Applying the Ergodicity Defect to Flows

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Linear Shallow Water Model

Introduction

Ergodicity Defect[3], d, Defined

- Measures how far away a flow is from being ergodic
- Ergodic ↔ time average = space average
  - Birkhoff's characterization

d = norm factor*(time average - space average)

$$d = norm\ factor * \left[\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f\left(T^k(x)\right) - \int_X f(x)\, dx \right]$$

❖ Where T is a map of x indicating the trajectory

- Ranges from 0-1 where 0 is ergodic and 1 is the identity map
- **Indicates complexity of a trajectory**
  - **How well it samples its space**

Linear Shallow Water Equations[1]

- Assume length ≫ depth
- For the solution we consider two modes superimposed:
  - A time-independent mode that creates a four cell structure
  - A time-periodic mode that perturbs the cells

$$u(x,y,t) = -2\pi \sin(2\pi x)\cos(2\pi y)\, u_0 + \cos(2\pi y)\, u_1(t)$$

$$v(x,y,t) = 2\pi \cos(2\pi x)\cos(2\pi y)\, u_0 + \cos(2\pi y) v_1(t)$$

$$h(x,y,t) = \sin(2\pi x)\cos(2\pi y)\, u_0 + \sin(2\pi y)\, h_1(t)$$

$$\dot{u}_0 = 0 \qquad \dot{u}_1 = v_1 \qquad \dot{v}_1 = -u_1 - 2\pi h_1 \qquad \dot{h}_1 = 2\pi v_1$$

Objective

The purpose of the project is to increase our understanding of the linear shallow water equations using the ergodicity defect and to investigate the use of the defect as a diagnostic for data assimilation methods in the context of LSW model.

Methods

Gray Scale Pictures

- **Depict the ergodicity defects for trajectories in a flow**
- Color represents the ergodicity defect of the trajectory that starts at that point
- Color range based on range of ergodicity defects
  - Min defect = black, Max defect = white
- **Useful to see:**
  - **How the flow behaves overall**
  - **Groups of trajectories that behave similarly**
  - **Interesting trajectories**

Trajectory Movies

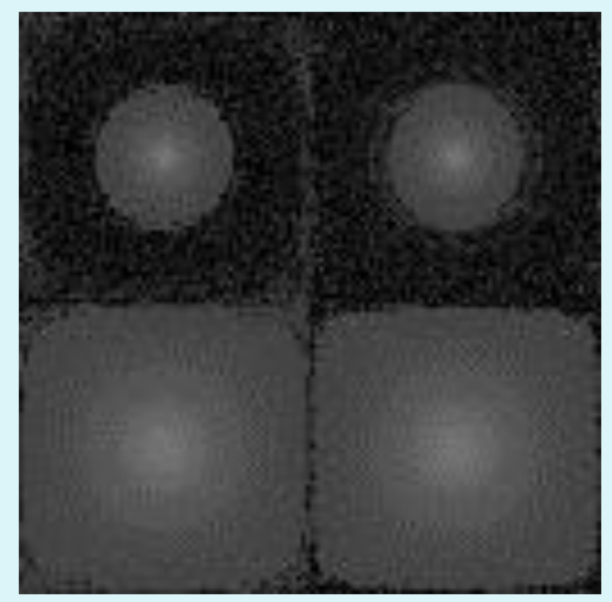
- Using gray scale picture information we chose individual trajectories and created movies of how they behave to further understand the flow.

Varying Parameters

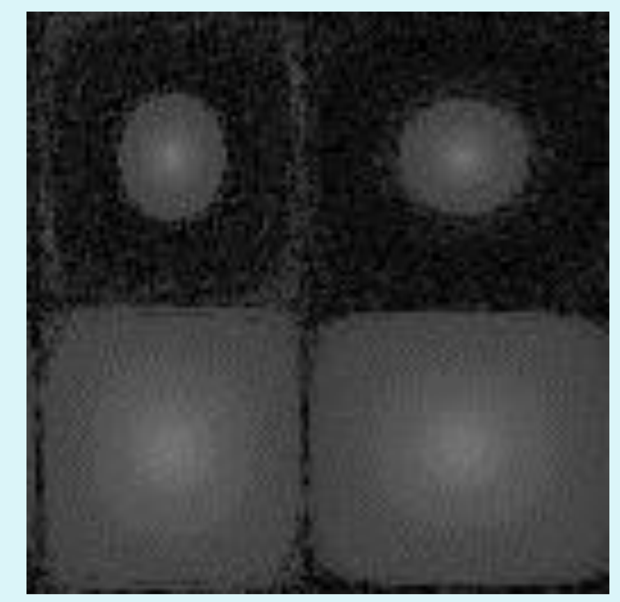
- We varied all u_0, $u_1(0)$ and $h_1(0)$ between -2 and 2.
- u_0 controls the strength of the underlying cellular flow
- u_1 and h_1 control the strength of the time-dependent perturbation of the flow

Results

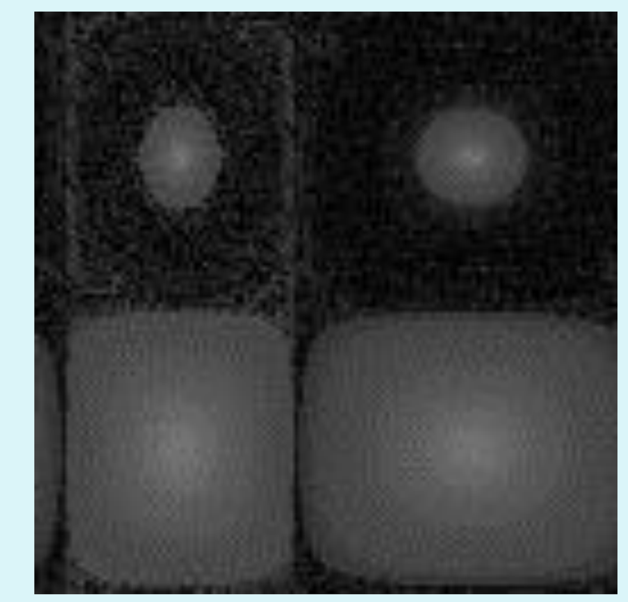
$u_1(0)=0$ $h_1(0)=1$ $v_1(0)=0$ $u_0=1$



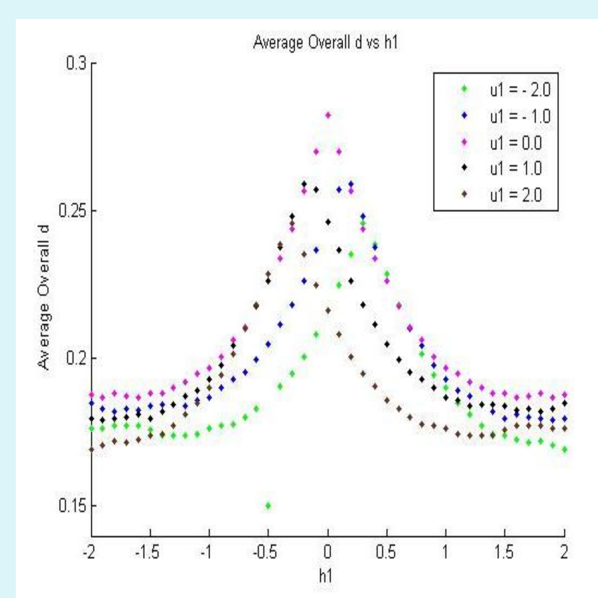
$u_1(0)=1$ $h_1(0)=1$ $v_1(0)=0$ $u_0=1$

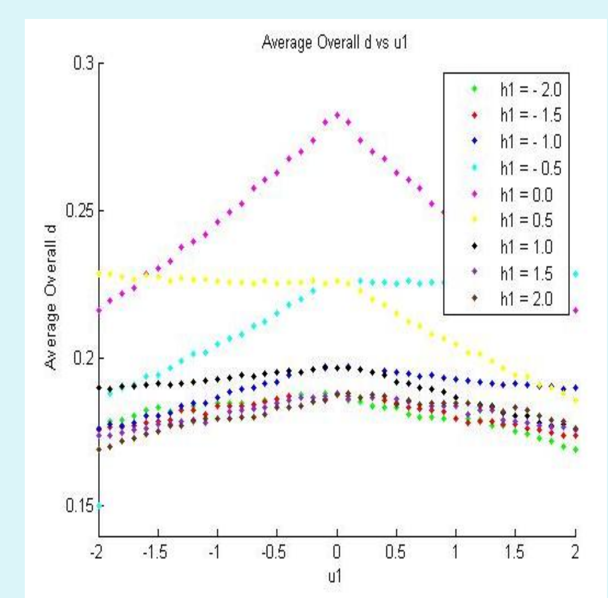


$u_1(0)=2$ $h_1(0)=1$ $v_1(0)=0$ $u_0=1$



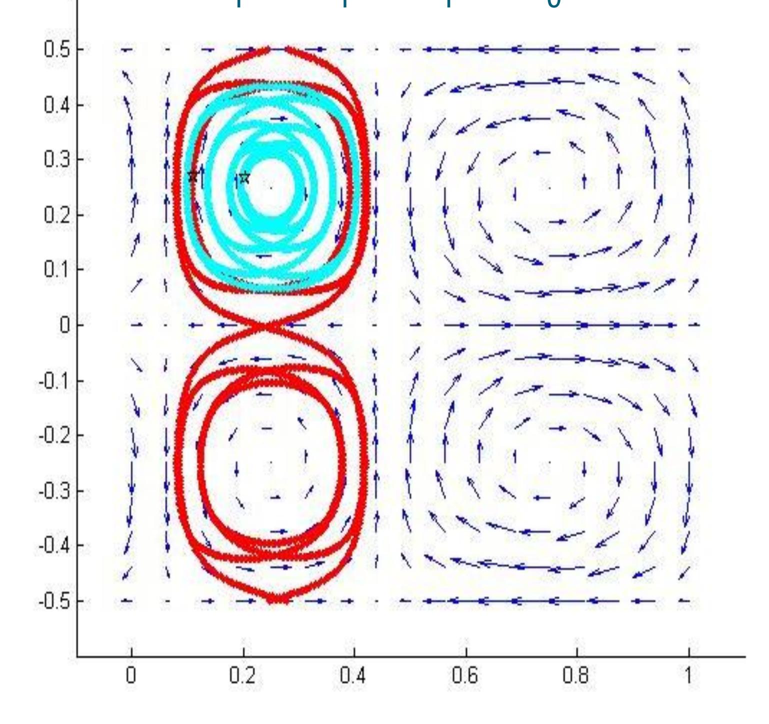
Darker color ↔ lower ergodicity defect ↔ more complex trajectory
The pictures above show the areas that behave similarly.
The light gray areas move little while the trajectories in the dark area sample the space well.





The average ergodicity defect for all of the trajectories of a flow changes as the initial parameters change

$u_1=2$ $h_1=1$ $v_1=0$ $u_0=1$



These two trajectories starting at the stars sample the space differently as show in the above gray scale

Conclusion

The ergodicity defect:

- **Is an indicator of complexity of flow**
- **Can be used to separate areas of differing complexity**
- **Gives a good overall picture of the flows behavior**

Further Work

- Use on other dynamical systems/flows
  - In progress
- Use with other analyzing functions
- Correlate ergodicity defect to other flow properties like mixing

References

1. Apte, A., Jones, C. K., & Stuart, A. M. (2008). A Bayesian approach to Lagrangian data assimilation. *Tellus*(60A), 336-347.
2. Bullitt, E., Gerig, G., Pizer, S. M., Lin, W., & Aylward, S. R. (2003). Measuring tortuosity of the intracerebral vasculature from MRA images. *IEEE Trans Med Imaging, 22*(9), 1163-1171.
3. Scott, S. E., Redd, T. C., Kuznetsov, L., Mezic, I., & Jones, C. K. (2009). Capturing deviation from ergodicity at different scales. *Physica D*(238), 1668-1679.

Acknowledgements

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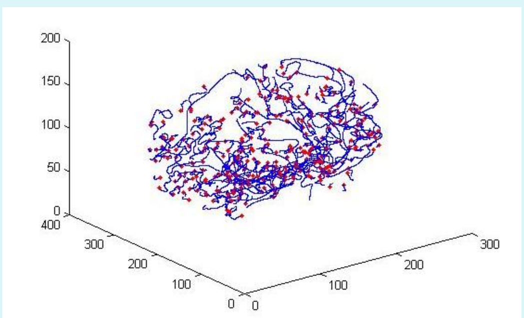
Measuring Tortuosity

Introduction

- Tortuosity describes how curvy a blood vessel is. Highly tortuous vessels are recognized as an important indicator of many cancers.
  - Decreasing tortuosity in tumors may be a sign of effective treatment
- Tortuosity is hard to quantify because there are many ways a vessel can be tortuous.
- Measuring the complexity of the 'trajectory' of a vessel can serve as good measure of its tortuosity.

Methods and Results

- Applied ergodicity defect to synthetic data in[2]
  - 2-D Sine waves and 3-D coils
  - Varying frequency and amplitude
- Ergodicity defect decreased with increasing amplitude, indicating increased complexity
- Currently applying ergodicity defect to real tumor blood vessel data



This image is of blood vessels in the brain

Conclusion and further work

- The ergodicity defect will be able to be used as a measure of tortuosity when the following questions have been addressed:
  - What domain should be used and how should it be mapped for uniformity?
  - What scales would best capture the complexity of the vessels?
  - How can groups of vessels be compared?
  - How to weight the vessels to account for height?
- Apply the ergodicity defect to real data to find the best way to use it to measure tortuosity