4) *Comparative Relational;* students were able to identify differences in problems without using a computational approach (e.g. "30 is 2 more than 28, so x=2")

Question 6

(0) *unclear;* yes or no with no explanation or unclear explanation (e.g. "it would be x" or "you cannot solve an expression")

(1) *rigid operational;* students were unable to make connections between the two equations and would view the equation with the need that equations should always be in the operations equal answer format

(2) *flexible operational;* students were able to recognize some properties of equality and were able to recognize and accept operations = operations number sentences (e.g. solving the first question but not knowing what to do for the second)

(3) *Comparative Computational;* Students were able to identify differences in problems using a computational approach i.e. solving the equations.

(4) *Comparative Relational;* Students were able to identify differences in problems with out using a computational approach (e.g. 16 is one greater than 15 so the answer would be one greater)

Question 7

(0) *unclear;* yes or no with no explanation

(1) *rigid operational;* students were unable to make connections between the two equations and would view the equation with the need that equations should always be in the operations equal answer format

(2) flexible operational; students were able to recognize some properties of equality and were able to recognize and accept operations = operations number sentences.

(3) Comparative Computational; Students were able to identify differences in problems using a computational approach i.e. solving the equations (e.g. "125=125")

(4) Comparative Relational; Students were able to identify differences in problems with out using a computational approach (e.g. "its just plus and minus one")

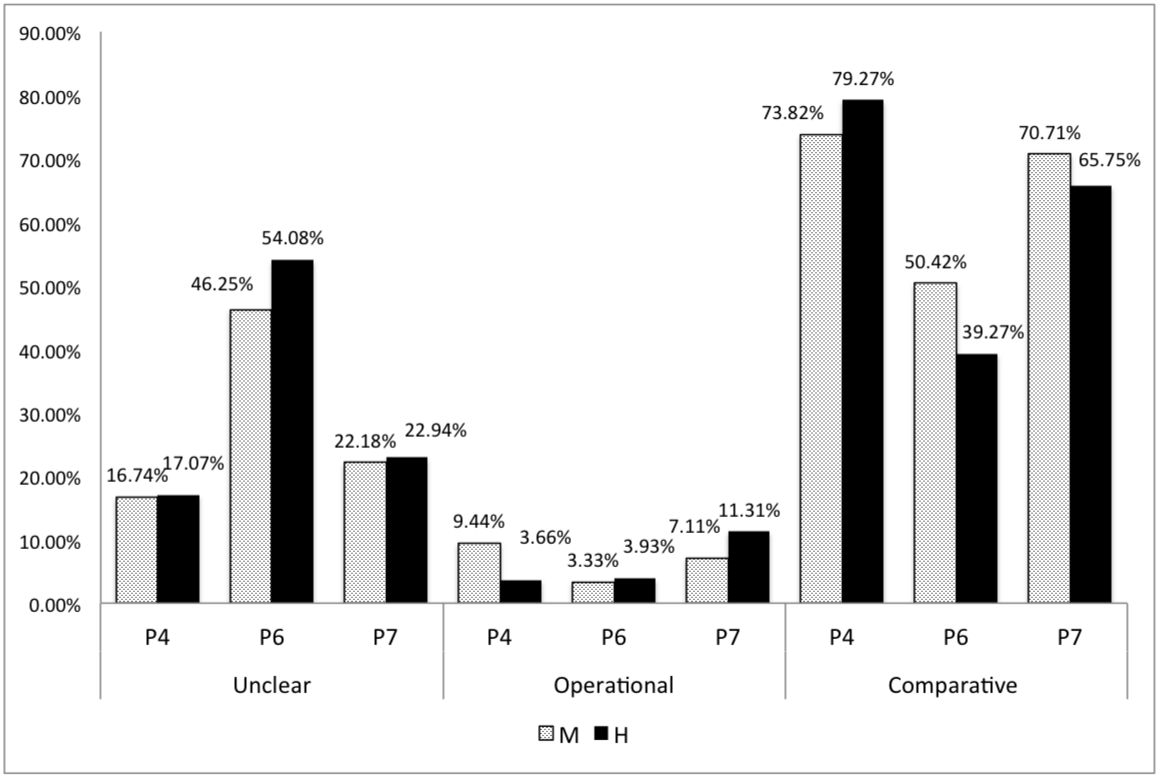

Figure 4: Comparing Quantities

Figure 4 gives a summary of students' responses to these three problems. Looking at students' responses to Problem 4 we can see that both middle and high school students interpreted problem 4 mostly comparatively (74% and 79% respectively).

For Problem 7, both middle and high school students predominantly gave comparative responses with 70% of middle school students and 66% of high school students answering in a comparative way.

For Problem 6 more high school students and middle school students gave unclear responses (about 50% in each group). Also, significantly more middle school students (50%) answered Problem 6 in a comparative way compared to the number of high school students (39%) who answered Problem 6 comparatively.

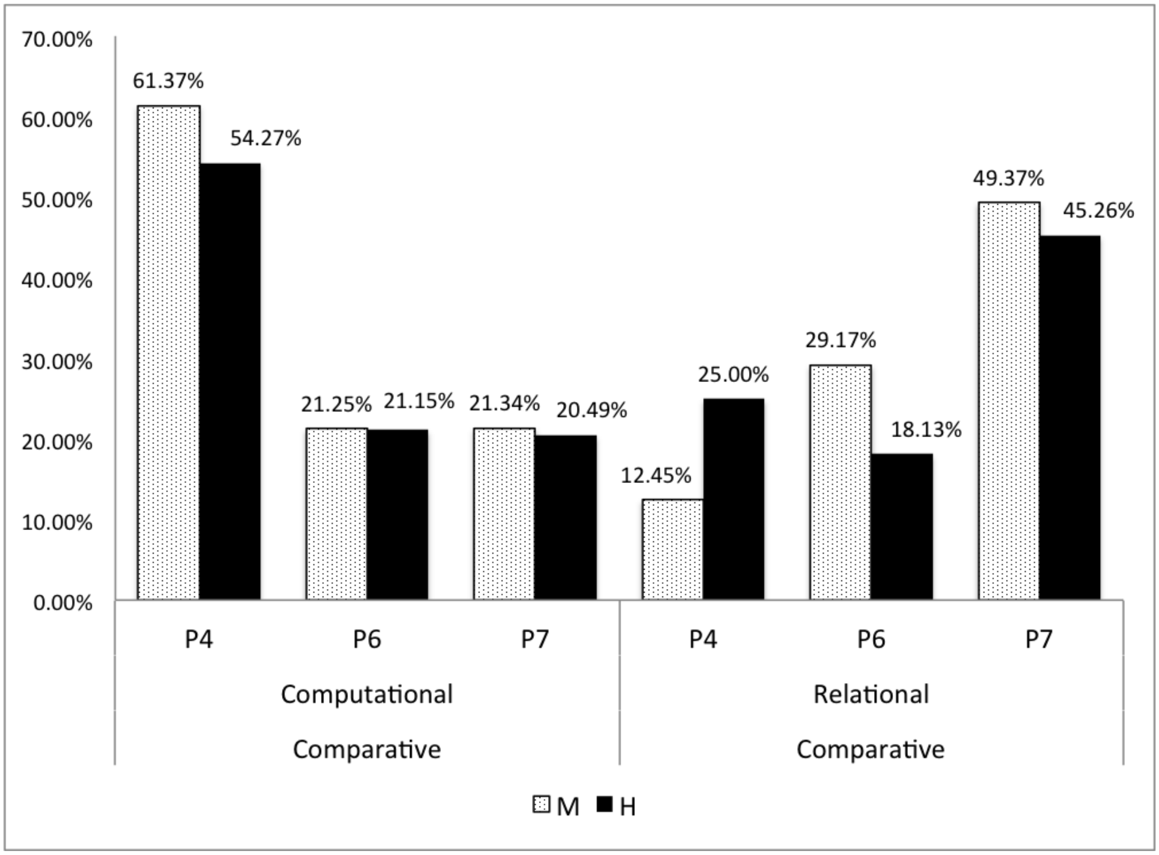

Figure 5: Comparing Quantities Comparative graph

We take a closer look at these comparative responses across the three problems (Problem 4, 6, and 7) in Figure 5. We can see that for Problem 6 and Problem 7 only about 21% of middle and high school students answered computationally. In contrast, students answered Problem 4 mostly computationally; 61% of middle school and 54% of high school students responded computationally. In contrast only 12% of middle school students and 25% of high school students viewed Problem 4 relationally. In Problem 7 we see that when giving a comparative answer, both middle school (49%) and high school (45%) students tended to give a comparative relational explanation. But only 29% of middle school students and 18% of high school students viewed Problem 6 relationally. This difference was significant with $z=3.04$ where $p<0.05$.

Discussion

Work on this research provided me with important insights into the understanding of equality by that middle and high school students. As a pre-service secondary mathematics teacher, through this work I learned why, as a future teacher, I should be sensitive and pay attention to the ways in which my students use and interpret the equal sign. Discussion about the importance of good understanding of equality and the equal sign is something that is not often focused upon in my education and math classes. Work on this research made me think how well I understand equality. This is a problem that is often carried over, in my experience, even into the college classroom. Thus it is important for pre-service teachers to evaluate their own understanding of the equal sign.

The results of this research document the need for the teachers to explicitly discuss equality in middle and high school classrooms. As a pre-service teacher I already look at this information and start thinking about how I can use this general understanding and implement explicit instruction of the equal sign in my own further work with students. My goal is to help my future students to consistently think about the equal sign and the equality in a comparative relational way. I also plan to share what I learned about the importance of the concept of equality and the equal sign, and what I discovered about students' knowledge of these issues in my own education classes at Benedictine College. I plan to discuss the results of my summer work with my classmates in an effort to bring these ideas into the classrooms we are observing as a part of our education.

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