On lattice isomorphisms of inverse semigroups

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Abstract

An $\mathcal{L}$-isomorphism between inverse semigroups S and T is an isomorphism between their lattices $\mathcal{L}(S)$ and $\mathcal{L}(T)$ of inverse subsemigroups. The author and others have shown that if S is aperiodic – has no nontrivial subgroups – then any such isomorphism Φ induces a bijection ϕ between S and T. We first characterize the bijections that arise in this way and go on to prove that under relatively weak 'archimedean' hypotheses, if ϕ restricts to an isomorphism on the semilattice of idempotents of S, then it must be an isomorphism on S itself, thus generating a result of Goberstein. The hypothesis on the restriction to idempotents is satisfied in many applications. We go on to prove theorems similar to the above for the class of completely semisimple inverse semigroups.

Over the past quarter-century, several authors have investigated the extent to which an inverse semigroup S is determined by its lattice $\mathcal{L}(S)$ of inverse subsemigroups (see the survey [8] and the monograph [12]): given an $\mathcal{L}$-isomorphism, that is, an isomorphism $\Phi : \mathcal{L}(S) \to \mathcal{L}(T)$ for some inverse semigroup T, how are S and T related? It is easily seen that since Φ restricts to an $\mathcal{L}$-isomorphism between their respective semilattices of idempotents, E_S and E_T, it induces a bijection ϕ_E between them. Following the lead of Goberstein [4] we focus here on the situation where ϕ_E is an isomorphism (see below for a rationale for this simplification).

It has long been known that ϕ_E extends to a bijection $\phi : E_S \cup N_S \to E_T \cup N_T$, where N_S denotes the set of elements that belong to no subgroup of S. In the aperiodic (or 'combinatorial') case where, by definition, all subgroups are trivial, ϕ is then a bijection between S and T. In turn, ϕ induces Φ in the obvious way. In this note we first characterize the bijections so obtained, in Theorem 2.3, and then in Theorem 4.3 find a general sufficient condition in order that this bijection should be an isomorphism, improving on some results of Goberstein [4].

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Since groups are not generally determined by their subgroup lattices, proving lattice determinability of nonaperiodic inverse semigroups must involve either some assumptions on the lattice determinability of the subgroups or some structural 'tying-in' of the subgroups into the overall form of the semigroup. It was shown by Ershova (see [12]) that as long as each nonaperiodic $\mathcal{D}$-class contains at least two idempotents (essentially a statement about the principal factors) then the partial bijection ϕ can be extended to a bijection θ between S and T. If S is also completely semisimple, θ again induces Φ. Again, we characterize the bijections that can arise in this way, in Theorem 2.5. We go on to show in Theorem 4.5 that, under similar hypotheses to the aperiodic case, θ is again an isomorphism. However, we shall see that assuming that Φ induces isomorphisms on both the semilattice of idempotents and the principal factors does not suffice to induce an isomorphism on the semigroup itself, without the extra hypothesis of Theorem 4.5.

The assumption that ϕ_E be an isomorphism is satisfied in many circumstances. For instance this occurs whenever S is simple or is E-unitary with a greatest $\mathcal{J}$-class which is nontrivial [7]. It also arises from external conditions. For instance, Goberstein showed (loc. cit.) that "most" isomorphisms between the inverse semigroups of partial automorphisms of two inverse semigroups induce an $\mathcal{L}$-isomorphism of this type. In [2], [3], K.H. Cheong and the author studied $\mathcal{C}o$-isomorphisms between inverse semigroups – isomorphisms between their lattices of convex inverse subsemigroups (which are not generally sublattices of the lattices of all inverse subsemigroups) – and reduced their study to those that induce an isomorphism between the respective semilattices of idempotents. In the completely semisimple case, they showed that $\mathcal{C}o$-isomorphisms with this property are equivalent to $\mathcal{L}$-isomorphisms of the same type. This equivalence is not true in general.

The two theorems on the bijections that induce $\mathcal{L}$-isomorphisms were inspired by Cheong's initial work on the corresponding problem for $\mathcal{C}o$-isomorphisms.

1   Preliminaries

The lattice $\mathcal{L}(S)$ of inverse subsemigroups of an inverse semigroup S has as its zero the empty inverse subsemigroup. If $U, V \in \mathcal{L}(S)$, their join is denoted $U \vee V$ and is the same as their join as subsemigroups. If $A \subseteq S$, we denote by $\langle A \rangle$ the inverse subsemigroup that it generates. The idempotents of S form a semilattice, denoted E_S. An inverse subsemigroup of S is *full* if it contains E_S. The filter $[E_S, S]$ in $\mathcal{L}(S)$ is denoted $\mathcal{LF}(S)$. Note that for a group G, $\mathcal{LF}(G)$ coincides with the subgroup lattice but the empty inverse subsemigroup must be adjoined to obtain $\mathcal{L}(G)$.

The *natural partial order* on an inverse semigroup S is defined by $a \leq b$ if $a = aa^{-1}b$; various useful equivalent properties may be found in [11], along with such basic semigroup concepts as Green's relations, ideals and principal factors. The notation $a||b$ means that a and b are incomparable with respect to that order. An inverse subsemigroup U is *convex*

if $a \in S$, $u, v \in U$ and $u \leq a \leq v$ imply that $a \in U$. The convex inverse subsemigroups form a lattice $\mathcal{C}o(S)$ which is not, in general, a sublattice of $\mathcal{L}(S)$. See [2], [3] for a comprehensive study of that lattice, with applications of the results contained herein.

An inverse semigroup is *aperiodic*, or *combinatorial*, if all of its subgroups are trivial; S is *completely semisimple* if each principal factor is completely 0-simple or is a group (equivalently, S contains no bicyclic subsemigroup – see below for the definition of the latter); S is *E-unitary* if whenever $a \geq e \in E_S$ then $a \in E_S$.

We review the basics of $\mathcal{L}$-isomorphisms of inverse semigroups. For surveys see [8] and [12]. Let $\Phi : \mathcal{L}(S) \to \mathcal{L}(T)$ be an isomorphism. Since $E_S\Phi = E_T$, Φ restricts to an isomorphism $\mathcal{L}(E_S) \to \mathcal{L}(E_T)$. It also follows that Φ restricts to an isomorphism $\mathcal{LF}(S) \to \mathcal{LF}(T)$.

Any $\mathcal{L}$-isomorphism Φ between semilattices E and F induces a bijection $\phi : E \to F$ by the rule $\langle e\phi \rangle = \langle e \rangle \Phi, e \in E$, and is in turn induced by ϕ, in the sense that $A\Phi = A\phi$ for every subsemilattice A. The bijection ϕ is characterized by the *weak isomorphism* property: if $e, f \in E$ then $e||f$ if and only if $e\phi||f\phi$, in which case $(ef)\phi = e\phi f\phi$.

If Φ is an $\mathcal{L}$-isomorphism between inverse semigroups S and T, we shall denote the weak isomorphism between E_S and E_T by ϕ_E. Note that the class of groups is closed under $\mathcal{L}$-isomorphisms, since groups are characterized among inverse semigroups by the property that their semilattices of idempotents are trivial, and ϕ_E is bijective. Hence if $e \in E_S$ then Φ restricts to an $\mathcal{L}$-isomorphism between the maximal subgroups H_e and $H_{e\phi_E}$. In particular, if S is aperiodic, so is T.

Let N_S denote the set of elements of S that do not belong to a subgroup. For each $a \in N_S$, by [7] there is a unique element b of T such that $\langle a \rangle \Phi = \langle b \rangle$, $(aa^{-1})\phi_E = bb^{-1}$ and $(a^{-1}a)\phi_E = b^{-1}b$. Rephrasing this statement, setting $\{b, b^{-1}\} = \{a\phi_N, a^{-1}\phi_N\}$ yields choice functions $\phi_N : N_S \to N_T$ that are bijections (since the same process may be applied to Φ^{-1}), and satisfy $a^{-1}\phi_N = (a\phi_N)^{-1}$, $\{aa^{-1}, a^{-1}a\}\phi_N = \{a\phi_N(a\phi_N)^{-1}, (a\phi_N)^{-1}a\phi_N\}$ and $\langle a \rangle \Phi = \langle a\phi_N \rangle$ for every $a \in N_S$. Since ϕ_E also satisfies the above properties, its union with any ϕ_N yields a bijection $\phi : E_S \cup N_S \to E_T \cup N_T$ with the same properties. The uniqueness of b implies that any bijection that induces Φ is of this type. Note that by always choosing $a\phi_N$ to be b itself, the partial bijection ϕ will then uniquely preserve $\mathcal{L}$ and $\mathcal{R}$. We follow [12] in terming this the 'base partial bijection'. While it will be convenient to make that assumption when necessary, we shall not always do so. We summarize the above discussion.

RESULT 1.1 *Let Φ be an $\mathcal{L}$-isomorphism between inverse semigroups S and T. There is a bijection $\phi : E_S \cup N_S \to E_T \cup N_T$, unique up to possible interchange of $a\phi$ and $a^{-1}\phi$ for each $a \in S$, such that ϕ restricts to a weak isomorphism from E_S to E_T, $\langle a \rangle \Phi = \langle a\phi \rangle$ and $\{aa^{-1}, a^{-1}a\}\phi = \{a\phi(a\phi)^{-1}, (a\phi)^{-1}a\phi\}$ for every $a \in E_S \cup N_S$.*

There is a unique choice of ϕ that preserves $\mathcal{L}$ and $\mathcal{R}$, namely the base partial bijection.

If S is aperiodic, ϕ becomes a bijection of S upon T that induces Φ, the properties of which we now summarize. The final statement below is a compilation of results of Ershova and the author that may be found in §42 of [12].

RESULT 1.2 *Let S be an aperiodic inverse semigroup. Any $\mathcal{L}$-isomorphism Φ from S to a (necessarily aperiodic) inverse semigroup T is induced by a bijection $\phi : S \to T$, unique up to possible interchange of $a\phi$ and $a^{-1}\phi$ for each $a \in S$, such that ϕ restricts to a weak isomorphism from E_S to E_T and $\{aa^{-1}, a^{-1}a\}\phi = \{a\phi(a\phi)^{-1}, (a\phi)^{-1}a\phi\}$ for every $a \in S$. There is a unique choice of ϕ that preserves $\mathcal{L}$ and $\mathcal{R}$, namely the base bijection. That choice of ϕ restricts to an isomorphism $\langle a \rangle \to \langle a\phi \rangle$ for each $a \in S$.*

We now turn to the nonaperiodic situation, keeping in mind the remarks in the introduction. Ershova showed (see [12, §43.7]) that the base partial bijection $\phi : E_S \cup N_S \to E_T \cup N_T$ may be extended to a bijection $\theta : S \to T$ as long as each nonaperiodic $\mathcal{D}$-class contains at least two idempotents, as follows. For each idempotent e in such a $\mathcal{D}$-class, choose an element r_e of $N_S \cap R_e$. Each $a \in H_e$ may be expressed uniquely in the form $a = r_e s^{-1}$ for some $s \in H_{r_e}$; put $a\theta = r_e \phi(s\phi)^{-1}$. (Define $\theta = \phi$ on N_S.) It follows from the construction that since ϕ preserves $\mathcal{L}$ and $\mathcal{R}$, so does θ. It was also shown that if Φ is induced by an isomorphism $\kappa : S \to T$, then $\kappa = \theta$. Again following [12], when θ is defined it is termed the *base bijection* associated with Φ.

We conclude these preliminaries with some further background on inverse semigroups.

According to [11, Theorem IX.3.11], each monogenic inverse semigroup is defined by exactly one of the following relations, where k, l are positive integers: (i) $a^k = a^{-1}a^{k+1}$; (ii) $a^k a^{-1} = a^{-1}a^k$; (iii) $a^k = a^{k+l}$; (iv) $a = a$. Each has a *type* associated with it: those in (i) are of type (k, ∞^+) and possess a bicyclic kernel (least ideal); those in (ii) are of type (k, ∞) and have an infinite cyclic group kernel; those in (iii) are of type (k, l) and have a finite cyclic group kernel; that in (iv) is free. In the first three cases, if $k = 1$ then the semigroup itself is bicyclic, infinite cyclic or finite cyclic, respectively. If $k \geq 2$, then it is an extension of its kernel by the quotient of the free monogenic inverse semigroup modulo the ideal generated by a^k (the quotient being a semigroup of type $(k, 1)$).

A semigroup is *group bound*, or an *epigroup*, if some power of each element belongs to a subgroup. From the classification above, it can be seen that a monogenic inverse semigroup is group bound if and only if its semilattice of idempotents is finite. In general, therefore, an inverse semigroup is group bound if and only if it contains no free monogenic nor bicyclic inverse subsemigroup. All periodic inverse semigroups are group bound.

We review from [11, page 47] the construction of retract ideal extensions. Let B be an inverse semigroup and A an inverse semigroup wih zero. Put $A^* = A - \{0\}$. Suppose $\psi : A^* \to B$ is a partial homomorphism, that is, $a\psi a'\psi = (aa')\psi$ whenever $a, a', aa' \in A^*$. Then the set $S = A^* \cup B$ becomes an inverse semigroup under the operation that extends that in B; extends the partial operation in A^*; is given by $aa' = a\psi a'\psi$ when $a, a' \in A^*$

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but $aa' = 0$ in A; and by $ab = a\psi b$ when $a \in A^*, b \in B$, and similarly for ba. It is the *retract* (and ideal) extension of B by A defined by ψ. (Taking the union of ψ with the identity map on B yields a retraction of S.)

2 Characterizing the induced bijection

LEMMA 2.1 *Let S and T be inverse semigroups and suppose $\phi : S \to T$ is a bijection that induces an isomorphism $\mathcal{LF}(S) \to \mathcal{LF}(T)$. If $a, b \in S$, $b \notin E_S$ and $b < a$ then $b\phi \leq (a\phi)^n$ for some nonzero integer n.*

Proof. If $b < a$ then $b = bb^{-1}a \in E_S \vee \langle a \rangle$, the full inverse subsemigroup of S generated by a, so $b\phi \in E_T \vee \langle a\phi \rangle$. Since $E_S\phi = E_T$, if $b \notin E_S$ then $b\phi \notin E_T$. Thus $b\phi$ can be expressed as the product of an idempotent and a nonzero power of $a\phi$, that is, $b\phi \leq (a\phi)^n$ for some nonzero integer n.

Following the terminology of [9], a product ab in an inverse semigroup is *restricted* if $a^{-1}a = bb^{-1}$, in which case $a\mathcal{R}ab\mathcal{L}b$.

The following lemma is well known, in that it may also be deduced from the standard constructions of the multiplication on an inverse semigroup by way of the restricted products and the partial order (for example, see [9]).

LEMMA 2.2 *Let S be an inverse semigroup. A subset A of S is an inverse subsemigroup if and only if (a) it is closed under restricted products and inverses, (b) E_A is a subsemilattice of E_S and (c) if $b \in S$, $bb^{-1} \in A$ and $b < a \in A$, then $b \in A$.*

Proof. Necessity of (a) and (b) is obvious; that of (c) follows from the equation $b = bb^{-1}a$. To prove sufficiency, let $a, b \in A$. Then $ab = (abb^{-1})(a^{-1}ab)$, where the product on the right hand side is restricted. It suffices, then, to show that $a^{-1}ab \in A$ (that $abb^{-1} \in A$ follows dually – observe that condition (c) is actually self dual, since the natural partial order respects inverses). Now $a^{-1}ab \leq b \in A$ and $(a^{-1}ab)(a^{-1}ab)^{-1} = (a^{-1}a)(bb^{-1})$ so it remains by (c) to show that $(a^{-1}a)(bb^{-1}) \in A$. The product $a^{-1}a$ is restricted, so $a^{-1}a \in A$, and similarly for bb^{-1}. Then (b) implies that the product belongs to A.

THEOREM 2.3 *Let S and T be aperiodic inverse semigroups. Any $\mathcal{L}$-isomorphism Φ from S to T is induced by a unique bijection $\phi : S \to T$ satisfying*

1. *ϕ restricts to a weak isomorphism from E_S to E_T;*

2. *ϕ preserves $\mathcal{L}$ and $\mathcal{R}$;*

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3. if $b < a$ in S, $b \notin E_S$, then $b\phi \leq (a\phi)^n$ for some integer n, and similarly for ϕ^{-1}.

Conversely, any bijection $\phi : S \to T$ that satisfies 1, 2 and 3 induces an $\mathcal{L}$-isomorphism from S to T.

The condition 3 can be replaced by 3': ϕ induces an isomorphism of $\mathcal{LF}(S)$ on $\mathcal{LF}(T)$.

Proof. The existence of the unique bijection $\phi : S \to T$ satisfying 1 and 2 was shown in §1. Clearly, since ϕ induces Φ, it induces the restriction of Φ to $\mathcal{LF}(S)$. Thus 3 follows from Lemma 2.1.

To prove the converse, assume 1,2 and 3 hold. (The final statement is immediate from Lemma 2.1.) It follows that $(a\phi)^{-1} = a^{-1}\phi$ for all $a \in S$. It is easily seen that ϕ^{-1} also satisfies 1 and 2. Hence, by symmetry it suffices to show that if $A \in \mathcal{L}(S)$ then $A\phi \in \mathcal{L}(T)$. We apply Lemma 2.2. Clearly $A\phi$ is closed under inverses. Since weak isomorphisms are precisely the bijections between semilattices that induce $\mathcal{L}$-isomorphisms between them, $E_{A\phi}$ is a subsemilattice of E_T. Let $a, b \in A$, with $(a\phi)^{-1}a\phi = b\phi(b\phi)^{-1}$, so that $a\phi \mathcal{R} a\phi b\phi \mathcal{L} b\phi$. Then since ϕ preserves $\mathcal{L}$ and $\mathcal{R}$, $a^{-1}a = bb^{-1}$, so $a\mathcal{R}ab\mathcal{L}b$ and therefore $a\phi \mathcal{R}(ab)\phi \mathcal{L} b\phi$. By aperiodicity, $a\phi b\phi = (ab)\phi \in A\phi$. So $A\phi$ is closed under restricted products. Finally, suppose $b \in S, a \in A$, with $b\phi(b\phi)^{-1} \in A\phi$ and $b\phi < a\phi$. Since ϕ preserves $\mathcal{R}$, $b\phi(b\phi)^{-1} = (bb^{-1})\phi$, so $bb^{-1} \in A$ and by 3, applied to ϕ^{-1}, $b \leq a^n$ for some integer n. But $a^n \in A$ so, by Lemma 2.2, $b \in A$ and $b\phi \in A\phi$, as required.

By the observations preceding Result 1.1, the condition 2 can be replaced by 2' $\{aa^{-1}, a^{-1}a\}\phi = \{a\phi(a\phi)^{-1}, (a\phi)^{-1}a\phi\}$ for every $a \in S$ as long as uniqueness of ϕ is replaced by uniqueness up to possible interchange of $a\phi$ and $a^{-1}\phi$ for each $a \in S$. This result then characterizes the bijections that can induce an $\mathcal{L}$-isomorphism between aperiodic inverse semigroups.

We now turn to the nonaperiodic situation and characterize the bijections that induce $\mathcal{L}$-isomorphisms. In general, of course, an $\mathcal{L}$-isomorphism between groups is not induced by a bijection. To obtain a complete analogue to the previous theorem we shall need to assume that, in the semigroups under consideration, every nonaperiodic $\mathcal{D}$-class contains at least two idempotents, so that an $\mathcal{L}$-isomorphism Φ from S to T induces the base bijection $\theta : S \to T$ defined in §1.

It seems that little is known of the properties of θ outside the completely semisimple situation. However, it was shown in [12, Lemma 44.5] that if S is a Brandt semigroup that is not a group with zero, then θ induces Φ and the question was raised when this occurs in general. We may extend the lemma to all completely semisimple inverse semigroups of the appropriate type, without difficulty. It is known [7] that the class of such semigroups is closed under $\mathcal{L}$-isomorphisms.

LEMMA 2.4 *Let S be a completely semisimple inverse semigroup in which each non-aperiodic $\mathcal{D}$-class contains at least two idempotents. If Φ is an $\mathcal{L}$-isomorphism from S to an inverse semigroup T then it is induced by the base bijection $\theta : S \to T$.*

Proof. Let $a \in S$. If $a \in E_S \cup N_S$ then $\langle a \rangle \Phi = \langle a\phi \rangle = \langle a\theta \rangle$. So suppose $a \in H_e$, $a \notin E_S$, $e \in E_S$. Put $r = r_e$, $a = rs^{-1}$ and $a\theta = r\phi(s\phi)^{-1} \in H_{e\phi}$. Following the same argument as in the lemma cited above, based on the fact that the principal factor associated with e is a Brandt semigroup (see [12, Exercise 7b, page 126]), $\langle a \rangle = \langle r, s \rangle \cap H_e$ and $\langle a\theta \rangle = \langle r\phi, s\phi \rangle \cap H_{e\phi} = \langle r, s \rangle \Phi \cap H_e \Phi = (\langle r, s \rangle \cap H_e)\Phi = \langle a \rangle \Phi$. Hence θ induces Φ.

THEOREM 2.5 *Let S and T be any completely semisimple inverse semigroups. An $\mathcal{L}$- and $\mathcal{R}$-preserving bijection $\theta : S \to T$ induces an $\mathcal{L}$-isomorphism Φ between S and T if and only if*

1. *θ restricts to a weak isomorphism from E_S to E_T;*

2. *if $b < a$ in S, $b \notin E_S$, then $b\theta \leq (a\theta)^n$ for some integer n, and similarly for θ^{-1};*

3. *for each $e \in E_S$, θ induces an $\mathcal{L}$-isomorphism between the subgroups H_e and $H_{e\theta}$;*

4. *if ab is a restricted product in S and $a, b \in N_S$ then (i) if $ab \in N_S$ then $(ab)\theta = a\theta b\theta$ and (ii) otherwise $(ab)\theta$ and $a\theta b\theta$ generate the same subgroup of T.*

If each nonaperiodic $\mathcal{D}$-class of S and T contains at least two idempotents, then any $\mathcal{L}$-isomorphism between them is induced by a unique $\mathcal{L}$- and $\mathcal{R}$-preserving bijection, namely the base bijection θ.

Proof. The last statement follows from Lemma 2.4. Otherwise, we follow the general outline of the proof of the previous theorem. Necessity of 1 has already been noted. That of 2 follows from the same argument as in the previous theorem. Since Φ restricts to an $\mathcal{L}$-isomorphism between H_e and $H_{e\phi_E} = H_{e\theta}$, 3 is immediate. To prove 4 we shall need to use complete semisimplicity in a similar manner to that of the proof of Lemma 2.4.

If $x \in N_S$ then $\langle x \rangle \cap D_x$ consists of the four distinct elements $x, x^{-1}, xx^{-1}, x^{-1}x$ (since its principal factor is a Brandt semigroup). Similarly, if ab is a restricted product in S and a, b, ab all belong to N_S, then since $ab \in R_a \cap L_b$, $\langle a, b \rangle \cap D_a = (\langle a \rangle \cup \langle b \rangle \cup \langle ab \rangle) \cap D_a$. (See [12, Exercise 7(a), page 126].) Since θ preserves $\mathcal{L}$ and $\mathcal{R}$, the product $a\theta b\theta$ is also restricted in T and $a\theta, b\theta, a\theta b\theta \in N_T$. Hence $\langle a\theta, b\theta \rangle \cap D_{a\theta} = (\langle a\theta \rangle \cup \langle b\theta \rangle \cup \langle a\theta b\theta \rangle) \cap D_{a\theta}$. Since $\langle a, b \rangle \theta = \langle a\theta, b\theta \rangle$, and since $(ab)\theta \in R_{a\theta} \cap L_{b\theta}$ then, in view of the description of $\langle x \rangle \cap D_x$ given above, this element belongs to $\langle a\theta b\theta \rangle$ and is $\mathcal{H}$-related to $a\theta b\theta$. Applying that same description, we obtain $(ab)\theta = a\theta b\theta$.

Suppose ab is a restricted product where $a, b \in N_S$ but $ab \notin N_S$. In that case $a \in H_{b^{-1}}$ and $ab \in H_f$, where $f = aa^{-1} = b^{-1}b$. Since θ induces an $\mathcal{L}$-isomorphism between H_f and $H_{f\theta}$, $\langle (ab)\theta \rangle = \langle ab \rangle \theta$. Because the principal factor containing a is a Brandt semigroup, $\langle ab \rangle = \langle a, b \rangle \cap H_f$ (see [12, Exercise 7(b), page 126]). Because θ induces Φ,

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$(\langle a, b \rangle \cap H_f)\theta = \langle a\theta, b\theta \rangle \cap H_{f\theta}$. Finally, since θ preserves $\mathcal{L}$ and $\mathcal{R}$, $\langle a\theta, b\theta \rangle \cap H_{f\theta} = \langle a\theta b\theta \rangle$, similarly.

To prove the converse, assume $1-4$ hold. It is again easily seen that θ^{-1} also satisfies these conditions. Hence it suffices to prove that if $A \in \mathcal{L}(S)$ then $A\theta \in \mathcal{L}(T)$. That $A\theta$ satisfies (b) and (c) of Lemma 2.2 proceeds as in the proof of the previous theorem. It therefore remains to show that it is closed under inverses and restricted products. To show the former property, let $a \in A$. If $a \notin N_S$, then $(a\theta)^{-1} \in \langle a\theta \rangle = \langle a \rangle \theta \subseteq A\theta$, using 3. If $a \in N_S$ then $a^{-1} \in N_S$ and the product aa^{-1} is restricted. By (4)(ii), $\langle a\theta a^{-1}\theta \rangle = \langle (aa^{-1})\theta \rangle = \{(aa^{-1})\theta\} = \{a\theta(a\theta)^{-1}\}$, whence since θ preserves $\mathcal{L}$ and $\mathcal{R}$, $(a\theta)^{-1} = a^{-1}\theta$.

Now suppose $a\theta b\theta$ is a restricted product in $A\theta$. Since θ and θ^{-1} preserve $\mathcal{L}$ and $\mathcal{R}$, ab is a restricted product in A. There are various cases to consider. Suppose $a, b \in N_S$. That $a\theta b\theta \in A\theta$ is immediate from the two cases of 4. If neither a nor b belongs to N_S then since $a^{-1}a = bb^{-1}$, both lie in a subgroup H_e, $e \in E_S$, say, and the requisite conclusion follows from 3.

If $a \notin N_S, a \in H_e$, say, and $b \in N_S$ then $a\mathcal{R}b$ and $ab\mathcal{H}b$, so that the product $(ab)b^{-1}$ is restricted, with $ab, b^{-1} \in N_S$. Similarly, $a\theta \in H_{e\theta}, b\theta \in N_T$ and $a\theta b\theta \mathcal{H} b\theta$. Let $c = (a\theta b\theta)\theta^{-1}\mathcal{H}b$. Then the product cb^{-1} in S is restricted, with $cb^{-1} \in H_e$. By (4)(ii), $\langle (cb^{-1})\theta \rangle = \langle c\theta b^{-1}\theta \rangle$ in $H_{e\theta}$. Now since $b \in N_S$, $b^{-1}\theta = (b\theta)^{-1}$, as proved above. Thus $\langle c\theta b^{-1}\theta \rangle = \langle a\theta b\theta(b\theta)^{-1} \rangle = \langle a\theta \rangle$. But from (3), $\langle a\theta \rangle = \langle a \rangle\theta$ and $\langle (cb^{-1})\theta \rangle = \langle cb^{-1} \rangle\theta$, so $\langle a \rangle = \langle cb^{-1} \rangle$. Therefore $cb^{-1} = a^n$ for some integer n and $c = a^n b \in A$. Hence $a\theta b\theta = c\theta \in A\theta$. The case where $a \in N_S, b \notin N_S$ is dual.

Similar comments to those following the previous theorem apply, regarding the condition that θ be $\mathcal{L}$- and $\mathcal{R}$-preserving.

Remark: the second paragraph of the proof is essentially that of the corresponding case of Lemma 4.4 below, and of the result of Ershova cited there (see [12, 44.6.3(a)]).

3 Archimedean properties

In this section we treat the appropriate hypotheses under which we shall eventually prove that every $\mathcal{L}$-isomorphism induces an isomorphism. Goberstein [4] introduced the notion of a 'shortly connected' inverse semigroup and proved (Theorem 5) that any $\mathcal{L}$-isomorphism between such semigroups, with the property that the weak isomorphism ϕ_E is actually an isomorphism, is induced by a unique isomorphism. He also introduced a slightly stronger property, 'shortly linked', which is more natural and more easily verified. (He showed in the sequel [5] that these two properties are distinct.) For that reason, we prefer to work with generalizations of the latter property.

An inverse semigroup S is *shortly linked* if for any idempotent e of S and any element

a of S such that $e < aa^{-1}$, the set $F_{e,a} = \{f \in E_{\langle a \rangle} : e < f \leq aa^{-1}\}$ is finite. (Goberstein actually introduced this property in a different form, then showed it to be equivalent to the above in [4, Proposition 3].) Many inverse semigroups turn out to be shortly linked in one of two ways: by virtue of the property that they contain only finitely many idempotents above any given one; or by virtue of being group bound.

PROPOSITION 3.1　　*1. An inverse semigroup is shortly linked if and only if no idempotent is strictly below infinitely many idempotents of any monogenic inverse subsemigroup;*

　　2. every group bound, and hence every periodic and every finite inverse semigroup, is shortly linked;

　　3. every free inverse semigroup and monogenic inverse semigroup is shortly linked.

Proof. 1. Let S be an inverse semigroup and $e \in E_S$, $a \in S$. Since every idempotent of $\langle a \rangle$ is below either aa^{-1} or $a^{-1}a$, $\{f \in E_{\langle a \rangle} : e < f\} = F_{e,a} \cup F_{e,a^{-1}}$, from which the stated equivalence is clear.

2. A group bound inverse semigroup is characterized by the property that each of its monogenic inverse subsemigroups has finite semilattice of idempotents.

3. By [10] every free inverse semigroup S is "finite $\mathcal{J}$-above", that is, for any $x \in S$ there are only finitely many elements y such that $SxS \subseteq SyS$. Hence no idempotent can be strictly below infinitely many others. A similar argument applies to each monogenic inverse semigroup.

An inverse semigroup S is *pseudo-archimedean* if no idempotent of S is strictly below every idempotent of a free monogenic or bicyclic inverse subsemigroup. Clearly, by 1 of the proposition, every shortly linked inverse semigroup is pseudo-archimedean. If an idempotent is below infinitely many idempotents of a bicyclic subsemigroup then it must be below all of them. However, this is not obviously so for the free monogenic inverse subsemigroups and we now quote an example to show that the pseudo-archimedean property is strictly weaker than that of being shortly linked.

EXAMPLE 3.2 *[3, Example 3.11]. There is an aperiodic, E-unitary, pseudo-archimedean inverse semigroup that is not shortly linked. The semigroup is generated as an ideal by a single nonidempotent.*

We may weaken the pseudo-archimedean hypothesis still further. Let us call S *faintly archimedean* if whenever an idempotent e of S is strictly below every idempotent of a bicyclic or free monogenic inverse subsemigroup $\langle a \rangle$, then $e < a$. Adjoining a zero to a free monogenic inverse semigroup yields a faintly archimedean inverse semigroup that is not

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pseudo-archimedean. (K.H. Cheong [1] called S *weakly archimedean* if this implication is required to hold for all monogenic inverse subsemigroups $\langle a \rangle$. She showed that any inverse semigroup whose lattice of convex inverse subsemigroups is lower semimodular has this property. The term 'archimedean' alludes to its use in earlier work by the author [6].)

Note that if a monogenic inverse subsemigroup $\langle a \rangle$ is of type (k, ∞^+) for some positive integer k, then $a^{k+1}a^{-k}$ generates its bicyclic kernel, where $a^{k+1}a^{-k} \leq a$. Thus S is faintly archimedean if and only if whenever e is below every idempotent of $\langle a \rangle$, where $E_{\langle a \rangle}$ is infinite, then $e < a$.

While this property will be an adequate hypothesis in the aperiodic case, it needs to be strengthened slightly to cover the general situation. Let us call S *quasi-archimedean* if whenever an idempotent e is (not necessarily strictly) below every idempotent of $\langle a \rangle$, where $a \in N_S$, then $e < a$.

PROPOSITION 3.3 *The following are equivalent for an inverse semigroup S:*

1. S *is quasi-archimedean;*

2. *if $a \in N_S$, $b < a$ and bb^{-1} is below every idempotent of $\langle a \rangle$, then $b \in E_S$;*

3. S *is faintly archimedean and $\langle a \rangle$ is aperiodic for each $a \in N_S$;*

4. *for each $a \in N_S$, $\langle a \rangle$ is aperiodic and whenever $b < a$, $bb^{-1} < a^n a^{-n}$ and $b^{-1}b < a^{-n}a^n$ for every positive integer n, then $b \in E_S$.*

Proof. Throughout the proof, e denotes an idempotent of S and a an element of N_S.
$1 \Rightarrow 2$. With a, b as stated, then by 1, $bb^{-1} < a$. Thus $b = bb^{-1}a = bb^{-1}$.

$2 \Rightarrow 3$. If e is strictly below each idempotent of $\langle a \rangle$ then $b = ea < a$ and $bb^{-1} = e$. So by 2, $bb^{-1} = b < a$. In particular, this holds whenever $\langle a \rangle$ is bicyclic or free. Now if $\langle a \rangle$ has no least idempotent then from the discussion in §1 it is aperiodic. In the alternative case it has a group kernel with identity e, say, which is clearly below every idempotent of $\langle a \rangle$ and so is below a. Again setting $b = ea$, by a similar argument to the above we obtain $b \in E_S$. But the kernel is generated by b and so is trivial, that is, $\langle a \rangle$ is aperiodic.

$3 \Rightarrow 1$. Suppose e is below every idempotent of $\langle a \rangle$. On the one hand, if $E_{\langle a \rangle}$ is infinite then, by hypothesis, $e < a$. On the other hand, if $E_{\langle a \rangle}$ is finite, then $\langle a \rangle$ is aperiodic and its least idempotent is $a^{n+1} = a^n$ for some positive integer n. Then $e \leq a^n = a^n a < a$.

$2 \Rightarrow 4$. Under the assumption of 4, $bb^{-1} = b(b^{-1}b)b^{-1} \leq a(a^{-n}a^n)a^{-1} \leq a^{-(n-1)}a^{n-1}$ for $n > 1$, so in fact bb^{-1} is below every idempotent of $\langle a \rangle$ and 2 applies. Aperiodicity follows as in $2 \Rightarrow 3$.

$4 \Rightarrow 3$. Suppose e is below every idempotent of $\langle a \rangle$, where $\langle a \rangle$ is bicyclic or free. Let $b = ea$. Then $b < a$, $bb^{-1} = e < a^n a^{-n}$ and $b^{-1}b = a^{-1}ea < a^{-1}a^{-n}a^n a$ for every positive

integer n. By 4, $ea \in E_S$, that is, $e = ea < a$.

COROLLARY 3.4 *An aperiodic inverse semigroup S is quasi-archimedean if and only if it is faintly archimedean.*

4 $\mathcal{L}$-determinability

In this section we generalize the result of Goberstein, cited earlier, on aperiodic inverse semigroups, and prove a similar result on completely semisimple nonaperiodic inverse semigroups.

LEMMA 4.1 *Let S and T be inverse semigroups and suppose $\phi : S \to T$ is a bijection that induces an isomorphism $\mathcal{LF}(S) \to \mathcal{LF}(T)$, restricts to an isomorphism on E_S and satisfies $\{aa^{-1}, a^{-1}a\}\phi = \{a\phi(a\phi)^{-1}, (a\phi)^{-1}a\phi\}$ for every $a \in S$. If $a \in S$, $e \in E_S$ and $e < a$, then $e\phi < a\phi$.*

Proof. Since the conclusion is unaffected by interchanging $a\phi$ with $a^{-1}\phi$, we may assume that ϕ preserves $\mathcal{L}$ and $\mathcal{R}$. Now $e < aa^{-1}$, so that since ϕ restricts to an isomorphism on E_S, $e\phi < (aa^{-1})\phi = (a\phi)(a\phi)^{-1}$. Thus $e\phi\mathcal{R}e\phi a\phi$ and so if we put $c = (e\phi a\phi)\phi^{-1}$, then $c\mathcal{R}e$. Hence $e\phi a\phi \in E_T$, that is, $e\phi < a\phi$. For otherwise, applying Lemma 2.1 to ϕ^{-1} yields $c \leq a^k$ for some nonzero integer k, so that $c = ea^k = e$, a contradiction.

LEMMA 4.2 *Let S be any quasi-archimedean inverse semigroup and suppose that ϕ is a bijection of S upon an inverse semigroup T that (i) restricts to an isomorphism on E_S, (ii) preserves $\mathcal{L}$ and $\mathcal{R}$, (iii) restricts to an isomorphism on $\langle a \rangle$ for each $a \in N_S$ and (iv) induces an isomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(T)$. Then whenever $b < a$ in S and $a \in E_S \cup N_S$, $b\phi < a\phi$.*

Proof. Since ϕ restricts to an isomorphism on E_S, the conclusion holds if $a \in E_S$, so assume $a \in N_S$. If $b \in E_S$ then Lemma 4.1 applies. Otherwise, according to Lemma 3.3(4), either there is a greatest positive integer n such that $bb^{-1} < a^n a^{-n}$ or the (left-right) dual of that statement holds. In the latter case, a dual argument applies, so consider the former possibility. Note that in conjunction with $b < a$, the stated inequality is equivalent to $b < a^n a^{-(n-1)}$. By Lemma 2.1, $b\phi \leq (a^n a^{-(n-1)})^k \phi$ for some nonzero integer k.

Suppose $k = 1$. Since ϕ restricts to an isomorphism on $\langle a \rangle$, $(a^n a^{-(n-1)})\phi = (a\phi)^n (a\phi)^{-(n-1)} \leq a\phi$, yielding the desired conclusion.

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Next suppose $k > 1$. Then by a simple inductive argument, $(a^n a^{-(n-1)})^k = a^{n+k-1} a^{-(n-1)}$. Hence $(bb^{-1})\phi \leq ((a^{n+k-1} a^{-(n-1)})(a^{n+k-1} a^{-(n-1)})^{-1})\phi = (a^{n+k-1} a^{-(n+k-1)})\phi$. Now ϕ is an isomorphism on E_S, so $bb^{-1} \leq a^{n+k-1} a^{-(n+k-1)}$. Since, for $k > 1$, $a^{n+k-1} a^{-(n+k-1)} \leq a^{n+1} a^{-(n+1)}$, the maximality assumption on n yields $bb^{-1} = a^{n+1} a^{-(n+1)}$. From $b < a$ we then obtain $b = a^{n+1} a^{-n}$ and so $b\phi = (a\phi)^{n+1} (a\phi)^{-n} \leq a\phi$, as required.

Finally, suppose $k < 0$. Now, by similar arguments, $b^{-1}b \leq a^{n-k-1} a^{-(n-k-1)} \leq a^n a^{-n}$ whence, since $b < a$, $bb^{-1} \leq a^{n+1} a^{-(n+1)}$ once more and the concluding argument of the preceding case applies.

4.1   Aperiodic inverse semigroups

THEOREM 4.3 *Let S be any aperiodic, quasi-archimedean (equivalently, faintly archimedean) inverse semigroup and let Φ be an $\mathcal{L}$-isomorphism from S to an inverse semigroup T such that the weak isomorphism $\phi_E : E_S \to E_T$ is actually an isomorphism. Then Φ is induced by a unique isomorphism, namely the base bijection ϕ defined in §1.*

Proof. Let $\phi : S \to T$ be as stated. According to [9, Theorem 3.1.5], if a mapping between two inverse semigroups is order preserving, restricts to a homomorphism on the semilattice of idempotents and preserves restricted products, then it is a homomorphism.

Let ab be a restricted product. Then since ϕ respects Green's relations, $a\phi b\phi$ is also such a product and the resulting element is $\mathcal{H}$-related to $(ab)\phi$, whence equal, by aperiodicity.

Since ϕ induces Φ, it induces its restriction $\mathcal{LF}(S) \to \mathcal{LF}(T)$. By Lemma 4.2, it is order preserving. Hence it is an isomorphism.

In [4, Proposition 9], Goberstein constructed nonisomorphic aperiodic, completely semisimple inverse semigroups S and T whose partial automorphism semigroups, and hence their lattices of inverse subsemigroups, are isomorphic. It follows from his construction that the induced bijection restricts to an isomorphism between their semilattices of idempotents.

As discussed in the introduction, the restriction that the weak isomorphism $\phi_E : E_S \to E_T$ be an isomorphism is not a strong one. As remarked there, if S is E-unitary, then ϕ_E is an isomorphism on the idempotents of any ideal generated by a nontrivial $\mathcal{J}$-class. We may apply this fact to the semigroup in Example 3.2, which is E-unitary and is generated as an ideal by a single nonidempotent. Recalling that every pseudo-archimedean inverse semigroup is faintly archimedean, we may apply Theorem 4.3.

Other applications, some of which were noted by Goberstein, are to new proofs of determinability of free inverse semigroups and of simple inverse semigroups whose lattices

of full inverse subsemigroups are modular.

4.2   Nonaperiodic, completely semisimple inverse semigroups

For Brandt semigroups with exactly two nonzero idempotents it is apparently unknown whether the base bijection θ is always an isomorphism (see [13, Problem 7.11]). However, Ershova proved the remarkable result [12, Theorem 44.6] (see also [13, Corollary 2.7]) that for Brandt semigroups with at least three nonzero idempotents, θ is always an isomorphism. In a similar fashion to the proof of the above lemma, this result may be incorporated into a result for completely semisimple inverse semigroups by noting that since θ preserves $\mathcal{L}$ and $\mathcal{R}$, if $ab = 0$ in a Brandt semigroup then $a\theta b\theta = 0$ in the image semigroup. The products that are nonzero are then precisely the restricted ones (those for which $a^{-1}a = bb^{-1}$) and the proof that $(ab)\theta = a\theta b\theta$ for such products in Brandt semigroups can simply be interpreted in the appropriate completely semisimple inverse semigroups, as follows. (Another proof may be obtained by inducing a lattice isomorphism between the corresponding principal factors of S and T.)

LEMMA 4.4 *Let S be a completely semisimple inverse semigroup with the property that each nonaperiodic $\mathcal{D}$-class contains at least three idempotents. If Φ is an $\mathcal{L}$-isomorphism from S to an inverse semigroup T then the bijection $\theta : S \to T$ preserves restricted products.*

THEOREM 4.5 *Let S be any completely semisimple, quasi-archimedean inverse semigroup in which each nonaperiodic $\mathcal{D}$-class contains at least three idempotents, and let Φ be any $\mathcal{L}$-isomorphism from S to an inverse semigroup T such that the weak isomorphism $\phi_E : E_S \to E_T$ is actually an isomorphism. Then Φ is induced by a unique isomorphism, namely the base bijection θ defined in §1.*

   Proof. By Lemma 4.4, θ preserves restricted products and since it extends ϕ, it restricts to an isomorphism on E_S. According to the first paragraph of the proof of Theorem 4.3, to prove it is an isomorphism it remains to show it is order preserving.

   By Lemma 2.4, θ induces Φ and so induces an isomorphism $\mathcal{LF}(S) \to \mathcal{LF}(T)$; and by Proposition 3.3, $\langle a \rangle$ is aperiodic for each $a \in N_S$, so the restriction of θ agrees with ϕ thereon and is therefore an isomorphism. Hence, if $b < a$ in S and $a \in E_S \cup N_S$, then $b\phi < a\phi$, by Lemma 4.2.

   To conclude the proof, suppose $b < a$, with $a \notin E_S \cup N_S, a \in H_f$, $f \in E_S$. Put $e = bb^{-1} < aa^{-1} = f$. Since D_a is nonaperiodic, R_f contains an element r of N_S and $a = rr^{-1}a = r(a^{-1}r)^{-1}$, where $a^{-1}r \in H_r \subset N_S$ (similarly to the definition of θ in §1). Then $b = eb = efb = err^{-1}b = (er)(b^{-1}r)^{-1}$, where this last product is easily verified to be restricted. According to the first sentence of the proof,

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$b\theta = (er)\theta(b^{-1}r)^{-1}\theta = (er)\theta((b^{-1}r)\theta)^{-1}$. But $er < r \in N_S$ and $b^{-1}r < a^{-1}r \in N_S$, so applying the previous paragraph we obtain $(er)\theta < r\theta$ and $(b^{-1}r)\theta < (a^{-1}r)\theta$, whence $((b^{-1}r)\theta)^{-1} < ((a^{-1}r)\theta)^{-1}$. Hence $b\theta < r\theta((a^{-1}r)\theta)^{-1} = a\theta$ (reversing the previous argument), as required.

We now give an example to show that the quasi-archimedean hypothesis is necessary for the above theorem to hold. We first provide a general construction of certain $\mathcal{L}$-isomorphic semigroups.

PROPOSITION 4.6 *Let B be an inverse semigroup and A an inverse semigroup with zero. Let S, S' be retract extensions of B by A, via partial homomorphisms ψ, ψ' that satisfy $\langle a\psi \rangle = \langle a\psi' \rangle$ for all $a \in A^*$. Then the identity map $I : \mathcal{L}(S) \to \mathcal{L}(S')$ is an isomorphism.*

Proof. Let $U \in \mathcal{L}(S)$. Let U' denote the set U, considered as an inverse subsemigroup of S'. In both S and S', A^* and B are closed under the inverse operation and under restricted products, since each is a union of $\mathcal{D}$-classes. By hypothesis, ψ and ψ' agree on the idempotents of A^*, so $E_{U'} \cong E_U$. Finally, let $u, v \in S'$, with $vv^{-1}, u \in U'$. If both u, v belong to either B or A^* then since the operations on U, U' coincide there, $v \in U'$. The remaining case is where $v \in B$ and $u \in A^*$. Now $v < u$ in S' if and only if $v \leq u\psi'$ in B. Since $\langle u\psi' \rangle = \langle u\psi \rangle$, $u\psi' \leq (u\psi)^n$ for some (without loss of generality) nonzero integer n. Now if $u^n \in A^*$ then $(u\psi)^n = u^n\psi$; if not, $(u\psi)^n = u^n$. In either event, $v \leq u^n \in U$ in S and so $u \in U$ in S, that is, $u \in U'$ in S'. By Lemma 2.2, $U' \in \mathcal{L}(S')$.

By symmetry, this map yields an order isomorphism between $\mathcal{L}(S)$ and $\mathcal{L}(S')$.

Now let $A = \langle a : a^5 = a^2 \rangle$ and let B be the Brandt semigroup $\mathcal{M}^0(X, G, X, I)$, where $G = \langle g : g^3 = 1 \rangle$ and $X = \{1, 2, 3\}$ (see [11] for this construction). Mapping a to g induces a partial homomorphism $\psi : A^* \to G$; a partial homomorphism ψ' may be defined similarly by mapping a to g^2. Here $\langle a\psi \rangle = G = \langle a\psi' \rangle$. Since G is isomorphic to the maximal subgroup $H_{(1,1,1)}$ of B, we may regard ψ and ψ' as mapping into B. The hypotheses of the proposition having been met, the identity map is an $\mathcal{L}$-isomorphism between the two semigroups S, S' defined there. Since the induced partial bijection ϕ is just the identity map it follows from the construction of θ (and the fact that the copies of B in S and in S' are isomorphic, by construction) that θ is also the identity map. However the identity map between S and S' is not an isomorphism, since $a^2 = (a\psi)^2 = g^2$ in S whereas $a^2 = (a\psi')^2 = g^4 = g$ in S'. As noted above, if Φ is induced by any isomorphism, that mapping must be θ. Hence we have exhibited the following.

EXAMPLE 4.7 *There exist two finite inverse semigroups, in each of which every non-aperiodic $\mathcal{D}$-class possesses at least three idempotents, and an $\mathcal{L}$-isomorphism between*

them that induces an isomorphism ϕ_E between their semilattices of idempotents and is induced by a bijection (namely θ) that preserves restricted products, but which is not induced by any isomorphism between the semigroups themselves. (In particular, the base bijection θ is not an isomorphism).

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