Marquette University
2006

COMPETITIVE SCHOLARSHIP EXAMINATION IN
MATHEMATICS

Do not open this booklet until you are directed to do so.

1. Fill out completely the following information about yourself.

PRINT

Last name
First name
Initial
Phone No.

ADDRESS

Street address
City
State
Zip

Your high school: Name __________________________ City ________________________

High School Counselor or Advisor: ____________________________________________

2. This examination consists of two parts. The time allowed for each will be approximately 60 minutes. Should you finish Part I early, you may proceed to Part II.

3. Part I consists of 20 objective-type questions. Each question has five possible answers marked: A., B., C., D., E. Only one answer is correct. You are to circle the letter corresponding to the correct response for as many problems as you can.

Example: If \( x = 5 \) and \( y = -2 \), then \( x + 4y \) is

\[ \begin{align*}
A & : \ -3 \\
B & : \ -2 \\
C & : \ -1 \\
D & : \ 0 \\
E & : \ +1
\end{align*} \]

4. Part II consists of 3 subjective-type questions. Show a summary of your work in this booklet for each question you attempt, whether or not you obtain a complete solution. Scratch paper is provided but be sure to show the essential steps of your work concisely in the space provided for each question. Only the work appearing in this booklet will be scored. You will be scored on your method of attack, ingenuity, insight, inventiveness, and logical developments as well as your solutions.

5. Pencils and scratch paper will be provided. No tables, rulers, compasses, protractors, slide rules, calculators, or other aids are permitted.

6. a. The scoring of questions in Part I has been devised to discourage random guessing and will be computed as follows:

\[ (\text{three times number correct}) - (\text{number wrong}) \]

b. The scoring for the three questions in Part II will be 13, 13, and 14 for a total of 40 points. Partial credit will be given so it will be to your advantage to do as much as you are able to do on each question.

7. For the scoring committee. Do not write in the box below.

<table>
<thead>
<tr>
<th>Part I:</th>
<th>Part II:</th>
<th>Score on Part I:</th>
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<tbody>
<tr>
<td>No. Correct:</td>
<td>Score on 1:</td>
<td>Score on Part II:</td>
</tr>
<tr>
<td>No. Wrong:</td>
<td>Score on 2:</td>
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</tbody>
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PART I

1. Let $n = x - y^{x-y}$. Find $n$ when $x = 1$ and $y = -3$.
   (A) 0
   (B) -80
   (C) 16
   (D) 82
   (E) 256

2. If $f(x) = 6^x$ then $f(x + 1) - f(x)$ equals:
   (A) 6
   (B) 2$f(x)$
   (C) 3$f(x)$
   (D) 5$f(x)$
   (E) 6$f(x)$

3. The area of the ring between two concentric circles is $12 \frac{1}{4} \pi$ square inches. The length of a chord of the larger circle tangent to the smaller circle, in inches, is:
   (A) 7
   (B) $5\sqrt{2}$
   (C) $7\sqrt{2}$
   (D) 6
   (E) 5

4. What is the value of $(1+i)^4$, where $i = \sqrt{-1}$?
   (A) 0
   (B) -2
   (C) 4$i$
   (D) 2
   (E) -4
5. What is the smallest prime number dividing $5^7 + 11^{10}$?

(A) 2
(B) 5
(C) 11
(D) $5^7 + 11^{10}$
(E) none of these

6. In the adjoining figure $\angle E = 40^\circ$ and the arcs $AB, BC$, and $CD$ all have equal length. Find the measure of $\angle ACD$

(A) $10^\circ$
(B) $15^\circ$
(C) $20^\circ$
(D) $\left(\frac{45}{2}\right)^\circ$
(E) $30^\circ$

7. If $K = \frac{c}{(x-1)^n}$, then $n$ equals:

(A) $\log \frac{c}{K} - \log(x-1)$
(B) $\frac{\log(c/K)}{\log(x-1)}$
(C) $\log K + \log c - \log x$
(D) $\log \left(\frac{c/K}{x-1}\right)$
(E) $\frac{\log(K-c)}{\log(x-1)}$

8. The number of distinct points common to the curves $x^2 + 4y^2 = 1$ and $x^2 - y = \frac{1}{2}$ is

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
9. A man makes a trip by automobile at an average speed of 60 mph. He returns over the same route at an average speed of 50 mph. His average speed for the entire trip is:

(A) 56
(B) 55
(C) 54
(D) $\frac{540}{11}$
(E) $54\frac{1}{2}$

10. The length $\ell$ of a tangent, drawn from a point $A$ to a circle of radius $r$ is $\frac{3}{4}r$. The (shortest) distance from $A$ to the circle is:

(A) $r$
(B) $\frac{3}{4}r$
(C) $\frac{1}{2}r$
(D) $\frac{1}{4}r$
(E) $\frac{1}{2}\ell$

11. Let $f(x) = ax^2 + bx + c$. If it happens that $c = \frac{b^2}{4a}$. Then the graph of $y = f(x)$ will certainly:

(A) have a maximum
(B) have a minimum
(C) be tangent to the y-axis
(D) be tangent to the x-axis
(E) lie in one quadrant only

12. The area of a square inscribed in a semicircle is to the area of a square inscribed in the entire circle as:

(A) 1 : 2
(B) 2 : 3
(C) 2 : 5
(D) 3 : 4
(E) 3 : 5
13. Given that $0 < x < 1$, and $a = x, b = x^a, c = x^b$, which of the following is true?

(A) $b < a < c$
(B) $c < a < b$
(C) $a < b < c$
(D) $b < a < c$
(E) $a < c < b$

14. Assume the following three statements are true:

I  All freshmen are human          II  All students are human          III  Some students study

Given the following 4 statements:

1) All freshmen are students
2) No freshmen study
3) Some humans study
4) Some freshmen study

Those which are logical consequences of I, II, and III are:

(A) 3
(B) 1 and 4
(C) 1
(D) 4
(E) None of them

15. If $(x + y)^4$ is expanded to the form $Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4$, what is the value of the coefficient $C$?

(A) 4
(B) 6
(C) 1
(D) 8
(E) 2

16. What is the value of $2006^2 - 2005^2$?

(A) 1
(B) 200
(C) 2001
(D) 4011
(E) 4030
17. Bob and Ray cut the grass at their house. It takes Bob alone 5 hours to cut the grass, but working together they finish the job in 3 hours. How many hours would it take Ray alone to cut the grass?

(A) 6
(B) 7
(C) 7\(\frac{1}{2}\)
(D) 8
(E) 8\(\frac{1}{2}\)

18. Triangles \(ABC\) and \(ADC\) are right triangles. Find the value of \(h\) if \(AB = 40\) and \(AC = 30\).

(A) 25
(B) 24
(C) 21
(D) 20
(E) 18

19. A plumber has to replace a large 24 inch diameter pipe with smaller 6 inch diameter pipes. In order to carry the same amount of water, how many smaller pipes must be used?

(A) 4
(B) \(4\pi\)
(C) 12
(D) 16
(E) 20

20. A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed three times, what is the probability of getting 3 heads?

(A) \(\frac{1}{9}\)
(B) \(\frac{1}{27}\)
(C) \(\frac{1}{8}\)
(D) \(\frac{2}{3}\)
(E) \(\frac{8}{27}\)
PART II

1. Suppose a cross is built by assembling two boards $2 \times 2 \times 10$ inches, as shown below. What is the total surface area of this cross?
2. In the trendy puzzle Sudoku, you have a $9 \times 9$ grid divided into nine $3 \times 3$ boxes (shown by darker lines). A solution will have each of the digits 1–9 appearing once in each row, column, and box. For this question, compute the number of ways that the nine 1’s can be arranged in a solution. That is, how many ways can you place nine 1’s in this grid so that exactly one appears in each row, column, and box? One example of such a placement is shown.

[13 POINTS]
3. Show that there are no integers $a$ and $b$ such that $a^2 - b^2 = 2006$. [14 POINTS]