Marquette University  
2003

COMPETITIVE SCHOLARSHIP EXAMINATION IN MATHEMATICS

Do not open this booklet until you are directed to do so.

1. Fill out completely the following information about yourself.

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<th>Last name</th>
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Your high school: Name ____________________________ City ____________________________

High School Counselor or Advisor: ____________________________

2. This examination consists of two parts. The time allowed for each will be approximately 60 minutes. Should you finish Part I early, you may proceed to Part II.

3. Part I consists of 20 objective-type questions. Each question has five possible answers marked: A., B., C., D., E. Only one answer is correct. You are to circle the letter corresponding to the correct response for as many problems as you can.

Example: If \( x = 5 \) and \( y = -2 \), then \( x + 4y \) is

- A. \( -3 \)  
- B. \( -2 \)  
- C. \( -1 \)  
- D. \( 0 \)  
- E. \( +1 \).

4. Part II consists of 3 subjective-type questions. Show a summary of your work in this booklet for each question you attempt, whether or not you obtain a complete solution. Scratch paper is provided but be sure to show the essential steps of your work concisely in the space provided for each question. Only the work appearing in this booklet will be scored. You will be scored on your method of attack, ingenuity, insight, inventiveness, and logical developments as well as your solutions.

5. Pencils and scratch paper will be provided. No tables, rulers, compasses, protractors, slide rules, calculators, or other aids are permitted.

6. a. The scoring of questions in Part I has been devised to discourage random guessing and will be computed as follows:

\[
\text{(three times number correct)} - \text{(number wrong)}.
\]

b. The scoring for the three questions in Part II will be 12, 14, and 14 for a total of 40 points. Partial credit will be given so it will be to your advantage to do as much as you are able to do on each question.

7. For the scoring committee. Do not write in the box below.

| Part I: | Part II: | Score on Part I: ❄️
|---------|---------|----------------|
| No. Correct: ❄️ | Score on 1: ❄️ | Score on Part II: ❄️
| No. Wrong: ❄️ | Score on 2: ❄️ |
| Score on 3: ❄️ | TOTAL: ❄️ |
PART I

1. The largest whole number such that seven times the number is less than 100 is
   
   (A) 12
   (B) 13
   (C) 14
   (D) 15
   (E) 16

2. A square and a circle have equal perimeters. The ratio of the area of the circle to the area of the square is
   
   (A) $\frac{4}{\pi}$
   (B) $\frac{\pi}{\sqrt{2}}$
   (C) $\frac{4}{1}$
   (D) $\frac{\sqrt{2}}{\pi}$
   (E) $\frac{\pi}{4}$

3. The fourth power of $\sqrt{1 + \sqrt{1 + \sqrt{1}}}$ is
   
   (A) $\sqrt{2} + \sqrt{3}$
   (B) $\frac{1}{2} \left( 7 + 3\sqrt{5} \right)$
   (C) $1 + 2\sqrt{3}$
   (D) 3
   (E) $3 + 2\sqrt{2}$

4. In the adjoining figure, $CDE$ is an equilateral triangle and $ABCD$ and $DEFG$ are squares. The measure of $\angle GDA$ is
   
   (A) $90^\circ$
   (B) $105^\circ$
   (C) $120^\circ$
   (D) $135^\circ$
   (E) $150^\circ$
5. The number of solutions to \( \{1, 2\} \subseteq X \subseteq \{1, 2, 3, 4, 5\} \), where \( X \) is a set, is

(A) 2
(B) 4
(C) 6
(D) 8
(E) None of these

6. A man walks \( x \) miles due west, turns \( 150^\circ \) to his left and walks 3 miles in the new direction. If he finishes at a point \( \sqrt{3} \) miles from his starting point, then \( x \) is

(A) \( \sqrt{3} \)
(B) \( 2\sqrt{3} \)
(C) \( \frac{3}{2} \)
(D) 3
(E) not uniquely determined by the given information

7. The smallest value of \( x^2 + 8x \) for real values of \( x \) is

(A) \(-16.25\)
(B) \(-16\)
(C) \(-15\)
(D) \(-8\)
(E) None of these

8. If \( 3^{2x} + 9 = 10 \cdot (3^x) \), then the value of \( x^2 + 1 \) is

(A) 1 only
(B) 5 only
(C) 1 or 5
(D) 2
(E) 10
9. If the ratio of \(2x - y\) to \(x + y\) is \(\frac{2}{3}\), what is the ratio of \(x\) to \(y\)?

(A) \(\frac{1}{5}\)

(B) \(\frac{4}{5}\)

(C) 1

(D) \(\frac{6}{5}\)

(E) \(\frac{5}{4}\)

10. If \(x = \frac{1 - i\sqrt{3}}{2}\), where \(i = \sqrt{-1}\), then \(\frac{1}{x^2 - x}\) is equal to

(A) \(-2\)

(B) \(-1\)

(C) \(1 + i\sqrt{3}\)

(D) 1

(E) 2

11. The supermarket has 128 crates of apples. Each crate contains at least 120 apples and at most 144 apples. What is the largest integer \(n\) such that there must be at least \(n\) crates containing the same number of apples?

(A) 4

(B) 5

(C) 6

(D) 24

(E) 25

12. The measures of the interior angles of a convex polygon are in arithmetic progression. If the smallest angle is \(100^\circ\) and the largest angle is \(140^\circ\), then the number of sides the polygon has is

(A) 6

(B) 8

(C) 10

(D) 11

(E) 12
13. The statement $x^2 - x - 6 < 0$ is equivalent to the statement:

(A) $-2 < x < 3$
(B) $x > -2$
(C) $x < 3$
(D) $x > 3$ and $x < -2$
(E) $x > 3$ or $x < -2$

14. A circle with area $A_1$ is contained in the interior of a larger circle with area $A_1 + A_2$. If the radius of the larger circle is 3, and if $A_1, A_2, A_1 + A_2$ is an arithmetic progression, then the radius of the smaller circle is

(A) $\frac{\sqrt{3}}{2}$
(B) 1
(C) $\frac{2}{\sqrt{3}}$
(D) $\frac{3}{2}$
(E) $\sqrt{3}$

15. A painting $18'' \times 24''$ is to be placed into a wooden frame with the longer dimension vertical. The wood at the top and bottom is twice as wide as the wood on the sides. If the frame area equals that of the painting itself, the ratio of the smaller to the larger dimension of the framed painting is:

(A) 1:3
(B) 1:2
(C) 2:3
(D) 3:4
(E) 1:1

16. A parabolic arch has a height of 16 inches and a span of 40 inches. The height, in inches, of the arch at a point 5 inches from the center $M$ is:

(A) 1
(B) 15
(C) $15\frac{1}{3}$
(D) $15\frac{1}{2}$
(E) $15\frac{3}{4}$
17. Let \( n \) be the number of ordered pairs \((x, y)\) of real numbers which satisfy \(5y - 3x = 15\) and \(x^2 + y^2 \leq 16\). Then \( n \) is:

(A) 0
(B) 1
(C) 2
(D) more than two, but finite
(E) greater than any finite number

18. In triangle \(ABC\), the median from vertex \(A\) is perpendicular to the median from vertex \(B\). If the lengths of sides \(AC\) and \(BC\) are 6 and 7 respectively, then the length of side \(AB\) is

(A) \(\sqrt{17}\)
(B) 4
(C) \(4\frac{1}{2}\)
(D) \(2\sqrt{5}\)
(E) \(4\frac{1}{4}\)

19. The remainder \( R \) obtained by dividing \(x^{100}\) by \(x^2 - 3x + 2\) is a polynomial of degree less than 2. Then \( R \) may be written as:

(A) \(2^{100} - 1\)
(B) \(2^{100}(x - 1) - (x - 2)\)
(C) \(2^{100}(x - 3)\)
(D) \(x(2^{100} - 1) + 2(2^{99} - 1)\)
(E) \(2^{100}(x + 1) - (x + 2)\)

20. Quadrilateral \(ABCD\) is inscribed in a circle with side \(AD\), a diameter of length 4. If sides \(AB\) and \(BC\) each have length 1, then side \(CD\) has length

(A) \(\frac{7}{2}\)
(B) \(\frac{5\sqrt{2}}{2}\)
(C) \(\sqrt{11}\)
(D) \(\sqrt{13}\)
(E) \(2\sqrt{3}\)
PART II

1. (a) Graph the solution set $R$ of the following system of inequalities:  

\[
\begin{align*}
& x - (1 + \sqrt{2})y \leq 1 \\
& x - y \geq 0 \\
& x^2 + y^2 \leq 1
\end{align*}
\]

(b) Calculate the area of $R$. 
2. Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other in 4 hours.  

(a) At what time should both candles be lighted so that, at 11 p.m., one stub is twice the length of the other?

(b) Assuming again that both candles are lighted at the same time, is it possible that, at 11 p.m., one stub is one hundred times the length of the other? Explain.
3. Let a sequence \( \{u_n\} \) be defined by \( u_1 = 5 \) and \( u_{n+1} - u_n = 3 + 4(n - 1), \ n = 1, 2, 3 \ldots \) \[\text{14 POINTS}\]

(a) Express \( u_n \) as a polynomial in the variable \( n \).
(b) Prove that \( u_n \leq n^3 + 4 \) for \( n = 1, 2, 3, \ldots \)