Marquette University  
2007

COMPETITIVE SCHOLARSHIP EXAMINATION IN MATHEMATICS

Do not open this booklet until you are directed to do so.

1. Fill out completely the following information about yourself.

   PRINT
   Last name               First name               Initial               Phone No.

   ADDRESS
   Street address          City               State               Zip

   Your high school:       Name __________________________  City ______________________

   High School Counselor or Advisor: ________________________________

2. This examination consists of two parts. The time allowed for each will be approximately 60 minutes. Should you finish Part I early, you may proceed to Part II.

3. Part I consists of 20 objective-type questions. Each question has five possible answers marked: A, B, C, D, E. Only one answer is correct. You are to circle the letter corresponding to the correct response for as many problems as you can.

   Example: If \( x = 5 \) and \( y = -2 \), then \( x + 4y \) is
   \( \text{A} \) \(-3\)  \( \text{B} \) \(-2\)  \( \text{C} \) \(-1\)  \( \text{D} \) \(0\)  \( \text{E} \) \(+1\).

4. Part II consists of 3 subjective-type questions. Show a summary of your work in this booklet for each question you attempt, whether or not you obtain a complete solution. Scratch paper is provided but be sure to show the essential steps of your work concisely in the space provided for each question. Only the work appearing in this booklet will be scored. You will be scored on your method of attack, ingenuity, insight, inventiveness, and logical developments as well as your solutions.

5. Pencils and scratch paper will be provided. No tables, rulers, compasses, protractors, slide rules, calculators, or other aids are permitted.

6. a. The scoring of questions in Part I has been devised to discourage random guessing and will be computed as follows:

   \[
   \text{(three times number correct)} - \text{(number wrong)}.
   \]

   b. The scoring for the three questions in Part II will be 10, 12, and 18 for a total of 40 points. Partial credit will be given so it will be to your advantage to do as much as you are able to do on each question.

7. For the scoring committee. Do not write in the box below.

<table>
<thead>
<tr>
<th>Part I:</th>
<th>Part II:</th>
<th>Score on Part I:</th>
<th>Score on Part II:</th>
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<tr>
<td>No. Correct:</td>
<td>Score on 1:</td>
<td>Score on 2:</td>
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<td>No. Wrong:</td>
<td>Score on 3:</td>
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PART I

1. If \( x \neq 0 \) or \( 4 \) and \( y \neq 0 \) or \( 6 \), then \( \frac{2}{x} + \frac{3}{y} = \frac{1}{2} \) is equivalent to

(A) \( 4x + 3y = xy \)

(B) \( y = \frac{4x}{6-y} \)

(C) \( \frac{x}{2} + \frac{y}{3} = 2 \)

(D) \( \frac{4y}{y-6} = x \)

(E) None of these

2. The sum of the distances from one vertex of a square with sides of length two to the midpoints of each of the sides of the square is

(A) \( 2\sqrt{5} \)

(B) \( 2 + \sqrt{3} \)

(C) \( 2 + 2\sqrt{3} \)

(D) \( 2 + \sqrt{5} \)

(E) \( 2 + 2\sqrt{5} \)

3. If \( x = 1 + 2^p \) and \( y = 1 + 2^{-p} \), then \( y \) in terms of \( x \) is

(A) \( \frac{x + 1}{x - 1} \)

(B) \( \frac{x + 2}{x - 1} \)

(C) \( \frac{x}{x - 1} \)

(D) \( 2 - x \)

(E) \( \frac{x - 1}{x} \)

4. The smallest value of \( x^2 + 8x \) for real values of \( x \) is

(A) \(-16.25\)

(B) \(-16\)

(C) \(-15\)

(D) \(-8\)

(E) None of these
5. The sum of all the integers between 50 and 350 which end in 1 is

(A) 5880
(B) 5539
(C) 5208
(D) 4877
(E) 4566

6. If 1 pint of paint is needed to paint a statue 6 ft. high, then the number of pints it will take to paint (to the same thickness) 540 statues similar to the original but only 1 ft. high, is

(A) 90
(B) 72
(C) 45
(D) 30
(E) 15

7. If the statement "All shirts in this store are on sale." is false, then which of the following statements must be true?

I. All shirts in this store are at non-sale prices.
II. There is some shirt in this store not on sale.
III. No shirt in this store is on sale.
IV. Not all shirts in this store are on sale.

(A) II only
(B) IV only
(C) I and III only
(D) II and IV only
(E) I, II and IV only

8. Find the area of the smallest region bounded by the graphs of \( y = |x| \) and \( x^2 + y^2 = 4 \).

(A) \( \frac{\pi}{4} \)
(B) \( \frac{3\pi}{4} \)
(C) \( \pi \)
(D) \( \frac{3\pi}{2} \)
(E) \( 2\pi \)
9. If \( r \) is positive and the line whose equation is \( x + y = r \) is tangent to the circle whose equation is \( x^2 + y^2 = r \), then \( r \) equals

(A) \( \frac{1}{2} \)
(B) 1
(C) 2
(D) \( \sqrt{2} \)
(E) \( 2\sqrt{2} \)

10. A store prices an item in dollars and cents so that when 4% sales tax is added no rounding is necessary because the result is exactly \( n \) dollars, where \( n \) is a positive integer. The smallest value of \( n \) is

(A) 1
(B) 13
(C) 25
(D) 26
(E) 100

11. Let \( f(t) = \frac{t}{1-t} \), \( t \neq 1 \). If \( y = f(x) \), then \( x \) can be expressed as

(A) \( f \left( \frac{1}{y} \right) \)
(B) \( -f(y) \)
(C) \( -f(-y) \)
(D) \( f(-y) \)
(E) \( f(y) \)

12. Each of the three circles in the adjoining figure is externally tangent to the other two, and each side of the triangle is tangent to two of the circles. If each circle has radius three, then the perimeter of the triangle is

(A) \( 36 + 9\sqrt{2} \)
(B) \( 36 + 6\sqrt{3} \)
(C) \( 36 + 9\sqrt{3} \)
(D) \( 18 + 18\sqrt{3} \)
(E) 45
13. How many pairs \((m, n)\) of integers satisfy the equation \(m + n = mn\)?

(A) 1
(B) 2
(C) 3
(D) 4
(E) more than 4

14. A supermarket has 128 crates of apples. Each crate contains at least 120 apples and at most 144 apples. What is the largest integer \(n\) such that there must be at least \(n\) crates containing the same number of apples?

(A) 4
(B) 5
(C) 6
(D) 24
(E) 25

15. If \(\sin x + \cos x = \frac{1}{5}\) and \(0 \leq x < \pi\), then \(\tan x\) is

(A) \(-\frac{4}{3}\)
(B) \(-\frac{3}{4}\)
(C) \(\frac{3}{4}\)
(D) \(\frac{4}{3}\)
(E) Not completely determined by the given information.

16. A piece of string is cut in two at a point selected at random. The probability that the longer piece is at least \(x\) times as large as the shorter piece (where \(x \geq 1\)) is

(A) \(\frac{1}{2}\)
(B) \(\frac{2}{x}\)
(C) \(\frac{1}{x+1}\)
(D) \(\frac{1}{x}\)
(E) \(\frac{2}{x+1}\)
17. If the line \( y = mx + 1 \) intersects the ellipse \( x^2 + 4y^2 = 1 \) exactly once, then the value \( m^2 \) is

(A) \( \frac{1}{2} \)

(B) \( \frac{2}{3} \)

(C) \( \frac{3}{4} \)

(D) \( \frac{4}{5} \)

(E) \( \frac{5}{6} \)

18. In a geometric series of positive terms the difference between the fifth and fourth terms is 576, and the difference between the second and first terms is 9. What is the sum of the first five terms of this series?

(A) 1061

(B) 1023

(C) 1024

(D) 768

(E) None of these

19. If \( x \) is real and positive and grows beyond all bounds, then \( \log_3(6x - 5) - \log_3(2x + 1) \) approaches:

(A) 0

(B) 1

(C) 3

(D) 4

(E) No finite number

20. The length of the sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. The cosine of the smallest angle is

(A) \( \frac{3}{4} \)

(B) \( \frac{7}{10} \)

(C) \( \frac{2}{3} \)

(D) \( \frac{9}{14} \)

(E) None of these
PART II

1. In the multiplication table of the numbers 1 through 1,000 times 1 through 1,000, how many times does the number 5,184 appear? [10 POINTS]
2. Let $\gamma$ be a circle and let $P$ be a point outside of $\gamma$. Prove that

$$\delta = \{ M \mid M \text{ is the midpoint of } PQ \text{ for some } Q \text{ on } \gamma \}$$

is a circle, and determine the center and radius of $\delta$.  

[12 POINTS]
3. Let $\ell$ be the line in the plane with equation $x = 5$, let $O$ denote the origin $(0, 0)$, and let $d$ be the usual distance function. Let $\beta$ denote the set of all points $P$ such that $d(P, O) = d(P, \ell)$.

(a) Find a rectangular equation (in the variables $x$ and $y$) for $\beta$.

(b) Find a polar equation (in the variables $r$ and $\theta$) for $\beta$.

(c) If $P$ and $Q$ are any two points of $\beta$ such that segment $PQ$ contains $O$, show that

$$\frac{1}{d(P, O)} + \frac{1}{d(Q, O)} = 0.4$$