Semidistributive Inverse Semigroups

Katherine G. Johnston-Thom and Peter R. Jones

February 9, 1999

The study of subalgebra lattices of algebras has a long and fruitful history. In particular, the subsemigroup lattices of semigroups have been the subject of continued investigation, especially by L.N. Shevrin and his colleagues (see [10] for a recent survey). Inverse semigroups whose lattice $\mathcal{L}$ of inverse subsemigroups lattices were distributive, modular, etc. are extremely restricted in nature (see [1] and [10, Section 17]) and with this in mind the second author initiated the study of the lattice $\mathcal{LF}(S)$ (or just $\mathcal{LF}$) of *full* inverse subsemigroups (those containing all the idempotents) of an inverse semigroup S and described those inverse semigroups for which $\mathcal{LF}$ is distributive in [6]. The prototype for this class is the bicyclic semigroup. The authors of the current paper similarly described the inverse semigroups for which $\mathcal{LF}$ is modular in [4]. In the first paper on the topic, progress was made on describing the inverse semigroups for which $\mathcal{LF}$ is semimodular, (for instance, the free inverse semigroups have this property) but a full answer remains hidden from view at this point. See [7] for a survey on $\mathcal{LF}$ and $\mathcal{L}$.

In this paper we describe the inverse semigroups for which $\mathcal{LF}$ is either *meet semidistributive* or *join semidistributive*, terms that are generally referred to as $SD(\wedge)$ and $SD(\vee)$, respectively. A lattice satisfies $SD(\wedge)$ if the following implication holds: $a \wedge b = a \wedge c \Rightarrow a \vee (b \wedge c) = a \wedge b$; $SD(\vee)$ is just its lattice-theoretic dual. These important lattice-theoretic implications are satisfied by free lattices, for instance [2]. We shall show that in fact each of meet semidistributivity and join semidistributivity of $\mathcal{LF}$ is equivalent to distributivity. Sufficiency is trivial, but the proof of necessity is certainly not so. This leads us to study an interesting Munn semigroup T_Y, where the semilattice Y is shown in Figure 1, Section 3.

For groups, both $\mathcal{LF}$ and $\mathcal{L}$ coincide with the lattice of subgroups. It was shown by Shiryaev [11] that in this case, $SD(\wedge)$ and distributivity are equivalent, and by a well-known result, equivalent to being locally cyclic (see the monograph on subgroup lattices by Suzuki [12]). Equivalence of $SD(\vee)$ and distributivity for subgroup lattices of groups was shown by F. Napolitani [8]. (See Lemma 2.6 for a proof.) However, it was shown by Shiryaev in the same paper that for subsemigroup lattices of semigroups, $SD(\wedge)$ and distributivity are distinct (albeit closely related). In other work, the same author studied

$SD(\vee)$ for those lattices and demonstrated a rather different set of criteria (see [10] pp 37–38 for a summary). So on the one hand our result is not entirely surprising. But on the other, there is no reason, *a priori*, to assume that it should hold.

It is convenient to use the terminology suggested by the paper's title: we call an inverse semigroup a $\mathcal{P}$-inverse semigroup if the lattice $\mathcal{LF}$ has property $\mathcal{P}$.

In outline, our argument in Section 2 follows those in [5, 6, 4]: $\mathcal{LF}(S)$ is a subdirect product of the lattices of full inverse subsemigroups of its principal factors, which are either simple or 0-simple inverse semigroups. Since $SD(\wedge)$ and $SD(\vee)$, being implicational, are preserved by sublattices and direct products, we are reduced to considering [0]-simple inverse semigroups. Completely 0-simple inverse semigroups are easily dealt with. It is then shown that a 0-simple inverse semigroup S that is either meet or join semidistributive has no zero-divisors, and that $\mathcal{LF}(S) \cong \mathcal{LF}(S \setminus \{0\})$, so that only the case where S is simple need concern us. The remainder of Section 2 then shows that such a semigroup satisfies the sufficient conditions for distributivity given by our main preliminary result, Result 1.1, which follows some preliminaries. Section 3 investigates the properties of an interesting Munn semigroup, which plays a key role in the main theorem.

It was of some surprise to the authors that, despite the inherent non-duality of $\mathcal{LF}$, almost every step in our proofs applies equally well to either of the hypotheses $SD(\wedge)$ or $SD(\vee)$.

1 Preliminaries

The reader is referred to [9] for properties of inverse semigroups, especially to Chapter 9 for information on monogenic inverse semigroups. We standardize some notation, at this point. Let S be an inverse semigroup. For any subset A of S, E_A will denote the set of idempotents of S that belong to A. An inverse subsemigroup is *full* if it contains E_S. Following [4], we shall denote by $\langle A \rangle$ the *full* inverse subsemigroup of S generated by a set A. Since we shall have several occasions to refer to the inverse subsemigroup that A generates, we need an alternative notation: $\langle\langle A \rangle\rangle$. Hence $\langle A \rangle = \langle\langle A \cup E_S \rangle\rangle$.

One difference in terminology from [9] is that we use the more modern term *aperiodic*, instead of "combinatorial", to describe inverse semigroups in which all subgroups are trivial.

An element a of an inverse semigroup is said to be *right regular* if $aa^{-1} \geq a^{-1}a$ and *strictly right regular* if strict inequality holds. In the latter case it is well known that the inverse subsemigroup $\langle\langle a \rangle\rangle$ generated by a is bicyclic, with identity aa^{-1}. (The dual definition also yields a bicyclic semigroup, with dual identity element.) Following [6] we say that E_S is *Archimedean* in S if for every strictly right regular element a and idempotent e of S, $a^{-n}a^n \leq e$ for some $n \geq 1$.

RESULT 1.1 *[6] A simple inverse semigroup S (not a group) is distributive if and only if*

(I) it is aperiodic;

(II) its semilattice of idempotents is Archimedean in S and the idempotents of each $\mathcal{D}$-class of S are totally ordered;

(III) its maximum group quotient is locally cyclic.

It was also shown in [6] that in that case, $\mathcal{LF}$ is a certain "contracted" direct product of the lattice of ideals of its semilattice of idempotents with the subgroup lattice of its maximal group quotient. In [4, Theorem 5.3] the authors went on to completely determine the *bisimple* distributive inverse semigroups, in terms of subgroups of the rationals.

A consequence of Result 1.1 is that any simple distributive inverse semigroup S is E-unitary: the least group congruence σ identifies no nonidempotents with idempotents. See [9] for various conditions equivalent to E-unitariness. We shall prove early in the next section that a simple meet semidistributive inverse semigroup is necessarily E-unitary. The next preliminary result will then be useful. We note that Corollary 3.7 of [6] follows immediately.

PROPOSITION 1.2 *Let S be any E-unitary inverse semigroup and T any inverse subsemigroup. Then $\mathcal{LF}(T)$ embeds in $\mathcal{LF}(S)$.*

Proof. Map any full inverse subsemigroup U of T to its join $U \vee E_S$ in $\mathcal{L}(S)$. Clearly $U \vee E_S \in \mathcal{LF}(S)$ and, since joins in $\mathcal{LF}(S)$ are those in $\mathcal{L}(S)$, the map is join-preserving.

To show it is intersection-preserving, we need to show that $(U \vee E_S) \cap (V \vee E_S) \subseteq (U \cap V) \vee E_S$, for all $U, V \in \mathcal{LF}(T)$. So let x belong to the left-hand subsemigroup. It is easily seen that $x = eu = fv$, for some $e, f \in E_S$, $u \in U$, $v \in V$. It follows that $u\sigma v$ and $uu^{-1}v\sigma vv^{-1}u$. Since it is easily seen that $uu^{-1}v\mathcal{R}vv^{-1}u$, we obtain $uu^{-1}v = vv^{-1}u = w$, say, using the fact that E-unitary inverse semigroups are characterized by the property $\mathcal{R} \cap \sigma = 1$ [3]. Now $w \in U \cap V$, since U and V are full in T, and $x = xx^{-1}w$, whence x belongs to the right-hand subsemigroup.

Finally, if $U \in \mathcal{LF}(T)$, let $x \in (U \vee E_S) \cap T$, so that $x = eu = xx^{-1}u$, for some $e \in E_S$, $u \in U$. Since $x \in T$, $xx^{-1} \in T$. But U is full in T, so $x \in U$. Thus $(U \vee E_S) \cap T = U$ and this immediately implies injectivity of the given map.

The following lemma, on the generation of full inverse subsemigroups, will be used repeatedly. Its proof is easy and its essence may be found in [5].

LEMMA 1.3 *Let S be an inverse semigroup and $a \in S \setminus E_S$. If b is a nonidempotent in the full inverse subsemigroup $\langle a \rangle$ generated by a, then $b = (bb^{-1})a^n$, for some nonzero integer n; if $n < 0$, then $b^{-1} = (b^{-1}b)a^{-n}$.*

Only elementary lattice theory will be needed in this paper. It is clear that since $SD(\wedge)$ and $SD(\vee)$ are defined by implications, they are preserved by sublattices and direct products. Each is also clearly a consequence of distributivity itself. Less clear is the following preservation property of $SD(\wedge)$. Whether the analogue holds for $SD(\vee)$ we do not know, leading to the one step in the proof of the main theorem in which the arguments differ.

LEMMA 1.4 *Let L be a complete meet semidistributive lattice and $\phi : L \to M$ be a surjective lattice morphism that preserves complete joins. Then M is meet semidistributive.*

Proof. For each $m \in M$, let m' be the greatest element of $m\phi^{-1}$. Note that the map $m \to m'$ is order-preserving: for if $m \leq n$ in M, then we have $(m' \vee n')\phi = m'\phi \vee n'\phi = m \vee n = n$, whence $m' \vee n' \leq n'$, that is, $m' \leq n'$. It now follows that $(m \wedge n)' \leq m', n'$ and the reverse inequality follows from the definition, that is, the map is meet-preserving.

Now let $m, n, r \in M$, with $m \wedge n = m \wedge r$. Then $m' \wedge n' = m' \wedge r'$ and from $SD(\wedge)$ in L, we have $m' \wedge (n' \vee r') = m' \wedge n'$. Applying ϕ yields $m \wedge (n \vee r) = m \wedge n$.

The following technical lemma is at the heart of almost every argument that follows. The notation $a\|b$ means that a and b are incomparable.

PROPOSITION 1.5 *Suppose the lattice L contains a, b, c that satisfy: $a \wedge b = a \wedge c$, $b \vee a = b \vee c$ and $a\|b$: then L is neither meet nor join semidistributive. In particular, this is the case when L has a zero, $a, b \neq 0$ and $a \wedge b = 0 = a \wedge c$ and $b \vee a = b \vee c$.*

Proof. The final statement follows immediately from the first. Now suppose a, b, c are as in that first statement. Then $a \wedge (b \vee c) = a \wedge (b \vee a) = a \neq a \wedge b$, contradicting $SD(\wedge)$. But the statement is self-dual.

COROLLARY 1.6 *In a modular lattice, $SD(\wedge)$, $SD(\vee)$ and distributivity are equivalent.*

Proof. A modular lattice is nondistributive if and only if it contains a copy of the five-element bounded lattice $\{0, 1, a, b, c\}$ with atoms a, b, c. This sublattice satisfies the conditions in the proposition – and $SD(\wedge)$ and $SD(\vee)$ are inherited by sublattices.

2 The main results

The main result of the paper is the following.

THEOREM 2.1 *The following are equivalent for an inverse semigroup S:*
(i) S is meet semidistributive;
(ii) S is join semidistributive;
(iii) S is distributive.

The steps in the proof were outlined in the introduction. We first reduce the problem to the case of simple and 0-simple inverse semigroups.

RESULT 2.2 *Let S be an inverse semigroup. Then*
(i) [5] $\mathcal{LF}(S)$ is isomorphic to a subdirect product of the lattices of full inverse subsemigroups of its principal factors; and each of the latter lattices is isomorphic to an interval sublattice of the $\mathcal{LF}(S)$.
(ii) Hence S is meet semidistributive if and only if each of its principal factors is meet semidistributive, and similarly for join semidistributivity.

The case when S is a group was dealt with in the introduction. If $S = G^0$ is a group with adjoined zero, then clearly $\mathcal{LF}(S) \cong \mathcal{L}(G)$. Now suppose S is completely 0-simple, that is, a Brandt semigroup.

PROPOSITION 2.3 *Let S be a completely 0-simple inverse semigroup, not a group with adjoined zero. The following are equivalent:*
(i) S is meet semidistributive;
(ii) S is join semidistributive;
(iii) S is aperiodic with exactly two nonzero idempotents;
(iv) S is distributive.

Proof. Suppose S is either meet or join semidistributive and let e, f be distinct nonzero idempotents. Let $a \in R_e \cap L_f$. It is easily calculated that $\langle a \rangle = E_S \cup \{a, a^{-1}\}$. If $x \in H_e$ then $\langle xa \rangle = E_S \cup \{xa, (xa)^{-1}\}$, where $xa\mathcal{H}a$, and so $\langle x \rangle \cap \langle a \rangle = E_S = \langle x \rangle \cap \langle xa \rangle$. Now since $x = (xa)a^{-1}$, $\langle a \rangle \vee \langle x \rangle = \langle a \rangle \vee \langle xa \rangle$. By (the latter statement of) Proposition 1.5, $\langle x \rangle = E_S$, that is, $x \in E_S$.

Now suppose that g is third nonzero idempotent. Let $b \in R_f \cap L_g$ and $c = ab \in R_e \cap L_g$. Then $\langle b \rangle = E_S \cup \{b, b^{-1}\}$ and $\langle c \rangle = E_S \cup \{c, c^{-1}\}$. It follows that $\langle a \rangle \cap \langle b \rangle = E_S = \langle a \rangle \cap \langle c \rangle$. But from $a = cb^{-1}$ we also have $\langle b \rangle \vee \langle a \rangle = \langle b \rangle \vee \langle c \rangle$. Thus Proposition 1.5 yields a contradiction.

That (iii) implies (iv) was shown in [5].

It only remains to consider the [0]-simple inverse semigroups that are not completely [0]-simple. The next result reduces one step further, to the heart of the problem.

LEMMA 2.4 *Let S be a 0-simple inverse semigroup that is not completely 0-simple. If S is either meet or join semidistributive then it has no zero divisors and the simple inverse subsemigroup $S \setminus \{0\}$ is either meet or join semidistributive, respectively.*

Proof. Suppose S has zero divisors. Then there exist nonzero idempotents e, f whose product is zero. By 0-simplicity, (see [3, Lemma 5.7.1] there exists an idempotent less than e and $\mathcal{D}$-related to f, we may assume that $e\mathcal{D}f$. So there exists $a \in R_e \cap L_f$. Again, there exists an idempotent g, say such that $g\mathcal{D}e$ and $g < e$. Let $b \in R_e \cap L_g$ and $c = a^{-1}b$, an element of $R_f \cap L_g$.

We show $\langle b\rangle \cap \langle a\rangle = E_S = \langle b\rangle \cap \langle c\rangle$ which, together with the equation $\langle a\rangle \vee \langle b\rangle = \langle a\rangle \vee \langle c\rangle$, (based on $b = aa^{-1}b = ac$) contradicts Proposition 1.5.

To show the first of the two equalities, suppose that x is a nonidempotent in $\langle b\rangle \cap \langle a\rangle$. Then, applying Lemma 1.3, $x = (xx^{-1})b^k$, for some positive k, without loss of generality (by replacing x by its inverse, if necessary), so that $x \in eSg$; similarly, $x = (xx^{-1})a^n$, whence $x \in eSf$, if $n > 0$, or $x \in fSe$, if $n < 0$. If $x \in eSg \cup eSf$, then $x = xg = (xf)g = 0$, and likewise if $x \in eSg \cup fSe$, then $x = ex = e(fx) = 0$. The second equality follows from a dual argument, with gSf and fSg appearing in place of eSf and fSe.

The map $A \to A \cup \{0\}$ is easily seen to be an isomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(S \setminus \{0\})$.

In the remainder of this section we shall show that a meet or join semidistributive simple inverse semigroup (that is not a group) satisfies the conditions (I) - (III) of Result 1.1, completing the proof of Theorem 2.1.

LEMMA 2.5 *Let S be a simple inverse semigroup that is either meet or join semidistributive. Then it is E-unitary.*

Proof. Suppose S is not E-unitary. Hence S is not a group and it contains a nonidempotent a and an idempotent g such that $ga = g = ag$. Put $e = aa^{-1}$ and $f = a^{-1}a$. By simplicity, S contains an idempotent $\mathcal{D}$-related to e and below eg, so that we may assume that $g < e$ and $g\mathcal{D}e$. Let $b \in R_e \cap L_g$ and let $c = a^{-1}b$.

The proof is similar to that of the previous lemma: we show $\langle b\rangle \cap \langle a\rangle = E_S = \langle b\rangle \cap \langle c\rangle$, which together with the equation $\langle a\rangle \vee \langle b\rangle = \langle a\rangle \vee \langle c\rangle$, contradicts Proposition 1.5.

To show the the first of the two equalities, we once again suppose that x is a nonidempotent in $\langle b\rangle \cap \langle a\rangle$, so that $x = (xx^{-1})b^k$, for some $k > 0$, without loss of generality, and $x \in eSg$; and $x = xx^{-1}a^n$, for some $n \neq 0$. Since $a^n g = g$, we obtain $x = xg = xx^{-1}g \in E_S$, another contradiction. Again, the second equality follows by a dual argument.

LEMMA 2.6 *Let S be an E-unitary inverse semigroup that is either meet or join semidistributive. Then S/σ is locally cyclic (and so abelian).*

Proof. In either case we make use of [4, Proposition 1.6], which when specialized to E-unitary inverse semigroups states that the map $\mathcal{LF}(S) \to \mathcal{L}(S/\sigma)$ induced by the natural morphism $S \to S/\sigma$ is a lattice morphism. This morphism clearly preserves complete joins.

For $SD(\wedge)$ the claim now follows from Lemma 1.4 and the result of Shiryaev, for groups, cited in the introduction.

For $SD(\vee)$ we do not have the analogue of that lemma and proceed directly instead. We first prove that under the hypothesis on S, S/σ is abelian. So let $a, b \in S/\sigma$. Then there exist $x\mathcal{R}y$ in S such that $x\sigma = a$, $y\sigma = b$. (For instance, if $x'\sigma = a$ and $y'\sigma = b$, put $x = y'(y')^{-1}x'$ and $y = x'(x')^{-1}y'$.) Now since $x(x^{-1}y) = y$ and $(x^{-1}y)y^{-1} = x^{-1}$, we have $\langle x, y\rangle = \langle x^{-1}y\rangle \vee \langle x\rangle = \langle x^{-1}y\rangle \vee \langle y\rangle$ and so by $SD(\vee)$ in S, $\langle x, y\rangle = \langle x^{-1}y\rangle \vee (\langle x\rangle \cap \langle y\rangle)$. Now by [4, Proposition 1.6], the natural morphism $S \to S/\sigma$ induces a lattice morphism of $\mathcal{LF}(S)$ upon $\mathcal{L}(S/\sigma)$, and so $\langle a, b\rangle = \langle a^{-1}b\rangle \vee (\langle a\rangle \cap \langle b\rangle)$. But $\langle a\rangle \cap \langle b\rangle$ is a subgroup of the centre of $\langle a, b\rangle$ and thus a normal subgroup of $\langle a, b\rangle$, so that $\langle a, b\rangle = \langle a^{-1}b\rangle(\langle a\rangle \cap \langle b\rangle)$. Then the factor group $\langle a, b\rangle/\langle a\rangle \cap \langle b\rangle$ is isomorphic to the cyclic group $\langle a^{-1}b\rangle/(\langle a\rangle \cap \langle b\rangle \cap \langle a^{-1}b\rangle)$. Hence the factor group of $\langle a, b\rangle$ by its centre is also cyclic and so, by a well known theorem, $\langle a, b\rangle$ is abelian.

To show S/σ locally cyclic, again let $a, b \in S/\sigma$. Since S/σ is abelian we may assume, without loss of generality, that $\langle a, b\rangle$ is the internal direct product of $\langle a\rangle$ and $\langle b\rangle$, that is, that $\langle a\rangle \cap \langle b\rangle = 1$. Now choosing $x, y \in S$ and proceeding as above, we obtain $\langle a, b\rangle = \langle a^{-1}b\rangle$.

We observe that our proof includes a proof that $SD(\vee)$ implies distributivity for groups (but for $SD(\wedge)$ uses the result of Shiryaev).

LEMMA 2.7 *Let S be a simple inverse semigroup that is either meet or join semidistributive, but is not a group. Then it is aperiodic and E_S is Archimedean in S.*

Proof. On the one hand, suppose E_S is not Archimedean in S. Then there is a strictly right regular element a, with $aa^{-1} = e$, say, and an idempotent g such that $g \not\geq a^{-n}a^n$, for all $n > 0$. Since S is simple, we may assume without loss of generality that $g\mathcal{D}e$ and $g < e$. If, on the other hand, S is not aperiodic, choose a nonidempotent a in a subgroup H_e, say; then there exists an idempotent g such that $g\mathcal{D}e$ and $g < e$.

In either case, we therefore have an element a such that $e = aa^{-1} \geq a^{-1}a$ and an idempotent $g < e$, $\mathcal{D}$-related to e and such that $g \not\geq a^{-n}a^n$ for all $n > 0$.

Let $b \in R_e \cap L_g$. We show that $\langle a\rangle \cap \langle b\rangle = E_S$. Suppose otherwise: then, as above, we may assume a nonidempotent x such that $x = xx^{-1}b^k$, for some $k > 0$ and $x = xx^{-1}a^n$ for some $n \neq 0$. Then $b^k\sigma a^n$. Suppose $n > 0$. By hypothesis, both b^k and a^n lie in R_e and since S is E-unitary, they are equal. However, this yields the contradiction $g = b^{-1}b > b^{-k}b^k = a^{-n}a^n$. Alternatively, $n < 0$, in which case

$a^{-n}a^n = e$ and so $a^{-n}b^k \in R_e$ and by E-unitariness $a^{-n}b^k = e$, yielding the contradiction $g = b^{-1}b > b^{-k}b^k \geq e \geq a^{-1}a$.

Consider the idempotent $h = (ab)^{-1}ab$. Since $h \leq g$ and $ab \in R_e \cap L_h$, h has the same properties that g has and by the argument of the previous paragraph, $\langle a \rangle \cap \langle ab \rangle = E_S$.

But $a = ae = (ab)b^{-1}$ and so $\langle b \rangle \vee \langle a \rangle = \langle b \rangle \vee \langle ab \rangle$, contradicting Proposition 1.5.

It remains to prove that the idempotents of each $\mathcal{D}$-class form a chain in a simple inverse semigroup that is either meet or join semidistributive. This is by far the most difficult step. We show first that the *width* $w(E_D)$ (the maximum number of mutually incomparable idempotents) of the poset E_D, under the natural order, is at most two. The incomparability relation on E_S is denoted $\|$, as for lattices.

LEMMA 2.8 *Let S be a simple inverse semigroup that is either meet or join semidistributive. Then $w(E_D) \leq 2$ for each of its $\mathcal{D}$-classes D.*

Proof. By Lemma 2.5 S is E-unitary. Suppose e, f, g are mutually incomparable $\mathcal{D}$-related idempotents; let $a \in R_e \cap L_f$, $b \in R_f \cap L_g$ and $c = ab$, so that $ab \in R_e \cap L_g$.

Let $T = \langle\langle a, b \rangle\rangle$, the inverse subsemigroup generated by a and b. We shall show that T is neither meet nor join semidistributive, contradicting Proposition 1.2. Let us consider products w of a's and b's that lie in R_e (and so begin with a). Since $aa^{-1} \| a^{-1}a$, we have $a^2, aa^{-2} \notin R_e$ so that if $w \in \langle\langle a \rangle\rangle$ then $w \in \{a, aa^{-1}\}$. From the given incomparabilities we also see that $aa^{-1}b, aa^{-1}b^{-1}, ab^{-1}, ab^2 \notin R_e$: this leaves $T \cap R_e = \{aa^{-1}, a, ab = c\}$ (since $abb^{-1} = a$).

It follows that the principal factor for the $\mathcal{J}$-class J_a in T is completely 0-simple with three nonzero idempotents. By Proposition 2.3, T is neither meet nor join semidistributive.

The next lemma will complete the proof of our main theorem. However, all the real work is deferred to the next section of the paper. According to Theorem 3.4 and the results of the current section, if $w(E_D) = 2$ for some $\mathcal{D}$-class D of S then S contains an isomorphic copy of the monoid M with presentation $\langle\langle z, e | zz^{-1} \geq z^{-1}z,\ zz^{-1} \geq e,\ e^2 = e,\ ez^{-1}z = z^{-2}z^2 \rangle\rangle$. However, by Theorem 3.3 M is neither meet nor join semidistributive, contradicting Proposition 1.2.

LEMMA 2.9 *Let S be a simple inverse semigroup that is either meet or join semidistributive. Then $w(E_D) = 1$, that is, E_D is a chain, for each $\mathcal{D}$-class D. Hence S is distributive.*

Note that according to Theorem 3.2 and Theorem 3.3, the condition that the idempotents of each $\mathcal{D}$-class form a chain is not a consequence of the other conditions.

We conclude this section with a somewhat technical lemma that will be used in the next section. We include it at this point because in the case when S is *bisimple*, the results of the following section may be circumvented and the conclusion of the previous lemma drawn almost immediately, as in the corollary below.

LEMMA 2.10 *Let S be an E-unitary inverse semigroup, with S/σ abelian. Then*

(i) If b is right regular and $e \in E_S$ satisfies $e > bb^{-1}$ and $e\mathcal{D}bb^{-1}$, then R_e contains a right regular element c such that $c\sigma b$. Hence

(ii) If $aa^{-1} \| a^{-1}a$ and $g \in E_S$ satisfies $g < aa^{-1}$ and $g\mathcal{D}aa^{-1}$ then $(ga)(ga)^{-1} \| (ga)^{-1}(ga)$.

Proof. (i) Since b is right regular, $b^2b^{-2} = bb^{-1}$. Now choose $d \in R_e \cap L_{bb^{-1}}$ and put $c = dbd^{-1}$. Since S/σ is abelian, $c\sigma b$ (and so $c \notin E_S$). Now $cc^{-1} = dbd^{-1}db^{-1}d^{-1} = dbbb^{-1}b^{-1}d^{-1} = dbb^{-1}d^{-1} = dd^{-1}dd^{-1} = dd^{-1} = e$; and $c^{-1}c \leq dd^{-1} = e$.

(ii) Suppose, on the contrary and without loss of generality, that ga is right regular. Then, with $e = aa^{-1}$, (i) implies that R_e contains a right regular element c, σ-related to ga and hence to a. But E-unitariness gives $c = a$, contradicting the assumption on a.

COROLLARY 2.11 *If S is a bisimple inverse semigroup that is either meet or join semidistributive, then E_D is a chain for each $\mathcal{D}$-class D. Hence S is distributive.*

Proof. Suppose S contains incomparable $\mathcal{D}$-related idempotents e, f. Let $a \in R_e \cap L_f$. Since $aa^{-1} \| a^{-1}a$ and $a^2\mathcal{D}a$, we apply (ii) of the preceding lemma, with $g = a^2a^{-2}$, to obtain $a^2a^{-2} \| (aa^{-1})(a^{-1}a)$. However this forces the idempotents a^2a^{-2}, $(aa^{-1})(a^{-1}a)$ and $a^{-2}a^2$ to be mutually incomparable, contradicting Lemma 2.8.

3 The inverse monoid M

In this section we investigate the properties of the inverse monoid M that was defined in the proof of Lemma 2.9. In view of its representation as the Munn semigroup of a particularly interesting semilattice (see Figure 1 below), it has some independent interest, so the first part, at least, of this section can be read independently of the rest of the paper.

As in the rest of the paper, if S is an inverse semigroup and X a subset of S then the inverse subsemigroup of S generated by X is denoted $\langle\langle X \rangle\rangle$, while the full inverse subsemigroup that it generates is denoted $\langle X \rangle$. The notation $(X|R)$ stands for an inverse semigroup presentation, where X is a nonempty set and R is a subset of $FI_X \times FI_X$, FI_X being the free inverse semigroup on X. Then $\langle\langle X|R \rangle\rangle$ is the inverse semigroup defined by this presentation: the quotient of FI_X by the congruence generated by R. Since it should not cause confusion, we shall use the same notation for the elements of FI_X and their classes in the quotient. Relations of the form $ef = f$, where e, f are idempotents of FI_X, will often be abbreviated as $e \geq f$.

We use the notational conventions of formal languages to describe certain subsets of inverse semigroups S. Thus X^+ denotes the subsemigroup generated by X and, if S is a monoid, X^* denotes the submonoid so generated (and if $X = \{x\}$, we use the respective abbreviations x^+ and x^*).

To begin, we repeat the definition of the monoid under consideration. Let

$$M = \langle\langle z, e | zz^{-1} \geq z^{-1}z,\ zz^{-1} \geq e,\ e^2 = e,\ ez^{-1}z = z^{-2}z^2 \rangle\rangle.$$

From the first relation it follows that z generates (as an inverse subsemigroup) a quotient of the bicyclic monoid with zz^{-1} as identity. Hence $\langle\langle z \rangle\rangle = z^{-1^*}z^*$. Further, in combination with the second relation we see that M is a monoid with identity $zz^{-1} = 1$. From the final relation we have that $ez^{-1} = z^{-2}z^2z^{-1} = z^{-1}(z^{-1}z)(zz^{-1})$ and obtain:

$$ez^{-1} = z^{-2}z \qquad \text{and thus} \qquad ze = z^{-1}z^2 \tag{1}$$

From these equations it is clear that any product of z's and e's in M may be rewritten so that any noninitial e is preceded by z^{-1} and any nonterminal e is succeeded by z, so that, using the fact that $zz^{-1} = 1$, at most one e need appear. Hence we have the following description of the elements of M, which will turn out to be canonical.

PROPOSITION 3.1 $M = z^{-1^*}\{e, 1\}z^*$.

Consider the semilattice Y shown in Figure 1 below. Its Munn semigroup T_Y consists of the isomorphisms between principal ideals of Y, under composition of partial mappings. See [3] for more details and examples. Denote by $\gamma : Yk_0 \to Yk_1$ the "glide reflection " that maps k_i to k_{i+1} and g_i to g_{i+1}, for all $i \geq 0$; and by ϵ the identity automorphism 1_{Yg_0}. It is clear that the relations in the presentation of M are satisfied, with $z = \gamma$ and $e = \epsilon$. Thus $\langle\langle \gamma, \epsilon \rangle\rangle = \gamma^{-1^*}\{\epsilon, 1\}\gamma^*$ Now $\gamma^{-n}\gamma^n = 1_{Yk_n}$ and $\gamma^{-n}\epsilon\gamma^n = 1_{Yg_n}$, $n \geq 1$, whence $\gamma^{-n}\epsilon\gamma^{n+1} : Yg_n \to Yg_{n+1}$, $n \geq 1$. It follows easily that the elements in the set-theoretic description $\gamma^{-1^*}\epsilon\gamma^*$ of $\langle\langle \gamma, \epsilon \rangle\rangle$ are distinct. Hence $\langle\langle \gamma, \epsilon \rangle\rangle \cong M$.

But it is also clear that there are no nontrivial automorphisms of the principal ideals of Y (so that T_Y is aperiodic and also that since each k_i covers two elements but each g_i only one, these two types of elements generate nonisomorphic ideals. Thus T_Y is generated by γ and ϵ. Further T_Y has two $\mathcal{D}$-classes, one the bicyclic semigroup $D_\gamma = \langle\langle \gamma \rangle\rangle$, the other D_ϵ. Since Y is subuniform, T_Y is simple. The other properties in the following theorem are easily verified.

THEOREM 3.2 *The inverse monoid M defined above is isomorphic to the Munn semigroup T_Y of the semilattice in Figure 1. It has the following properties:*

(i) The elements of M are in bijection with the set $z^{-1^*}\{e, 1\}z^*$;

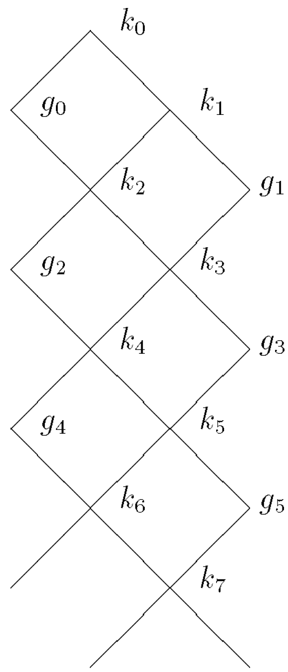

Figure 1: The semilattice Y

(ii) The element z generates a bicyclic inverse subsemigroup;
(iii) M is E-unitary, with M/σ infinite cyclic and $E_M \cong Y$;
(iv) M is aperiodic;
(v) E_M is Archimedean in M;
(vi) M is simple, with two $\mathcal{D}$-classes, one of width one, one of width two.

We remark that the E-unitariness of M may also be deduced directly, from the fact that it is an idempotent-pure quotient of the free product of the free inverse semigroup on z, which is E-unitary, and the trivial semigroup $\{e\}$.

THEOREM 3.3 *The inverse monoid M is neither meet semidistributive nor join semidistributive.*

Proof. Let $a = ez$, $b = ez^2$ and $c = a^{-1}b = z^{-1}ez^2$. Put $A = \langle a \rangle$, $C = \langle c \rangle$ and $B = (A \cap C) \vee \langle b \rangle$. We show that $C \cap A = C \cap B$ and $A \vee C = A \vee B$, but $A \| C$ in $\mathcal{LF}$, contradicting the first statement of Proposition 1.5.

We first need to establish some properties of a, b and c. The equations that are used in the proofs follow easily from equation (1) above, but are more transparent if viewed in the context of T_Y, using the relationships stated earlier, under which, for example, the idempotent $z^{-n}ez^n$ of E_M corresponds to g_n in Y. The corresponding isomorphisms map $Yg_0 \to Yg_1$, $Yg_0 \to Yg_2$ and $Yg_1 \to Yg_2$, respectively. Hence $a\mathcal{R}b\mathcal{L}c\mathcal{R}a^{-1}$.

(i) Since $aa^{-1} = e$ and $a^{-1}a = z^{-1}ez$, $aa^{-1}\|a^{-1}a$. Hence $D_a = \{aa^{-1}, a^{-1}a, a, a^{-1}\}$. Now $a^2a^{-1} = z^{-2}z^3$ is strictly right regular and therefore generates a bicyclic inverse subsemigroup with identity e. (In the terminology of [9], $\langle\langle a \rangle\rangle$ is of type $(2, \infty^-)$.) The remaining elements of $\langle\langle a \rangle\rangle$ form the $\mathcal{D}$-class $D_{a^2a^{-1}} = \langle\langle a^2a^{-1} \rangle\rangle$.

(ii) Similarly, since $cc^{-1} = z^{-1}ez$ and $c^{-1}c = z^{-2}ez^2$, $cc^{-1}\|c^{-1}c$ and $\langle\langle c \rangle\rangle$ consists of the two $\mathcal{D}$-classes $D_c = \{cc^{-1}, c^{-1}c, c, c^{-1}\}$ and $\langle\langle c^2c^{-1} \rangle\rangle$, where $c^2c^{-1} = z^{-3}z^4$ is strictly right regular and generates a bicyclic inverse subsemigroup with identity $z^{-1}ez$. Note also that $c^2c^{-1} = z^{-3}z^4 = a^{-1}a^2$.

(iii) Since $bb^{-1} = e$ and $b^{-1}b = z^{-2}ez^2$, b is strictly right regular and generates a bicyclic semigroup with identity e.

Now we show that $A \cap C = C \setminus \{c, c^{-1}\}$. From (i), $A \cap R_c = A \cap R_{a^{-1}} = \{a^{-1}, a^{-1}a\}$, so $c \notin A$. On the other hand, let x be a nonidempotent of C, so that $x = fy$ for some $f \in E_M$ and nonidempotent y of C. Suppose $y \in \langle\langle c^2c^{-1} \rangle\rangle$. By (ii), $c^2c^{-1} = a^{-1}a^2 \in A$ and $x \in A$ also. The alternative is that $y \in \{c, c^{-1}\}$. If $y = c$, suppose $f \geq z^{-1}ez = cc^{-1}$: then $x = c$; otherwise, using the isomorphism of E_M with Y, $f \leq e$. But then $x = fc = fec = fc^2c^{-1} = fa^{-1}a^2 \in A$. If $y = c^{-1}$, suppose $f \geq z^{-2}ez^2 = c^{-1}c$: then $x = c^{-1}$; otherwise $f \leq z^{-1}ez = cc^{-1}$ and $x = fc^{-1} = f(cc^{-2}) = f(a^{-2}a) \in A$. Hence $C \setminus \{c, c^{-1}\} \subseteq A$.

To show $A \cap C = B \cap C$, we observe that since $A \cap C \subseteq B$, the left hand side is contained in the right. From the preceding paragraph, it remains to show that $c \notin B = (A \cap C) \vee \langle b \rangle$. But neither $A \cap C$ nor $\langle b \rangle$ contains a nonidempotent in R_c (using (iii) for the latter), so no product in $(A \cap C) \vee \langle b \rangle$ can yield c.

The equation $A \vee C = A \vee B$ follows from $c = a^{-1}b$ and $b = a(a^{-1}b$. We have already seen that $c \notin A$; that $a \notin C$ is also clear, since $aa^{-1} \not\leq cc^{-1}, c^{-1}c$.

THEOREM 3.4 *Let S be an inverse semigroup that is simple, E-unitary with abelian maximal group quotient, aperiodic and simple, in which E_S is Archimedean and the maximum width of the posets E_D, over all $\mathcal{D}$-classes D is exactly two. Then S contains an inverse subsemigroup isomorphic to the monoid M defined above.*

Proof. Suppose f and h are such idempotents. Let $x \in R_f \cap L_h$.

Let us first consider $\langle\langle x \rangle\rangle$, using arguments similar to those in the proof of the previous theorem. Since $xx^{-1}\|x^{-1}x$, $D_x = \{xx^{-1}, x^{-1}x, x, x^{-1}\}$. Now since x^2x^{-2}, $(xx^{-1})(x^{-1}x)$

and $x^{-2}x^2$ are $\mathcal{D}$-related idempotents then, by hypothesis, at least two of them are comparable. By replacing x by its inverse, if necessary, we may assume that $x^2x^{-2} \geq (xx^{-1})(x^{-1}x)$, in which case it is also true that $(xx^{-1})(x^{-1}x) \geq x^{-2}x^2$. In fact, if any of these idempotents are equal, then x^2x^{-1} lies in the subgroup $H_{x^2x^{-2}}$ and is therefore idempotent by aperiodicity, contradicting E-unitariness, this element being in the same σ-class as x. Hence all the inequalities are strict, whence x^2x^{-1} is strictly right regular and generates a bicyclic inverse semigroup with identity x^2x^{-2} and satisfying $x^3x^{-1} = x^2$ (and so $x^ix^{-1} = x^{i-1}$ for $i \geq 3$).

By simplicity, S contains an idempotent g, say, such that $g\mathcal{D}x$ and $g \leq (xx^{-1})(x^{-1}x)$. Put $z = gxg$ and $e = (xgx^{-1})(x^{-2}gx^2)$. We will prove that $\langle\langle z, e\rangle\rangle \cong M$.

It will be convenient to introduce the idempotents $e_i = x^{-i}gx^i$, $i \geq 0$, (Here and throughout, the zeroth power refers to an adjoined identity element, so that $e_0 = g$). It will also be convenient to extend this notation by letting $e_{-1} = xgx^{-1}$. Clearly $x^{-1}e_{i-1}x = e_i$, for $i \geq 1$, but this equation also holds for $i = 0$, since $x^{-1}x > g$. Moreover, the inverse conjugation equations also hold, that is, $xe_ix^{-1} = e_{i-1}$ for $i \geq 0$. For $i = 0$ this is by definition and for $i = 1$, it follows from $xx^{-1} > g$. For $i \geq 2$, use the following: $gx^ix^{-1} = g(x^{-1}x)x^ix^{-1} = gx^{-1}x^{i+1}x^{-1} = gx^{-1}x^i = gx^{i-1}$. In fact these equations are equivalent to $xx^{-1} > e_i$, $i \geq -1$. Hence conjugation by x is an automorphism of the subsemilattice generated by $\{e_i : i \geq -1\}$. Note that we have also shown that all the idempotents e_i are $\mathcal{D}$-related.

From the definitions of e and z we have that:
(i) $e = (xgx^{-1})(x^{-2}gx^2) = e_{-1}e_2$,
(ii) $zz^{-1} = gxgx^{-1}g = g(xgx^{-1}) = e_0e_{-1}$,
(iii) $z^{-1}z = gx^{-1}gxg = g(x^{-1}gx) = e_0e_1$, and
(iv) $z^{-2}z^2 = (gx^{-1}g)e_0e_1(gxg) = g(x^{-1}(e_0e_1)x)g = e_0e_1e_2$.

Consider the three $\mathcal{D}$-related idempotents e_0, e_1 and e_2. According to Lemma 2.10, since $xx^{-1}\|x^{-1}x$, $e_0 = (gx)(gx)^{-1}\|(gx)^{-1}(gx) = e_1$, whence by the conjugation properties, $e_1\|e_2$. From the width assumption on $\mathcal{D}$-related idempotents it follows that e_0 and e_2 are comparable. Suppose $e_0 \leq e_2$, that is $g \leq x^{-2}gx^2$: then $g \leq x^{-2n}x^{2n}$ for all $n \geq 1$, contradicting the Archimedean property applied to the strictly right regular element x^2x^{-1}.

Hence $e_0 > e_2$ and so, by (ii) and (i), $zz^{-1} \geq e$. Further, by conjugation $e_{-1} > e_1$ and so, by (ii) and (iii), $zz^{-1} \geq z^{-1}z$; in fact this inequality is strict, for otherwise $z = gxg$ would be idempotent, contradicting E-unitariness. In addition, $e(z^{-1}z) = e_1e_2 = z^{-2}z^2$.

We have now shown that z and e satisfy the relations that define M. It remains to show that all the products in the set $z^{-1*}\{e, 1\}z^*$ are distinct. In fact, since M is aperiodic, it suffices to show that the (images of) the idempotents, namely the products $z^{-i}z^i$ and $z^{-i}ez^i$, $i \geq 0$, are distinct.

Since z generates a bicyclic inverse subsemigroup, the idempotents $z^{-i}z^i$, $i \geq 0$, are clearly distinct; and also since $z^iz^{-i} = zz^{-1} > e$, $ez^i\mathcal{R}e$, whence by E-unitariness and aperiodicity, $ez^i \notin H_e$ and $z^{-i}ez^i \neq e$, for all $i \geq 1$. Conjugation by z leads to the distinctness of the idempotents $z^{-i}ez^i$, $i \geq 0$.

Finally, suppose that $z^{-i}ez^i = z^{-j}z^j$, for some i, j. Then by conjugation, $e \in \langle\langle z \rangle\rangle$. If $e = zz^{-1}$, then $z^{-2}z^2 = e(z^{-1}z) = (zz^{-1})(z^{-1}z) = z^{-1}z$, which is impossible; similarly, $e \neq z^{-1}z$. Since $e \geq z^{-2}z^2 \geq z^{-i}z^i$ for $i \geq 2$, it follows that $e = z^{-2}z^2$. Using (i) and (iv) above, and $e_0 > e_2$, this becomes $e_{-1}e_2 = e_1e_2$. By conjugation we obtain $e_{k-3}e_k = e_{k-1}e_k$ for $k \geq 1$. Then $e_{k-5}e_k = e_{k-5}e_{k-3}e_k = e_{k-3}e_k = e_{k-1}e_k$ (using $e_{k-5} \geq e_{k-3}$) and by induction we obtain, for k even, at least, that $e_1e_k = e_{k-1}e_k$ (the odd k case not being needed).

To obtain a contradiction, we invoke the Archimedean principle, applied to the strictly right regular element gx^2: for some $n \geq 1$, $e_1 \geq (gx^2)^{-n}(gx^2)^n$. Now $(gx^2)^{-2}(gx^2)^2 = x^{-2}g(x^{-2}gx^2)gx^2 = x^{-2}e_0e_2x^2 = e_4$ and by induction, $(gx^2)^{-n}(gx^2)^n = e_{2n}$, for $n \geq 1$. So we have that $e_1 \geq e_{2n}$, that is $e_{2n} = e_1e_{2n}$, which by the equation at the end of the previous paragraph yields $e_{2n-1} \geq e_{2n}$. But then conjugation by x implies $e_0 \geq e_1$, and we have shown earlier that these idempotents are incomparable.

This completes the proof that all the given idempotents are distinct.

References

[1] T.I. Ershova, Inverse semigroups with certain types of inverse subsemigroups, *Mat. Zap. Ural. Univ., Sverdlovsk* **1**, no. 1 (1969), 62-76.

[2] R. Freese, J. Jezek. J.B. Nation, Free Lattices, Mathematical Surveys and Monographs, v. 42, American Mathematical Society, Providence, 1995.

[3] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.

[4] K. G. Johnston and P.R. Jones, Modular inverse semigroups, *J. Austral. Math Soc. (Series A)* **43** (1987), 47-63.

[5] P.R. Jones, Semimodular inverse semigroups, *J. London Math. Soc. (2)* **17** (1978), 446-456.

[6] P.R. Jones, Distributive inverse semigroups, *J. London Math. Soc. (2)* **17** (1978), 457-466.

[7] P.R. Jones, Inverse semigroups and their lattices of inverse subsemigroups, in Lattices, Semigroups and Universal Algebra, J. Almeida et al, eds., Plenum, New York, 1990.

[8] F. Napolitani, Elementi $\cup$-quasidistributivi nel reticolo dei sottogruppi di un gruppo, *Ricerche Mat.* **14** (1965), 93-101.

[9] M. Petrich, Inverse Semigroups, Wiley, New York, 1984.

[10] L.N. Shevrin and A.J. Ovsyannikov, Semigroups and Their Subsemigroup Lattices, Kluwer, Dordrecht, 1996.

[11] V.M. Shiryaev, Semigroups with $\wedge$-semidistributive subsemigroup lattice, *Semigroup Forum* **31** (1985), 47-68.

[12] M. Suzuki, Structure of a Group and the Structure of its Lattice of Subgroups, Springer-Verlag, Berlin, 1956.